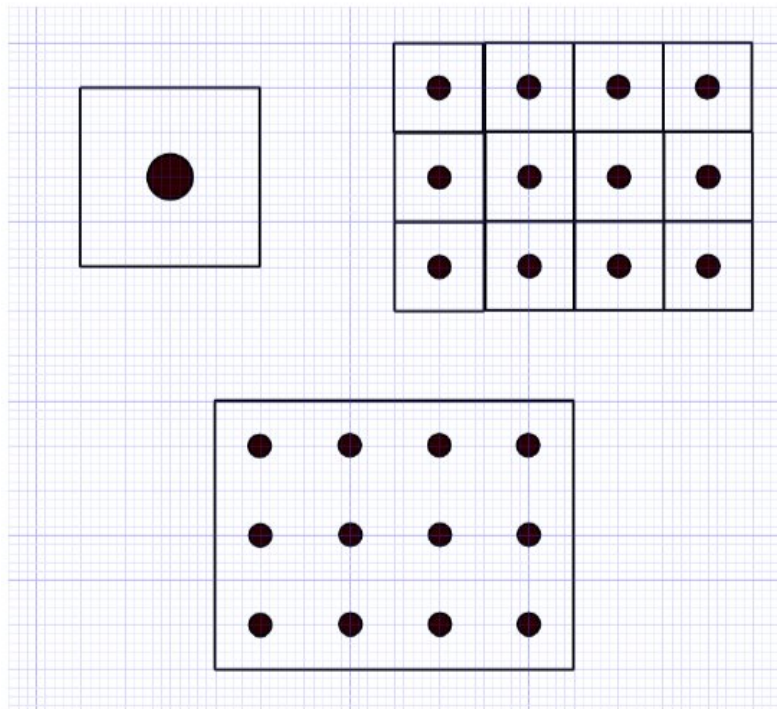


MIRROR CURVES

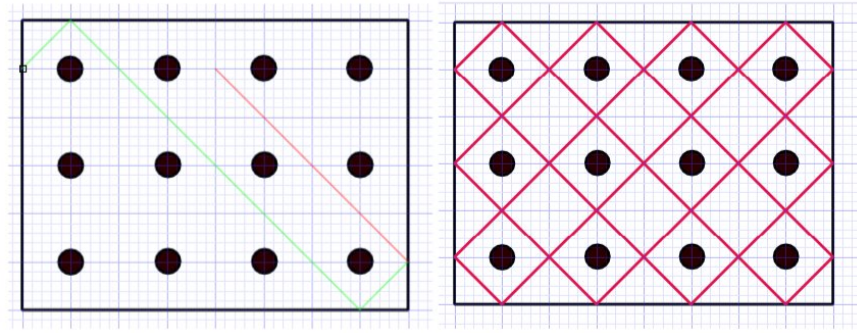
by Ljiljana Radovic

In the previous lecture we explained the term “mirror curves” (or Sona drawings). Mirror curves are trajectories of (imaginary) rays of light emitted from edge mid-points in the regular square grid $RG[a,b]$, with sides a and b , with one-sided mirrors on their external sides and two-sided mirrors placed between cells, coinciding with internal edges (cell borders) or perpendicular to them at their mid-points. After a series of reflections, a ray of light creates a closed path – a component of the mirror curve. If the complete $RG[a,b]$ is covered by a single mirror curve we have perfect (monolinear) Sona drawing. Otherwise, after closing the path of one component we start with a new ray of light emitted from another starting point and continue in this way until the whole $RG[a,b]$ is uniformly covered by mirror curves. How do we draw these curves? Usually we begin with a regular square grid $RG[a,b]$, but the same can be done with regular triangular or hexagonal grids. In the middle of a square we draw a black dot which will be surrounded by the mirror curve and multiply this square in order to obtain the regular square grid $RG[a,b]$ (Fig. 1).

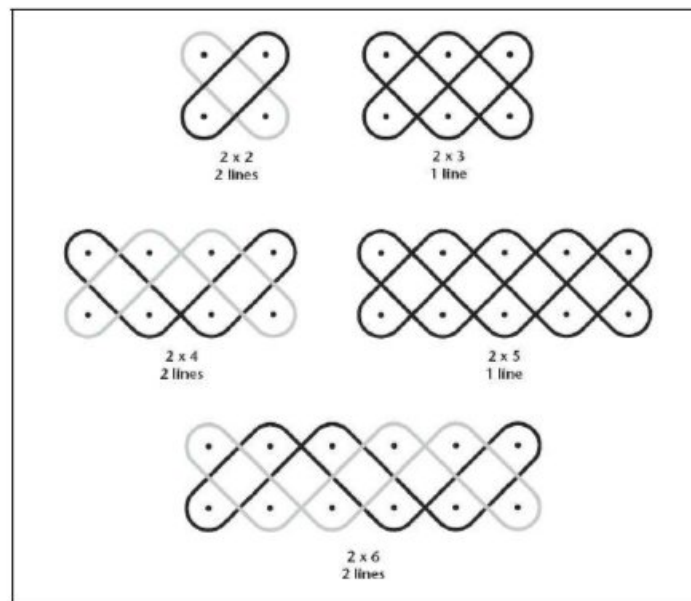


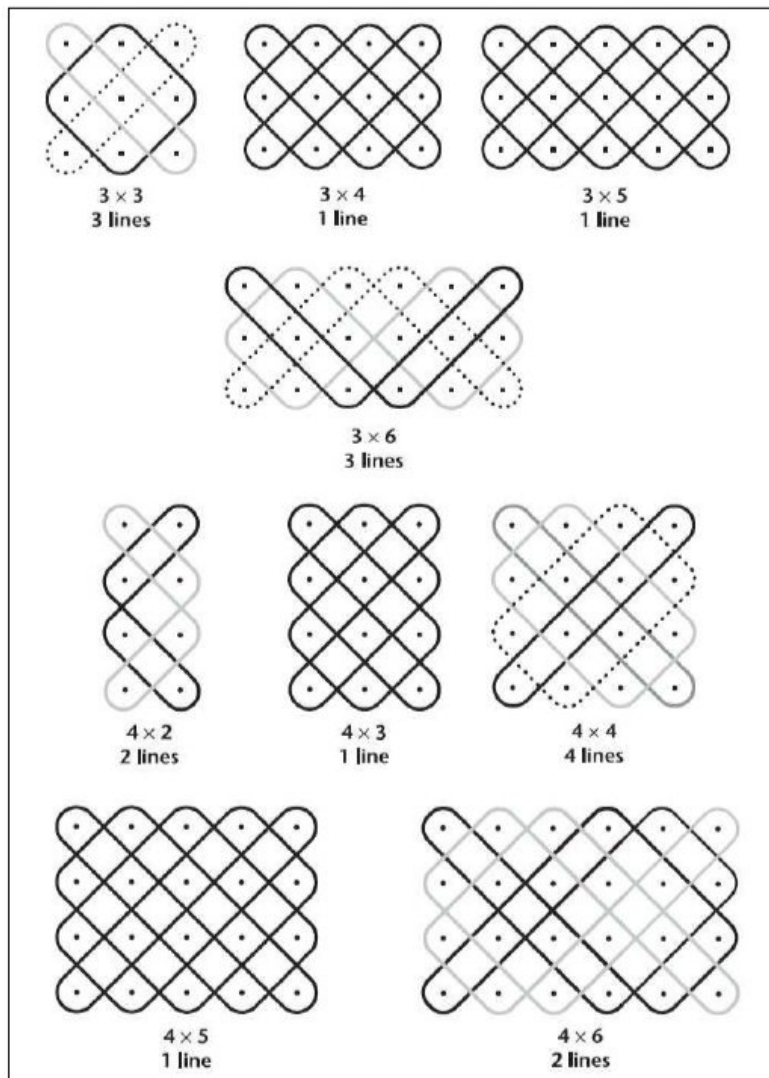
We begin with $RG[a,b]$ with no internal mirrors. The ray of light reflects and is emitted from the mid-point of an edge, reflects in the mid-points of the edges of bordering squares, and travels through the grid. In order to simplify, you can divide $RG[a,b]$ into small squares, without deleting the internal edges, and remove them after finishing your

work. If the numbers a and b are relatively prime, i.e., if $\text{GCD}(a,b)=1$ (as in our example, where $a=4$ and $b=3$, so $c=\text{GCD}(a,b)=1$), we obtain a single curve surrounding all dots and uniformly covering $\text{RG}[a,b]$. Otherwise, you obtain a multi-component mirror curve, where its number of components is $c = \text{GCD}(a,b)$. Instead of completely covering $\text{RG}[a,b]$ with dots you can use a system (a scheme) consisting only of dots, but in this case you need to consider the correct direction of the light-ray, especially when turning it in the border ends (Fig. 2).

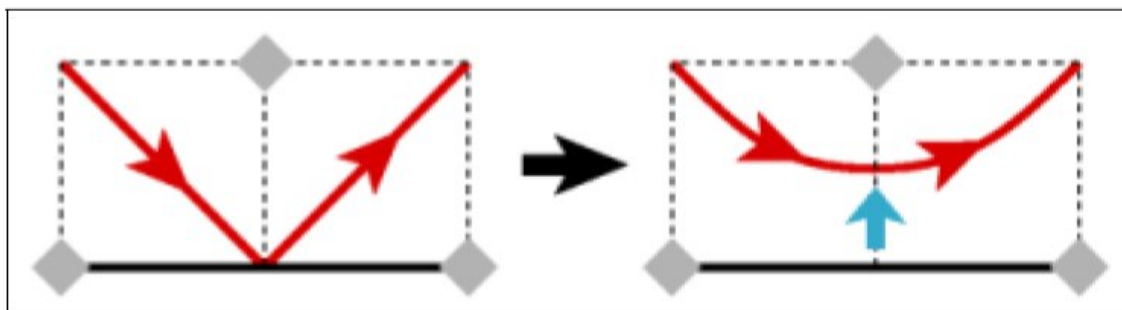


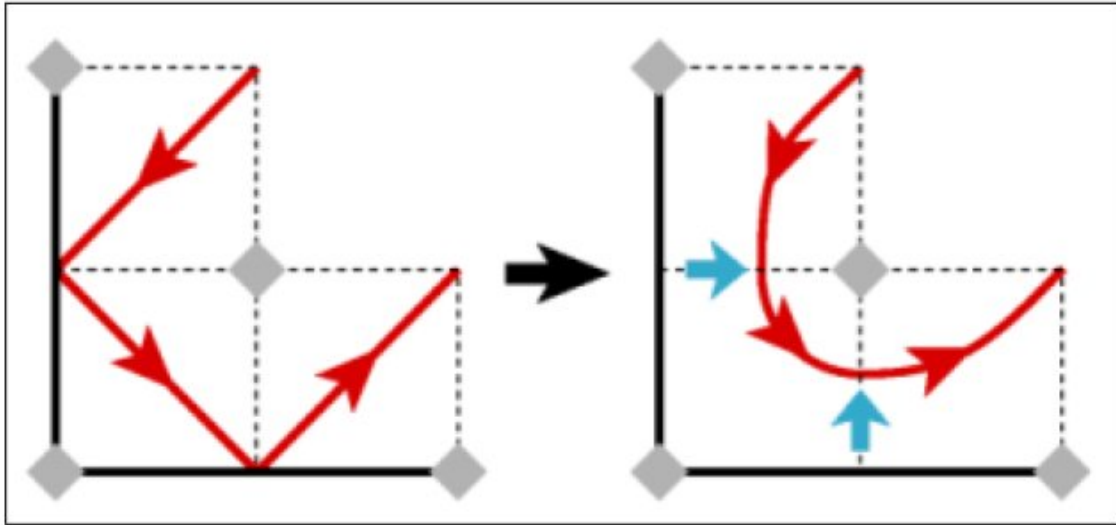
The next example shows different situations that can occur depending on the dimensions of $\text{RG}[a,b]$, i.e., $\text{GCD}(a,b)$. Try to prove that the number of components (it is useful to represent them with different colors, so they can be easily distinguished) is equal to $c=\text{GCD}(a,b)$. For example, for $a=2$ and $b=2$ the number of components is $c=\text{GCD}(2,2)=2$, for $a=4$ and $b=2$ the number of components is $c=\text{GCD}(4,2)=2$, for $a=5$ and $b=2$ the number of components is $c=\text{GCD}(5,2)=1$, for $a=6$ and $b=2$ the number of components is $c=\text{GCD}(6,2)=2$, for $a=6$ and $b=3$ it is $c=\text{GCD}(6,3)=3$ (Fig.3), etc.



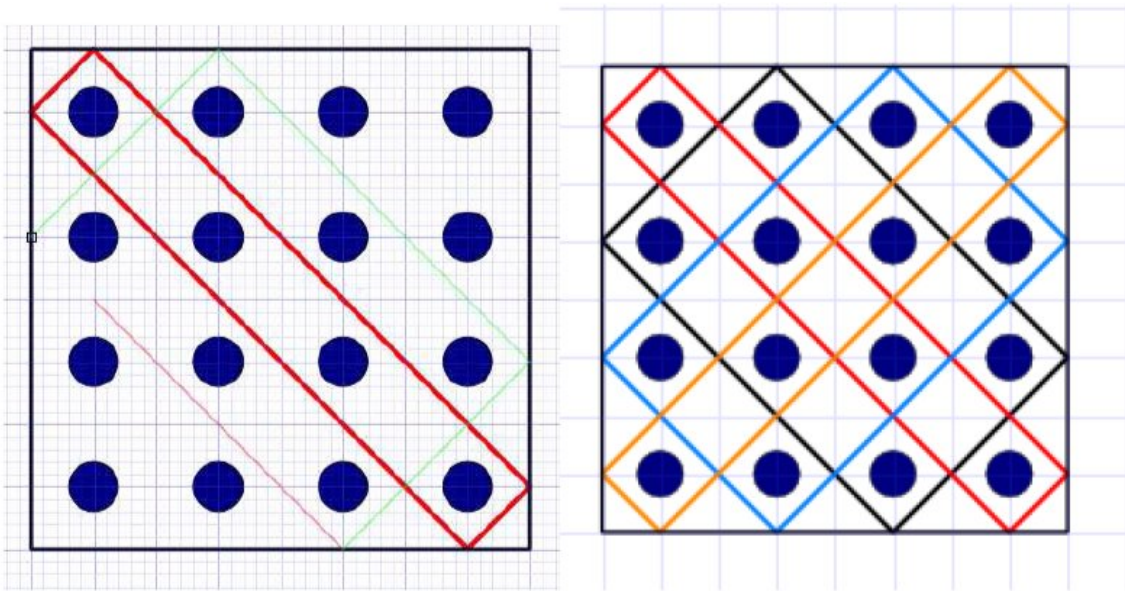


In order to draw curves with smoothed, rounded corners, for reflecting a ray of light in the external edges and in the corners of the grid, you can use the following rules shown in Fig. 4 (this can be done in *Adobe Illustrator* by the command **Filter>RoundCorners**, or by command **Effect>Stylize>RoundCorners**).



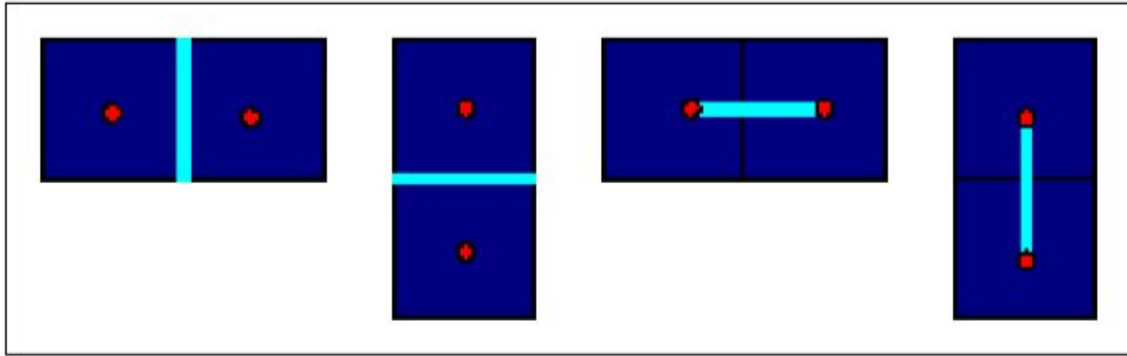


Let's see what will happen if the numbers a and b are not relatively prime. As before, we draw a rectangular grid, but now of dimensions 4×4 , and we place the rays of light (components) in it. In order to surround all dots and uniformly cover $RG[4,4]$ by the mirror curve, we need four components (Fig. 5).

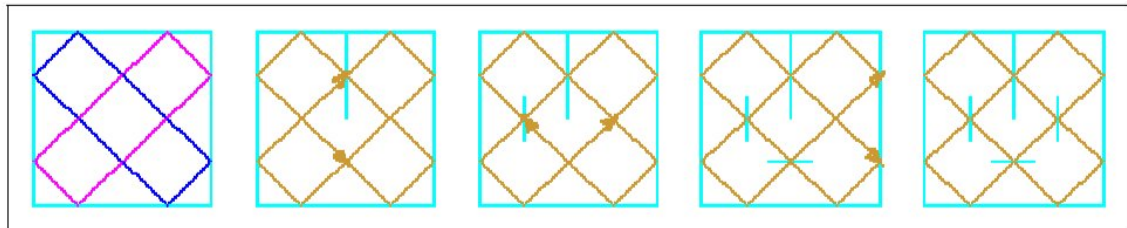


The next question is how to place internal two-sided mirrors within the existing scheme, in order to join our components into a single mirror curve, i.e. to get a single ray of light that traces it. When inserting an internal mirror, we have two possible choices (Fig. 6):

- place it between square cells, incident to the internal edge;
- place it perpendicular to the edge at its mid-point.

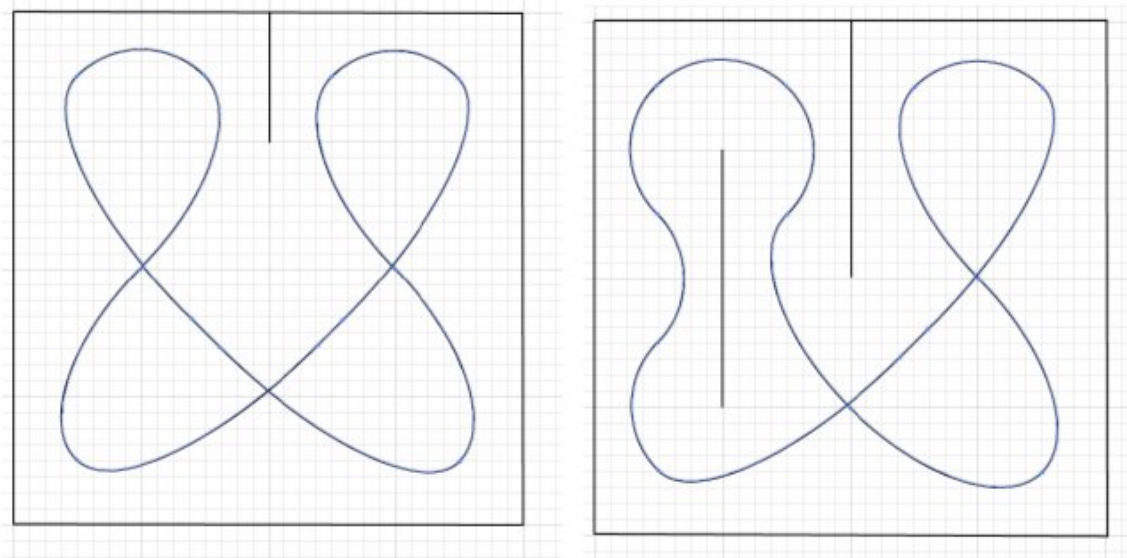


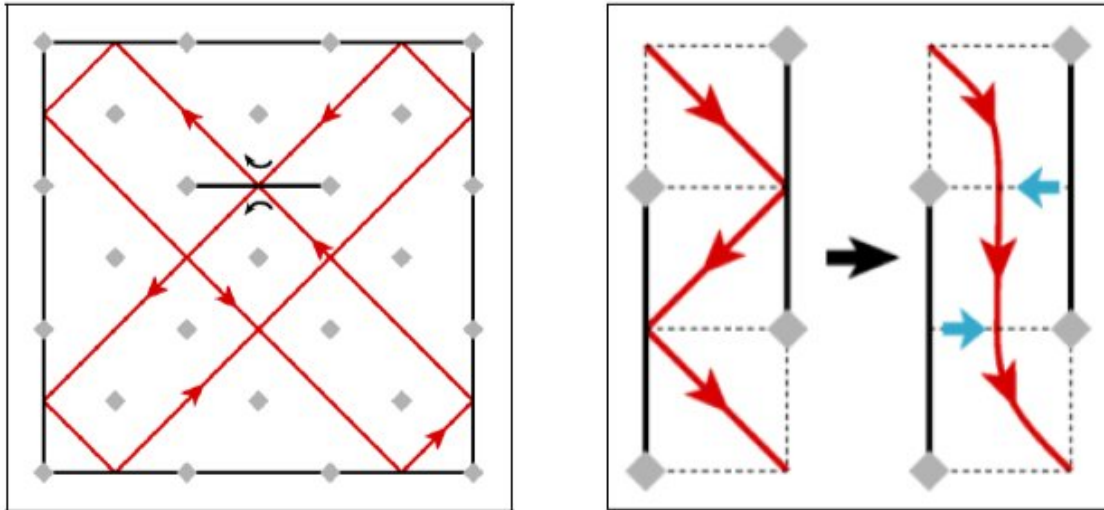
A mirror placed at the crossing point of two components (denoted by different colors) will join them into one. Placed at a self-crossing of an oriented component, a mirror placed between arrows breaks the component in two; otherwise, after inserting the new mirror, the number of curves remains unchanged (Fig. 7).



We repeat this algorithm until a single curve is obtained.

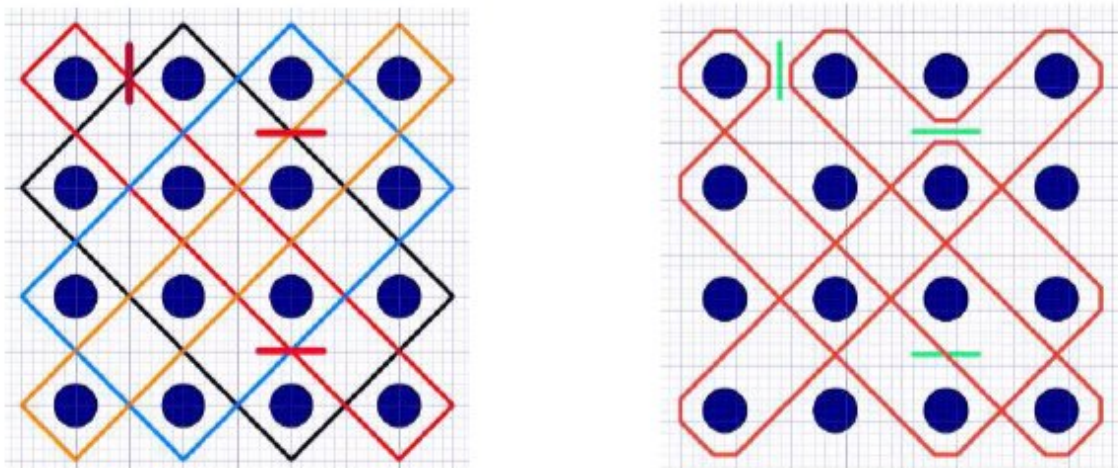
After inserting internal mirrors, we again make a smoothing of the mirror curve obtained (Fig. 8). The next image shows the smoothing of zig-zag lines (Fig. 9).





Notice that symmetrical placements of mirrors results in symmetrical mirror curves.

Let's return to the example of the 4×4 grid covered by four components. After inserting an internal mirror at the crossing point of two different curves (represented by different colors), the number of components reduces to three, since the two curves are joined into one (after this, you need to make a new coloring of the components, representing the joined curves by a single color). Continuing in this way by inserting new internal mirrors at the crossing point of two components, at each step the number of curves decreases by one, so that after four steps we obtain a single component – monolinear mirror curve uniformly covering $RG[4,4]$. Figure 10 shows one possible way to introduce four internal mirrors in order to get a single curve. Certainly, the solution of this problem is not unique; you can choose different arrangements of four internal mirrors each resulting in a monolinear mirror curve.



In Figure 10 the mirrors are not symmetric, so the resulting mirror curve is asymmetric as well. Figure 11 shows a symmetric arrangement of four internal mirrors according to a rotation of order 4, so the resulting drawing has the cyclic rotational symmetry of the group C_4 . However, we have not completely followed the rule about pairing components of different colors, and so we obtained a 4-component (imperfect) mirror curve. Try to make another symmetrical arrangement of internal mirrors in the same 4×4 grid resulting in a perfect (monolinear) mirror curve.

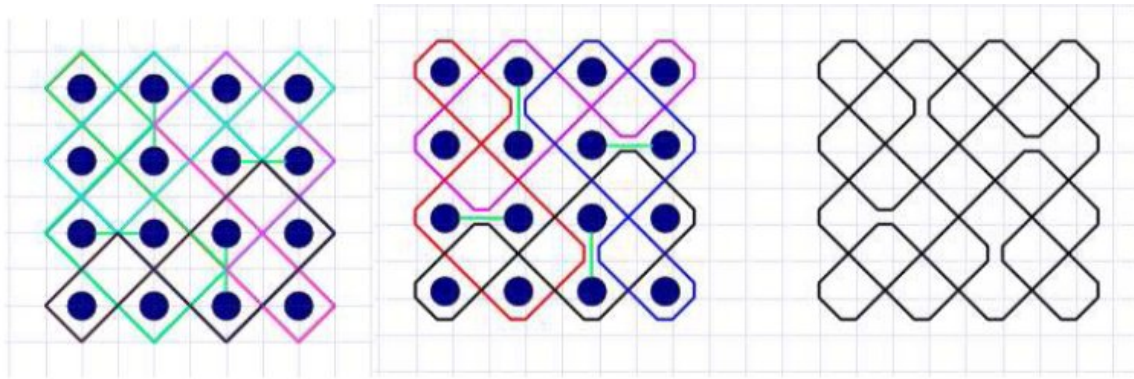


Figure 12 illustrates the same exercise within a 3×3 grid. At the beginning, we have 3 components, so it is necessary to insert at least 2 internal mirrors in order to obtain a single curve. By using two internal mirrors, placed asymmetrically, we obtain the drawing shown on the left. The right portion of Figure 12 shows the same curve turned into the diagram (projection) of an alternating knot, by introducing at every crossing the relation “over-under” in an alternating manner. You will find many similar examples in books describing Celtic knotworks or in computer programs for drawing Celtic knots.

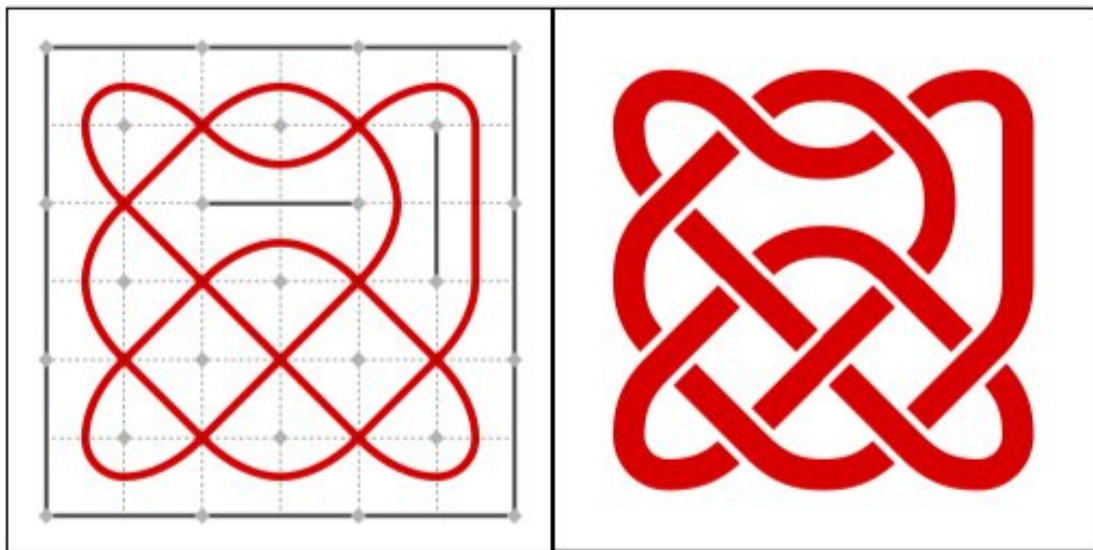
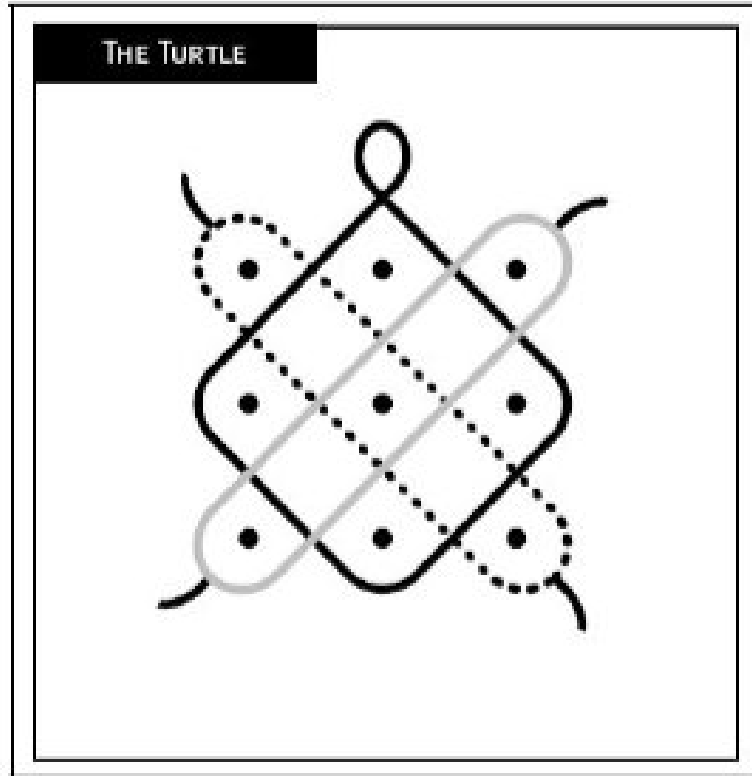
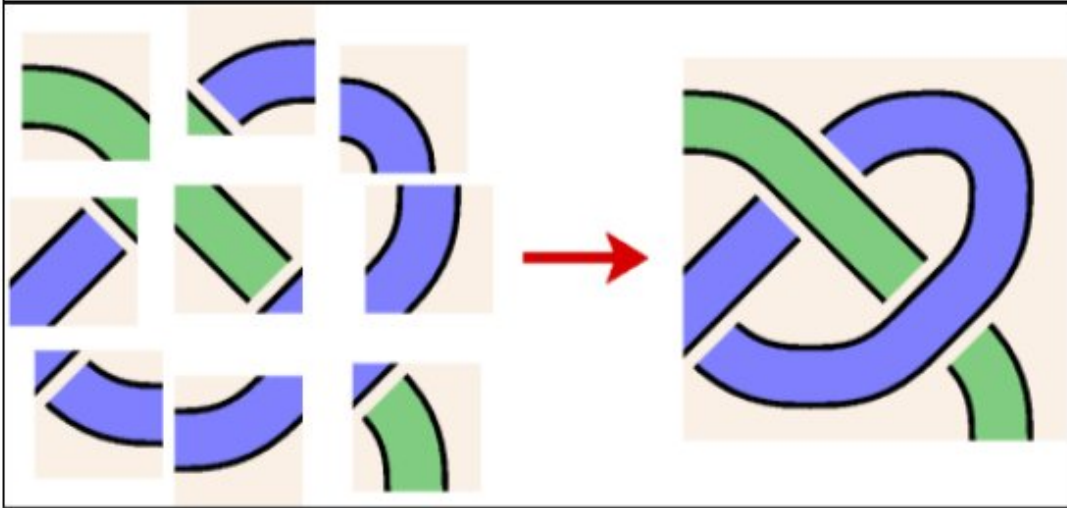
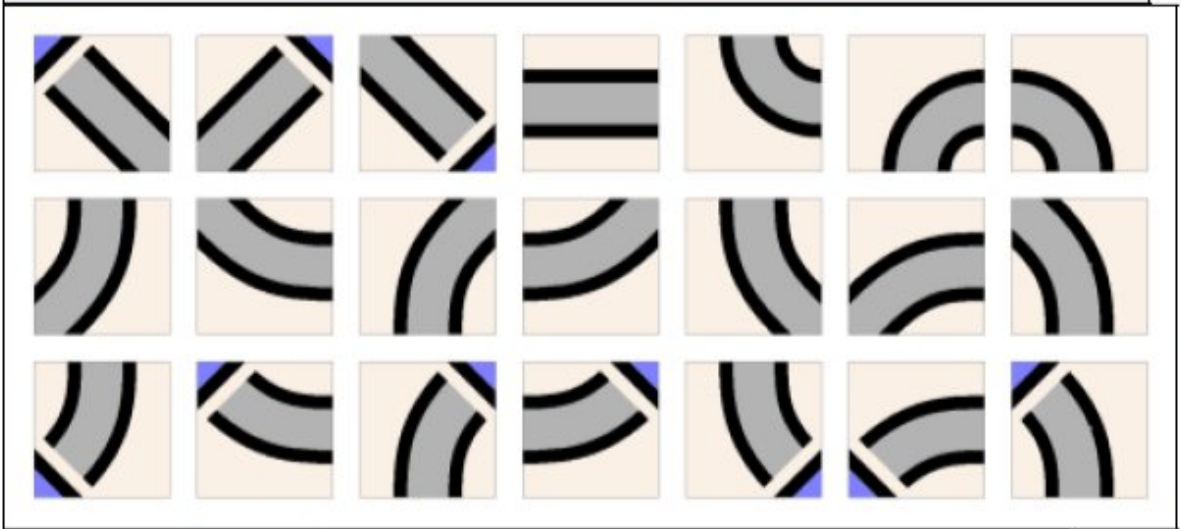
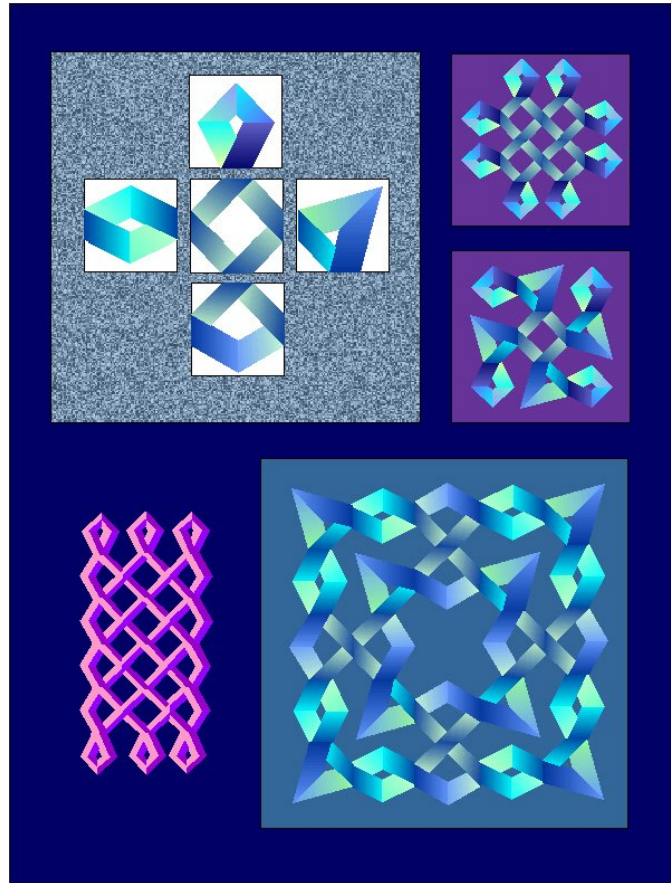


Figure 13 shows stylized drawing of a turtle, made by the Tchokwe people from Angola. Its geometrical basis is the 3×3 grid. By introducing internal mirrors (how many of them do you need for this?), try to transform this set of components into a monolinear curve.

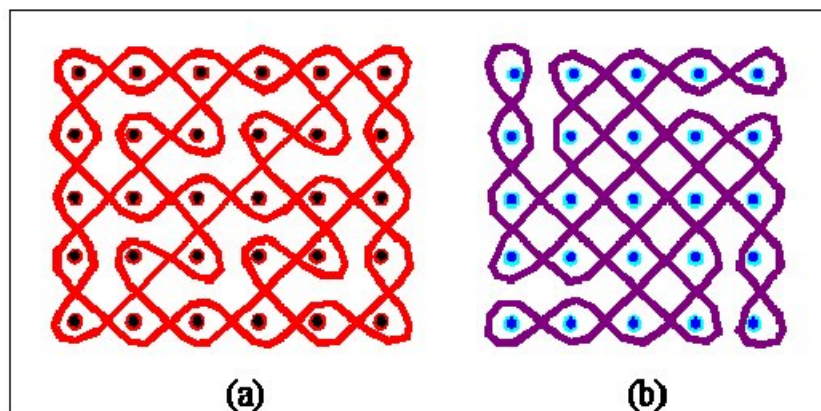


In order to obtain interlacing knot ornaments (mirror curves similar to Celtic designs) you can use a few basic tiles (modules) as shown in Fig. 14. These tiles represent all possible “states” of a single small square with regard to the portion of a mirror curve that it contains along with the relation “over-under”. By (re)combining them, you can obtain all possible mirror curves (or knot mosaics). The same can be done by using only 5 modules (“KnotTiles”) as shown in Fig. 15. For the knot game made from “KnotTiles”, please see <http://www.mi.sanu.ac.rs/vismath/op/tiles/kt/kt.htm>





Consider the possibility to insert internal mirrors into a regular square grid lacking internal mirrors and covered by a single curve (with $c = \text{GCD}(a, b) = 1$, i.e., when the sides of the grid are relatively prime numbers). We can try to plan the choice of positions for mirrors in advance, in order to obtain more interesting and visually pleasing single curves. In many cases this can be achieved by using symmetry (translational = repetitional, rotational, or any other). After some experimenting, you can obtain single curves similar to those created by the Tchokwe people (Fig. 16) from 6×5 square grids (initially containing one component) and 5×5 grids (initially containing 5 components).



We can discover the method of constructing single curves (Fig. 16) by beginning with 6×5 and 5×5 grids, and determining the positions of internal (two-way) mirrors, placed within the grids in symmetrical ways.

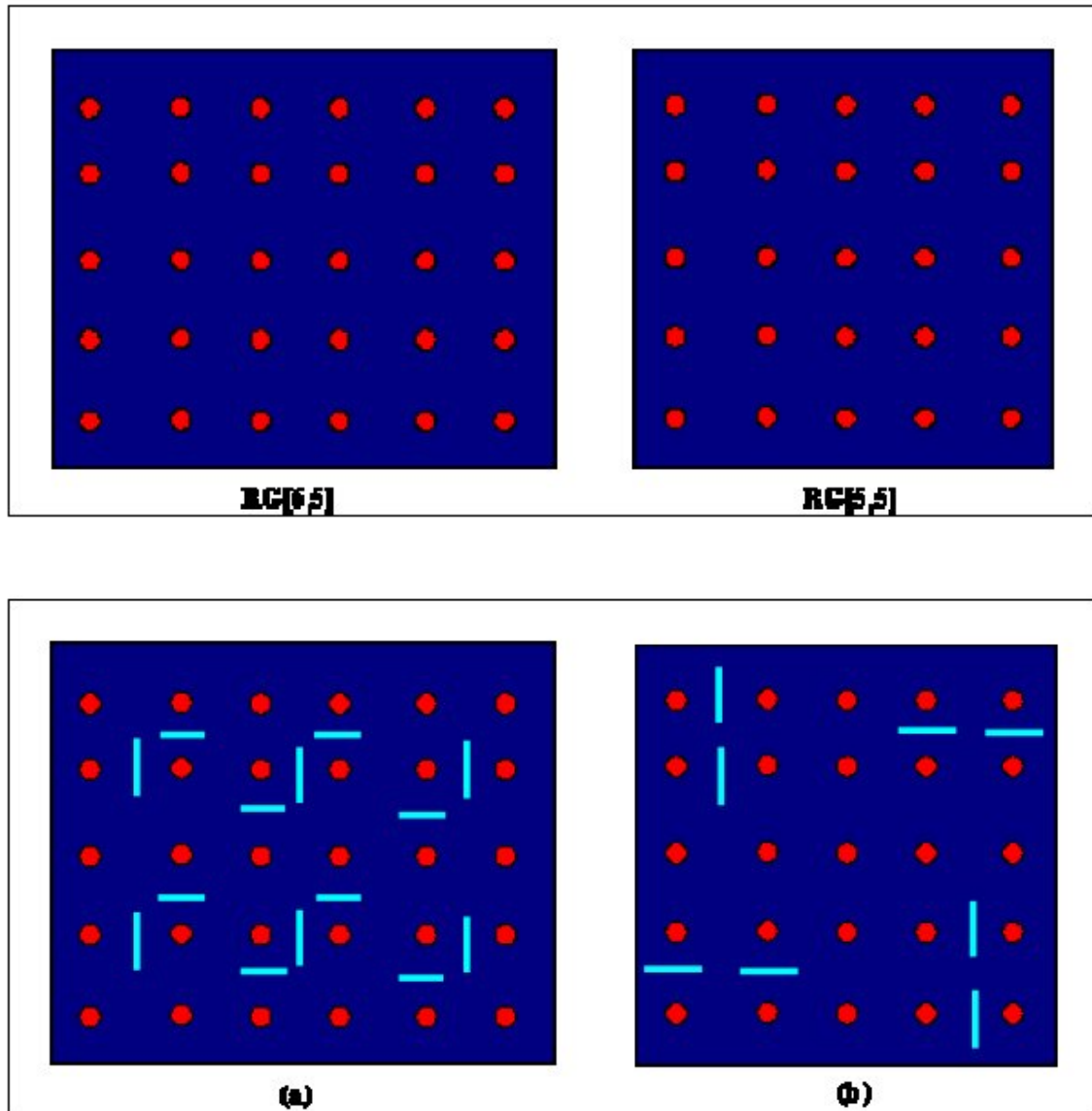


Figure 17 shows the 5×4 grid initially containing a 5-component mirror curve. After a few steps, by introducing six internal mirrors in appropriate (symmetrical) positions, the Celtic people once again obtained a monolinear mirror curve.

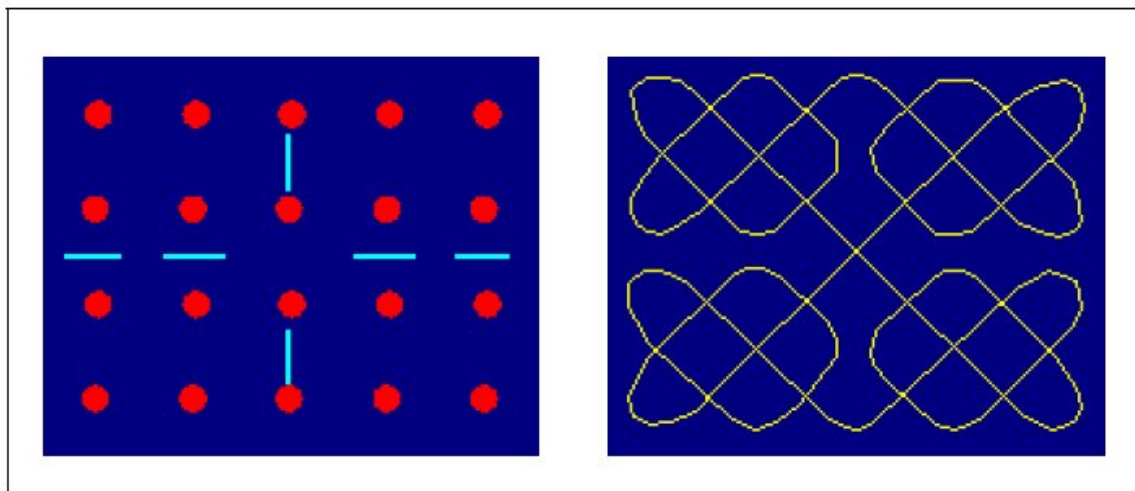
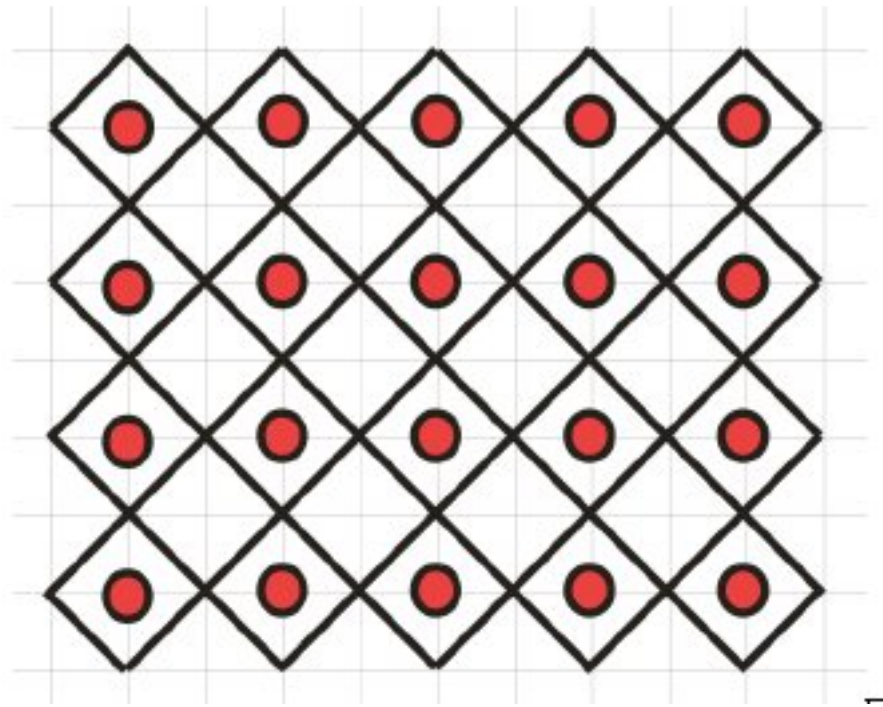
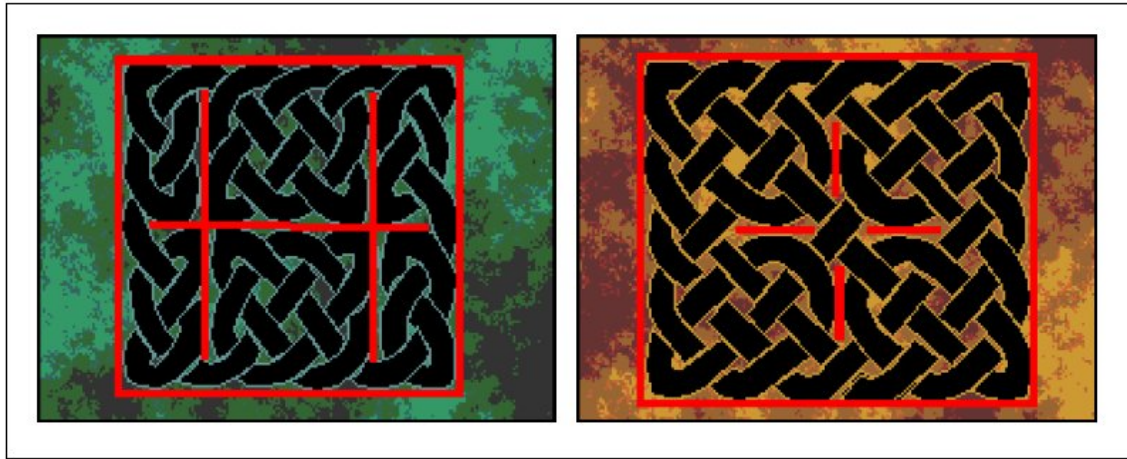
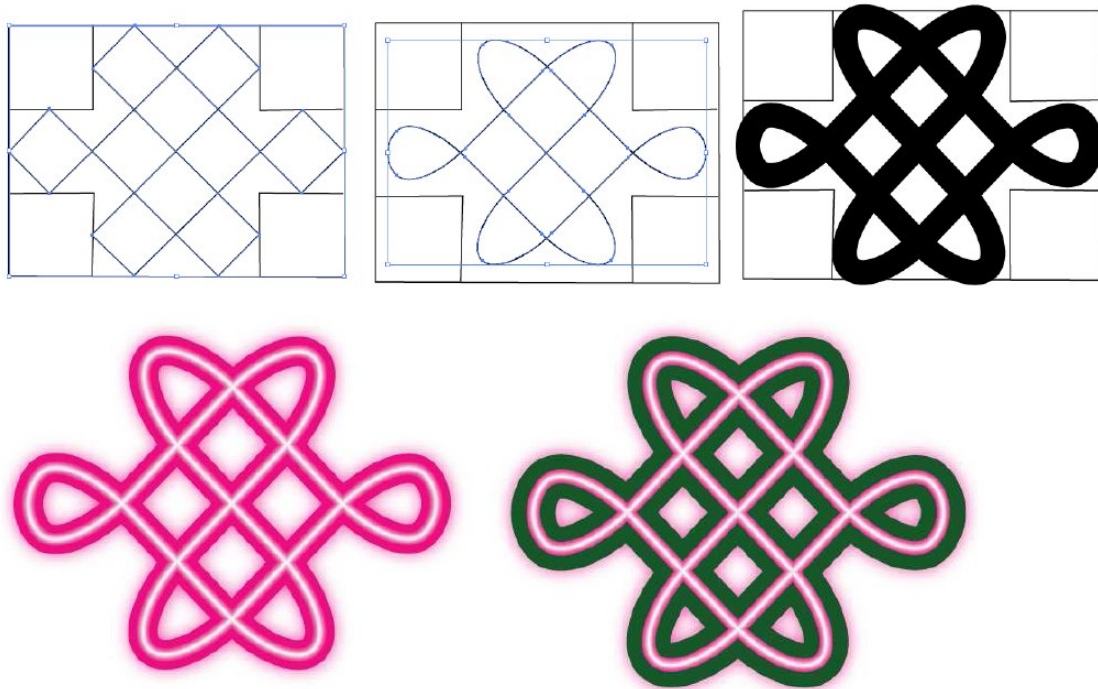


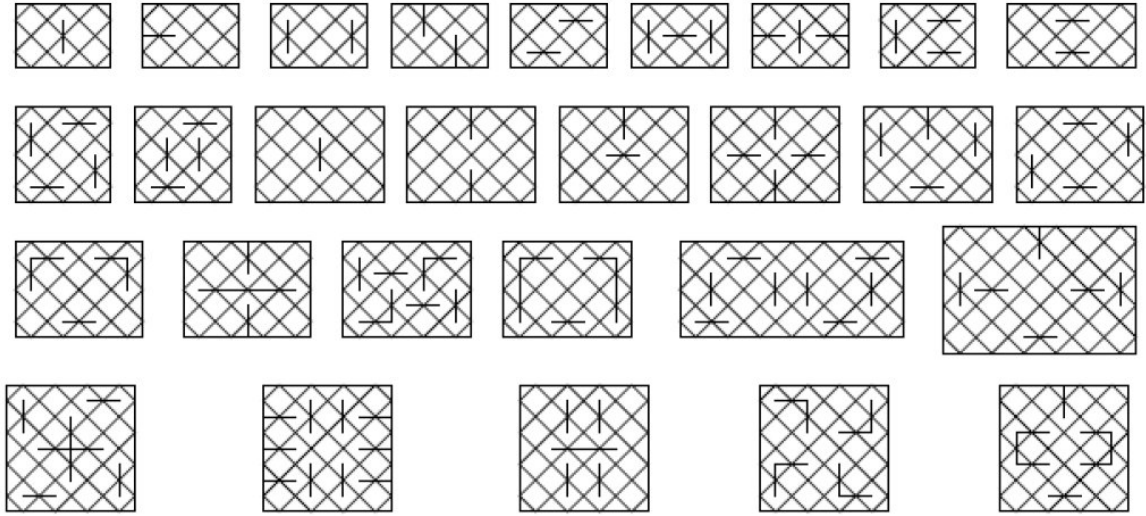
Figure 18 shows examples of Celtic knots from the cover the book “Celtic Art– the Methods of Construction” (Dower, New York, 1973) by G. Bain. All Celtic knots can be (re)constructed by taking a grid (part of a plane tessellation; in the simplest case a part of a square grid), placing internal mirrors in it, and introducing the relation “over-under” (interlacing) at every crossing of the mirror curve.



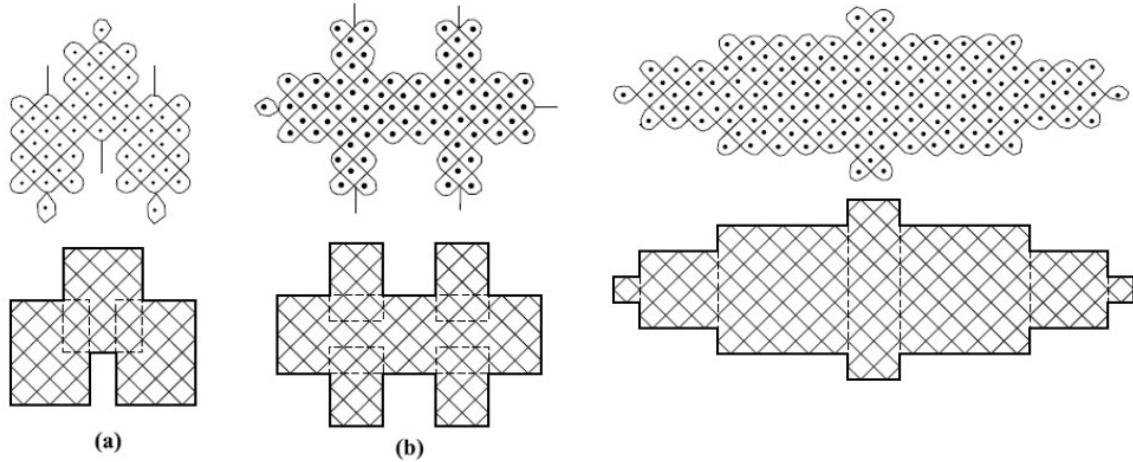
The next knot (Fig. 19) is created in *Adobe Illustrator*. Beginning with the 4×3 grid, we introduce four internal mirrors and create the mirror curve, drawn by a 20pt line with the use of neon effects. Isn't that simple?



In Figure 20, different mirror schemes are shown which can be used to create Celtic neon knots in the same manner as we constructed the knots in Fig. 19.



It is also possible to combine several rectangular schemes and extend our mirror-curve designs to obtain friezes or more complicated ornamental knots. By combining, joining, or overlapping parts of basic rectangular grids, we obtain composite mirror-curve designs (Sona drawings of animals made by the Tchokwe people, or monolinear Tamil threshold designs, called “pavitram” or “Brahma mudi”) (Fig. 21).



The same approach can be used in order to obtain composite Celtic knots (e.g., in the form of a cross) (Fig. 22).

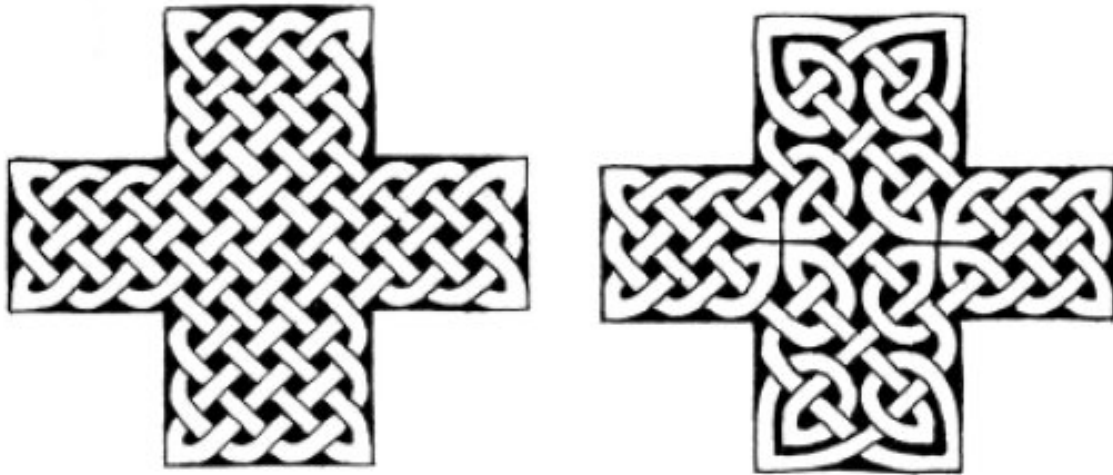


Figure 23 shows the basic elements (mirror schemes) used in Celtic knotwork, that we shall call “Celtic tangles” (since each of them can be treated as a tangle with four loose ends and connected to the others).

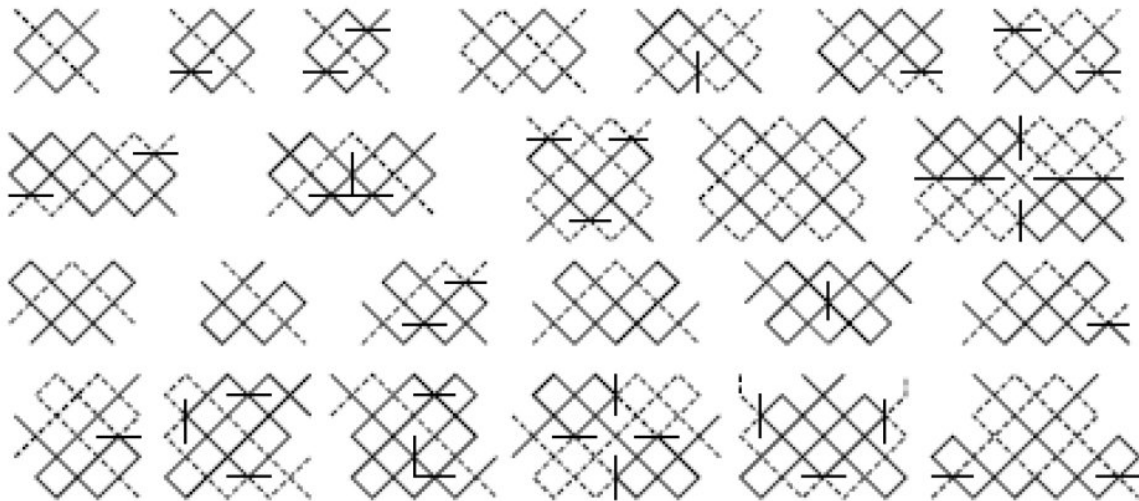
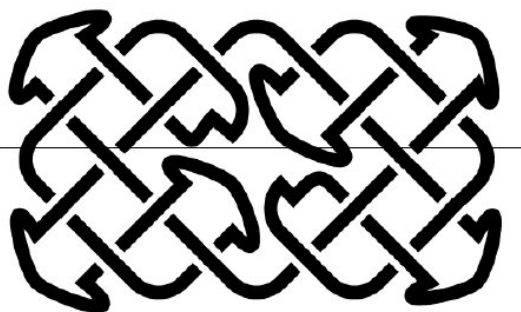
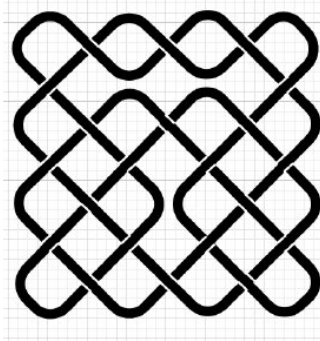
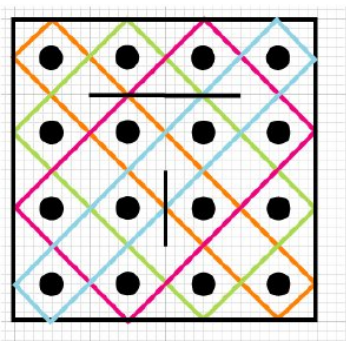
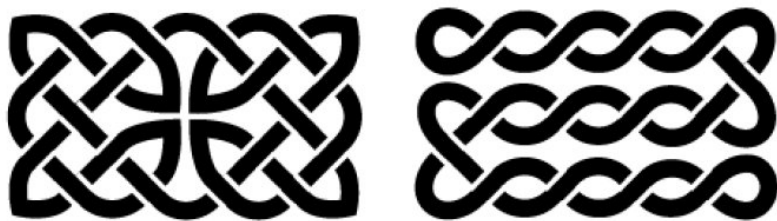
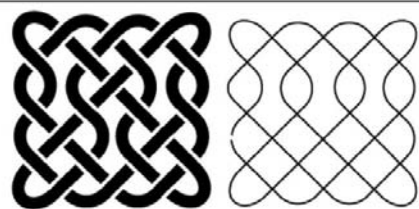
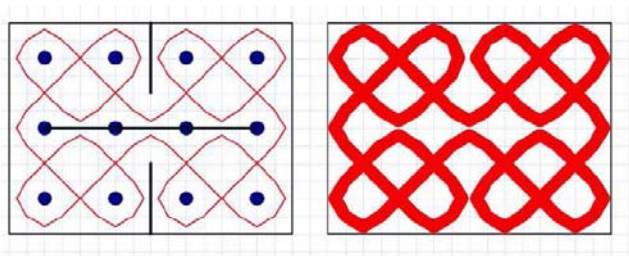
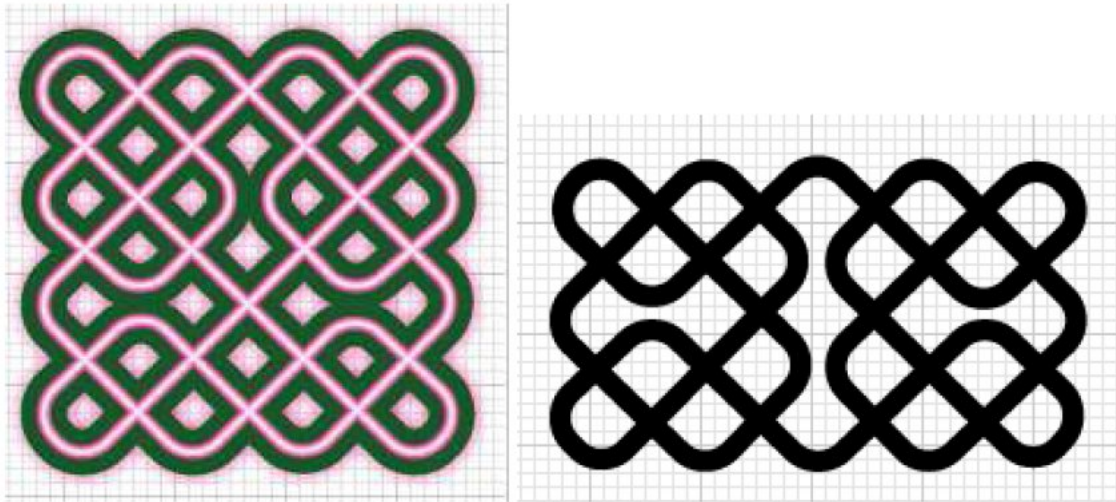


Figure 24 shows a few additional examples, created in *Inkscape* and *Adobe Illustrator*.





The homework is: “My first knot!” Try to untangle it!

Homework: By using the method described in previous examples of rectangular grids and mirror curves, construct a knot of your choice. Convert the image to JPG format, write a short description in *MS Word*, and describe all the phases of your work. Put the images into the document file Images.doc. Save your work as Name_Surname_MyKnot.doc and send it to my e-mail address (kappraff@adm.njit.edu). To draw, you can use any graphical computer program (*Inkscape, Adobe Illustrator, Corel Draw*).