CHAPTER 4

RATE OF CHANGE

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4.1 RATE OF CHANGE

Fig. 1 describes a trip that Mary is taking on her bike. The origin of the coordinate system is taken to be s = 0 at the location of Town Hall (TH). She starts her trip at her house (H) located at position $s_1 = 2$ miles at time $t_1 = 0$ hours and travels past her school (S) to the lake (L) located at $s_2 = 5$ miles reaching there at time $t_2 = 0.25$ hours (fifteen minutes). During this trip she occasionally slows down and then speeds up. To compute her average speed over the duration of the trip

$$\boxed{\text{speed}_{\text{avg}} = \frac{s_2 - s_1}{t_2 - t_1} = \frac{5-2}{0.25-0} = 12\,\text{mph}}$$

This expression is called a *difference quotient*. It measures the rate of change of distance with respect to time. So average speed is one example of a rate of change.

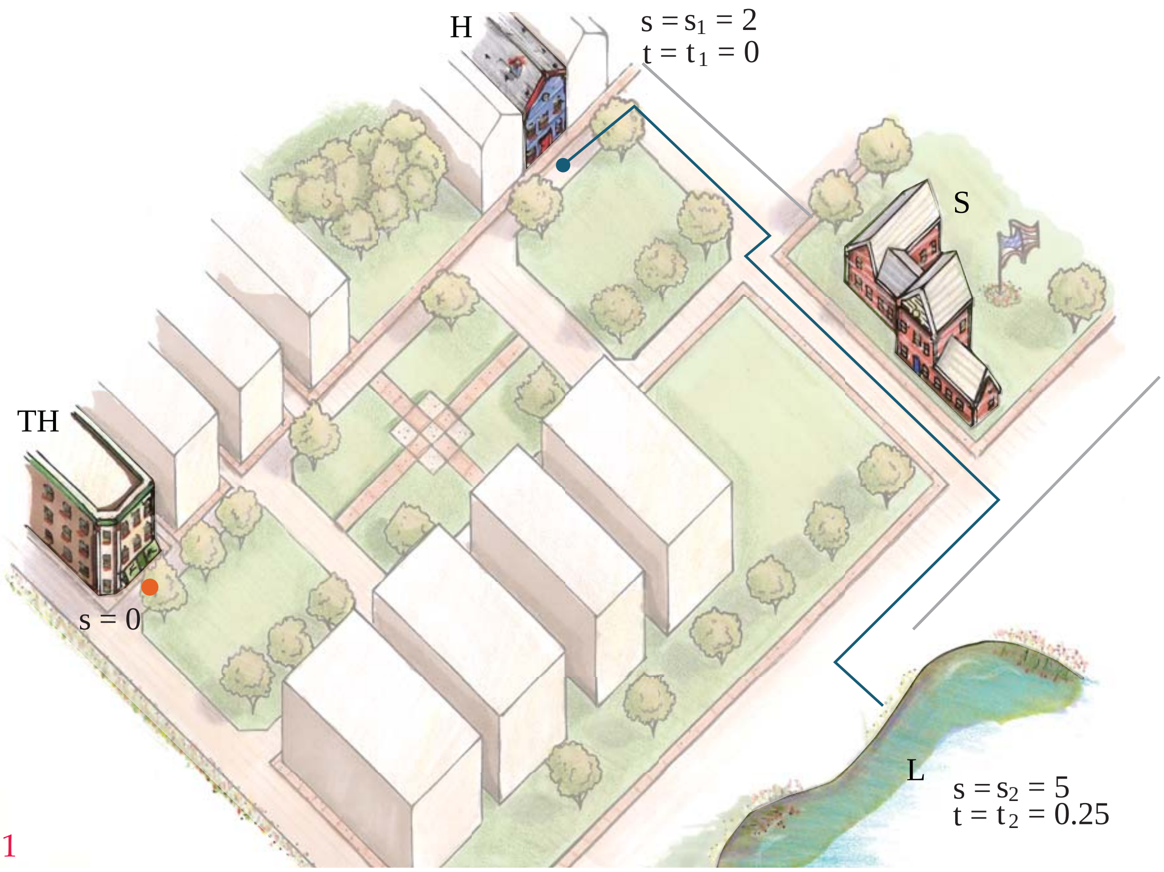


Fig.1

In general, the *average rate of change* of a function f over the interval from x = a to x = b of the graph of the function is given by the difference quotient :

$$\boxed{\frac{f(b)-f(a)}{b-a} = \frac{\Delta y}{\Delta x} = \text{change in y values/change in x values}} \qquad (1)$$

where y = f(x)

Notice that if you draw a line between the points: (a, f(a)) and (b, f(b)) as shown in fig. 2, then the difference quotient is the slope of the line.

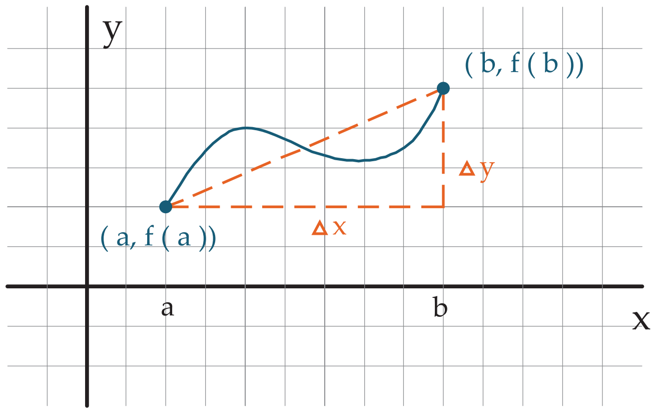


Fig. 2

The difference quotient is a measure of the rate at which the output values of a function change as the input values change. In fact, the difference quotient determines the average rate of change of the function. You can find many examples of rates of change in the problems at the end of this chapter. Here is one example:

Example 1: High levels of PCB in the environment affect pelican' eggs. Table 1 shows that as the concentration of PCB, C, in eggshells increases, the thickness, h, of the eggshell decreases, making the eggs more likely to break. Find the average rate of change in the thickness of the shell as the PCB concentration changes from 87 ppm to 452 ppm. Give units and explain why your answer is negative.

Table 1.

Concentration, C, in parts per million (ppm)	87	147	204	289	356	452
Thickness, h, in millimeters (mm)	0.44	0.39	0.28	0.23	0.22	0.14

Solution: Since we are looking for the average rate of change of thickness with respect to a change in PCB concentration, we have,

(Avg. rate of change between C = 87 and C= 452) = Change thickness/Change in the PCB level

$$\boxed{=\frac{\Delta h}{\Delta C}=\frac{0.14-0.44}{452-87}=-0.00082\frac{mm}{ppm}}$$

The units are thickness units (mm) over PCB concentration units (ppm), or millimeters over parts per million. The average rate of change is negative because the thickness of the eggshell *decreases* as the PCB concentration *increases*. The thickness of pelican eggs decreases by an average of 0.00082 mm for every additional part per million of PCB in the eggshell.

4.2 INSTANTANEOUS RATE OF CHANGE

Returning to Mary's trip to the lake, at any given time, t, her bike is traveling at a particular speed. Do you see that the closer that a later time, t_{later}, is to t the better estimate that the average speed will be to the actual or *instantaneous* speed at time t. Likewise, the average rate of change of the function f between a and b better reflects the instantaneous rate of change at x = a the closer that b is to a.

Example 2: From Table 1 we found that the avg. rate of change between C = 87 and C = 452 is 0.00082. We can find a better estimate to the instantaneous rate at C= 87 ppm by the difference quotient between C = 87ppm and C= 147ppm,

$$\text{Instantaneous rate of change at C = 87} \approx \frac{\Delta h}{\Delta C} = \frac{0.39-0.44}{147-87} = -0.0008333\frac{mm}{ppm}$$

If we let b = a + h then the difference quotient, (Eq. 1) can be written as :

$$\boxed{\frac{f(a+h)-f(a)}{h}} \qquad (2)$$

which we encountered in Expression 1 of Section 2.1. Looking ahead, the actual or instantaneous rate of change at x = a can be estimated by taking h to be small. In fact what happens if you let h approach 0? Try to do this for the results of Problem 2 of Capter 2.

Remark 1: While the average rate of change over some interval of x is the slope of the line drawn between the endpoints of the interval, the instantaneous rate of change at x = a will be the slope of the line tangent to the graph of y = f(x) at the point x = a.

Remark 2: We have seen that the average rate of change of the function y = f(x) can be represented by the difference quotient, $\frac{\Delta y}{\Delta x}$. Since the instantaneous rate of change is such an important quantity, it is represented at x = a by the notation, $\frac{dy}{dx}$, called the derivative of y = f(x) with respect to x. Geometrically the derivative is the slope of the line tangent to the graph of y = f(x) at x = a. In Chapters 9 and 10, we will introduce a computational method to determine derivatives.

Example 3: Apply Expression 1 to f(x) = x^2 to get,

$$\boxed{\text{Avg. rate of change} = \frac{(a+h)^2-a^2}{h} = \frac{2ah+h^2}{h} = 2a + h}$$

So the average rate of change = 2a + h. Now let h approach 0 and you find that,

$$\text{Instantaneous rate of change = 2a for any value of x = a, i.e. } \frac{dy}{dx} = 2x \text{ for any value of x .}$$

4.3 RATE OF CHANGE INFLUENCES THE NATURE OF THE CURVE

If the value of a function is changing we can identify the rate of change of the output as the input changes with the slope to the graph of the function.

a. Increasing and decreasing function

If the values of the function increase, as in Fig. 3, the rate of change or the slope of the curve will be positive. Therefore, positive rates of change indicate increasing functions. Alternatively, if the rate of change is negative then the values of the function will decrease .

Example 4: Table 2

x	0	5	10	15	20	25	30
y = f(x)	12.6	13.1	14.1	16.2	20.0	29.6	42.7
Δy		0.5	1.0	2.1	3.8	9.6	13.1

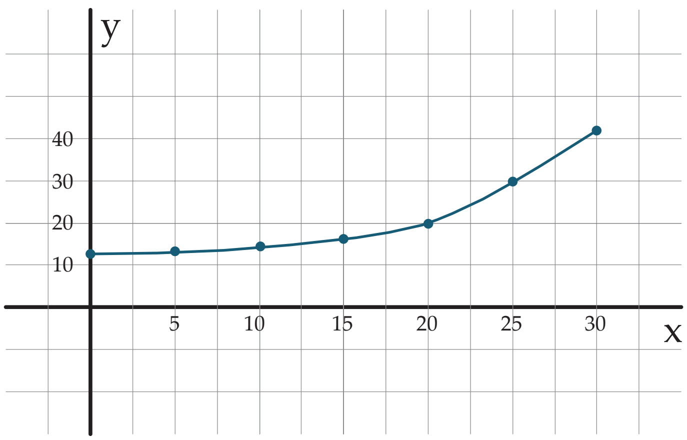


Fig. 3

Notice that the values of the function continuously increase as x increases (see Fig. 3), i.e., $\Delta y > 0$. . Now calculate the best approximation of the rate of change at any of the given values of this function by dividing Δy in Table 2 by $\Delta x = 5$. For example, the best approximation to the rate of change at x = 0 and 5 are,

$$\text{Rt. of Change at x = 0} \approx \frac{13.1-12.6}{5-0} = 0.1, \quad \text{Rt. of Change at x = 5} \approx \frac{14.1-13.1}{5-0} = 0.2$$

Likewise the rate of change at any of any other value of this function will be positive.

a. Convexity

If the rate of change of a function increases, then its slope increases and the curve will look like one of the curves in Fig. 4a below. These curves are said to be *concave up*. If the rate of change of the function decreases then the slope decreases and the curve is said to be *concave down*. It will look like one of the curves in Fig. 4b.

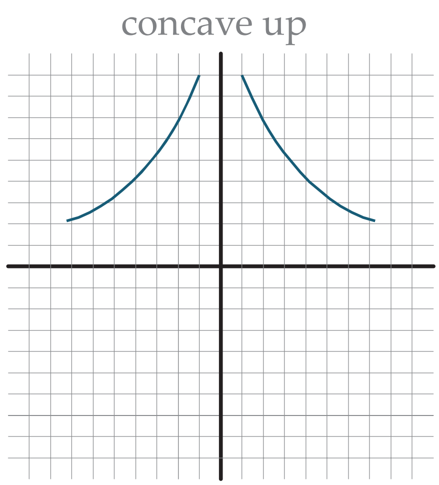


Fig. 4a

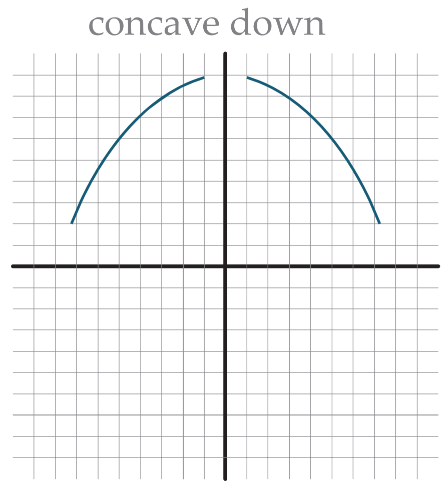


Fig. 4b

Example 5: Notice that the difference of the values of the function, Δy, in Table 2 continually increase so the rate of change increases and the curve is concave up as shown in Fig. 3.

So, by just looking at a table of data and evaluating Δy, you can see where its graph is increasing, decreasing or concave up or concave down. The following table lists the height of a grapefruit above the ground after it is thrown into the air as a function of time. In this case, rate of change is called the velocity. Notice that the height of the grapefruit first increases ($\Delta y > 0$) and then decreases ($\Delta y < 0$) as shown in Fig 5.

Table 3

t (sec)	0	1	2	3	4	5	6
y (feet)	6	90	142	162	150	106	30
Δy		84	52	20	-8	-44	-76

Remark 3: Since $\Delta x = 1$, Δy represents the approximate rate of change.

This is confirmed by calculation of the Δy between each pair of y values which is positive up until t = 3 and is then negative. Notice that Δy continually decreases in its value from t = 0 to t = 6, so the curve is concave down.

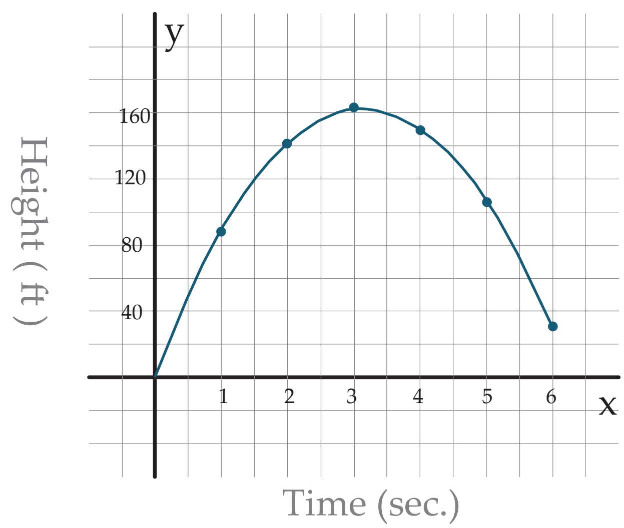


Fig. 5

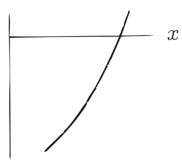
Problems

In Problems 1–4, decide whether the graph is concave up, concave down, or neither.

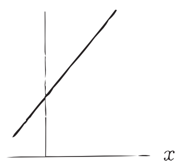
1.

2.

3.



4.



5. Table 1.9 gives values of a function $w = f(t)$. Is this function increasing or decreasing? Is the graph of this function concave up or concave down?

Table 1.9

t	0	4	8	12	16	20	24
w	100	58	32	24	20	18	17

6. Identify the x-intervals on which the function graphed in Figure 1.33 is:

(a) Increasing and concave up
(b) Increasing and concave down
(c) Decreasing and concave up
(d) Decreasing and concave down

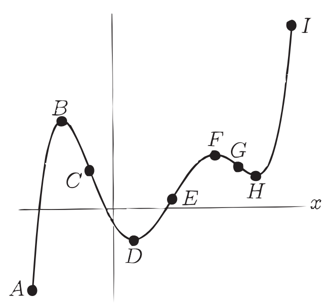


Figure 1.33

7. When a new product is advertised, more and more people try it. However, the rate at which new people try it slows as time goes on.

(a) Graph the total number of people who have tried such a product against time.
(b) What do you know about the concavity of the graph?

8. Graph a function $f(x)$ which is increasing everywhere and concave up for negative x and concave down for positive x.

9. Find the average rate of change of $f(x) = 2x^2$ between $x = 1$ and $x = 3$.

10. When a deposit of \$1000 is made into an account paying 8% interest, compounded annually, the balance, \$$B$, in the account after t years is given by $B = 1000(1.08)^t$. Find the average rate of change in the balance over the interval $t = 0$ to $t = 5$. Give units and interpret your answer in terms of the balance in the account.

11. Table 1.10 shows the production of tobacco in the US.[19]

(a) What is the average rate of change in tobacco production between 1996 and 2003? Give units and interpret your answer in terms of tobacco production.
(b) During this seven-year period, is there any interval during which the average rate of change was positive? If so, when?

Table 1.10 *Tobacco production, in millions of pounds*

Year	1996	1997	1998	1999	2000	2001	2002	2003
Production	1517	1787	1480	1293	1053	991	879	831

12. Do you expect the average rate of change (in units per year) of each of the following to be positive or negative? Explain your reasoning.

(a) Number of acres of rain forest in the world.
(b) Population of the world.
(c) Number of polio cases each year in the US, since 1950.
(d) Height of a sand dune that is being eroded.
(e) Cost of living in the US.

13. Figure 1.34 shows the length, L, in cm, of a sturgeon (a type of fish) as a function of the time, t, in years.[20]

(a) Is the function increasing or decreasing? Is the graph concave up or concave down?
(b) Estimate the average rate of growth of the sturgeon between $t = 5$ and $t = 15$. Give units and interpret your answer in terms of the sturgeon.

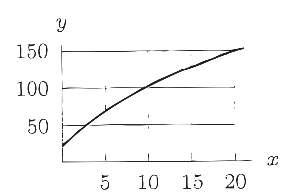


Figure 1.34

14. Table 1.11 shows the total US labor force, L. Find the average rate of change between 1940 and 2000; between 1940 and 1960; between 1980 and 2000. Give units and interpret your answers in terms of the labor force.[21]

Table 1.11 *US labor force, in thousands of workers*

Year	1940	1960	1980	2000
L	47,520	65,778	99,303	136,891

15. The total world marine catch[22] of fish, in metric tons, was 17 million in 1950 and 99 million in 2001. What was the average rate of change in the marine catch during this period? Give units and interpret your answer.

16. Figure 1.35 shows the total value of US imports, in billions of dollars.[23]

(a) Was the value of the exports higher in 1985 or in 2003? Approximately how much higher?

(b) Estimate the average rate of change between 1985 and 2003. Give units and interpret your answer in terms of the value of US imports.

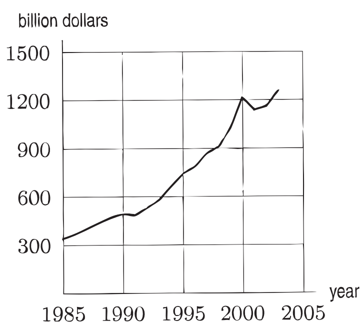


Figure 1.35

17. Table 1.12 gives sales of Pepsico, which operates two major businesses: beverages (including Pepsi) and snack foods.[24]

(a) Find the change in sales between 1999 and 2004.

(b) Find the average rate of change in sales between 1998 and 2004. Give units and interpret your answer.

Table 1.12 *Pepsico sales, in millions of dollars*

Year	1999	2000	2001	2002	2003	2004
Sales	20,367	20,438	23,512	25,112	26,971	29,261

18. Table 1.13 gives the revenues, R, of General Motors, the world's largest auto manufacturer.[25]

(a) Find the change in revenues between 1999 and 2004.

(b) Find the average rate of change in revenues between 1999 and 2004. Give units and interpret your answer.

(c) From 1999 to 2004, were there any one-year intervals during which the average rate of change was negative? If so, which?

Table 1.13 *GM revenues, billions of dollars*

Year	1999	2000	2001	2002	2003	2004
R	176.6	183.3	177.3	177.3	185.5	193.0

19. The number of US households with cable television[26] was 12,168,450 in 1977 and 73,365,880 in 2003. Estimate the average rate of change in the number of US households with cable television during this 26-year period. Give units and interpret your answer.

20. Figure 1.9 on page 5 shows the amount of nicotine $N = f(t)$, in mg, in a person's bloodstream as a function of the time, t, in hours, since the last cigarette.

(a) Is the average rate of change in nicotine level positive or negative? Explain.

(b) Find the average rate of change in the nicotine level between $t = 0$ and $t = 3$. Give units and interpret your answer in terms of nicotine.

21. Table 1.14 shows the concentration, c, of creatinine in the bloodstream of a dog.[27]

(a) Including units, find the average rate at which the concentration is changing between the

(i) 6^{th} and 8^{th} minutes. (ii) 8^{th} and 10^{th} minutes.

(b) Explain the sign and relative magnitudes of your results in terms of creatinine.

Table 1.14

t (minutes)	2	4	6	8	10
c (mg/ml)	0.439	0.383	0.336	0.298	0.266

Problems 22–23 refer to Figure 1.36 which shows the contraction velocity of a muscle as a function of the load it pulls against.

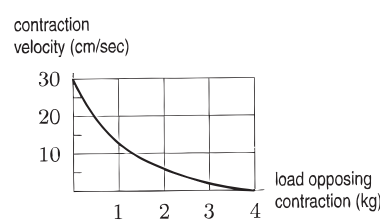


Figure 1.36

22. In terms of the muscle, interpret the

(a) Vertical intercept **(b)** Horizontal intercept

23. **(a)** Find the change in muscle contraction velocity when the load changes from 1 kg to 3 kg. Give units.

(b) Find the average rate of change in the contraction velocity between 1 kg and 3 kg. Give units.