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CONSTRUCTING SYMMETRIC CHOKWE SAND DRAWINGS

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Abstract: *The sona drawings of the Chokwe people of Angola/Zaire are a particularly attractive form of "mirror curves," visualizable as a curve bouncing through an arrangement of dots, bouncing off "mirrors" placed between some of the dots, and leaving a closed trail behind. The Chokwe artistic aesthetic strongly prefers mirror curves that: (1) separate each dot from the others; (2) generate a monolineal Eulerian circuit; and (3) are symmetric. In this paper we focus on the question of when a symmetric layout of dots will have a (symmetric) set of mirrors so that the resulting sona is not only monolineal, but is also symmetric. This appears to reflect the artistic values of the Chokwe people themselves in constructing such sona, and we argue that the sona produced by these constructions would appeal to the Chokwe artist within their cultural aesthetic, as well as being enjoyable to others.*

1. INTRODUCTION & TERMINOLOGY

The Chokwe people of Angola and Zaire have a drawing tradition, done both in sand drawings and on more permanent objects, that has attracted a substantial amount of mathematical attention (e.g. Ascher, 1991; Gerdes, 1994-2007, Jablan 2009, Kubik

2006, Schlatter 2004). Although their sona (singular "lusona") drawings arise in several different forms, one of the more common, and the most mathematical, consists of a grid of dots with a curve passing through the grid, "bouncing" off the external boundary of the grid, to form a single continuous curve. The "Leopard with Cubs" lusona shown in figure 1 is an example of such a drawing, with a few extra features added to the fundamental design to indicate the heads and tails of the mother leopard (horizontal) and her two cubs (vertical).

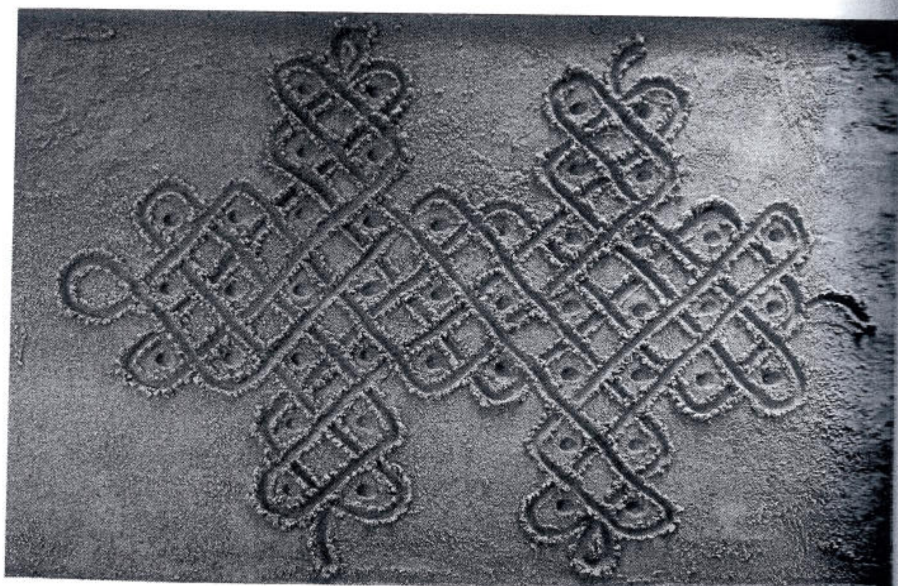

Figure 1: The "Leopard with Cubs" lusona, as it would be drawn in the sand, with additional heads and tails of the mother leopard (horizontal) and her two cubs (vertical).

We say that a *sona grid* is any layout of uniformly spaced dots, which we will assume to be a subset of the integer lattice, with the property that the collection of dots is connected with respect to adjacencies in the unit lattice. In discussing sona grids, we refer to the *bounding polygon* of a grid. This is the smallest connected rectilinear polygon one can construct using horizontal and vertical lines, that surrounds the sona grid with a spacing of ½ unit from all of the dots of the grid. For example, the bounding polygon for the principal part of "Leopard with Cubs" is shown as the heavy lines in figure 2. We may view the sona constructed from such a grid as being drawn by a curved line, drawn on the diagonals midway between the dots, and "bouncing" off the walls of this bounding polygon. While this is a useful metaphor, the curves do not actually reach the walls, bouncing off them like a strangely shaped pool table, but rather curve smoothly so as to just avoid hitting those edges. A related idea is the *bounding rectangle of the grid; the smallest horizontally aligned rectangle* that encloses this bounding polygon. This is shown as the lighter lines in figure 2.

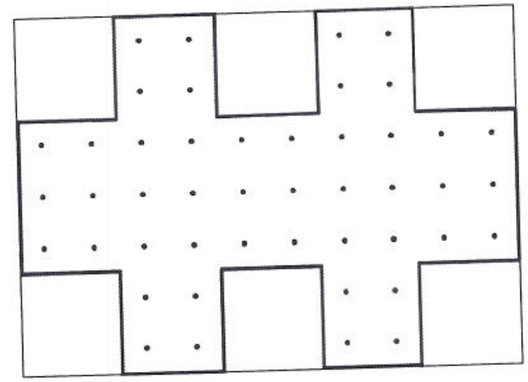

Figure 2: The bounding polygon (in dark) and the bounding rectangle (lighter lines) for the "Leopard with Cubs" lusona (without the added heads & tails).

Not all sona grids will give rise to acceptable sona drawings, because the Chokwe aesthetic generally requires three criteria for such a drawing. The first is that the line must be drawn as a single curve, passing "smoothly" through each intersection—e.g., not turning a corner where two parts of the curve cross. Secondly, they require that each of the dots in the sona grid must be surrounded, and separated from the other dots. Finally, they prefer that the primary drawing (excluding additional feet, tails, head, horns, etc. added afterwards) should be symmetric in some way.

For many of the more interesting sona, the curve drawn by the Chokwe artist includes turns within the grid layout itself. This is demonstrated by three traditional sona shown in figure 3. A common way to view such a sona is to imagine a series of walls inside the grid and view the curve as "bouncing" off of these walls. This sets these drawings within the mathematical framework of "mirror curves," such as studied by S. Jablan and others. (We should emphasize though that the Chokwe artists generally do not draw these walls.)

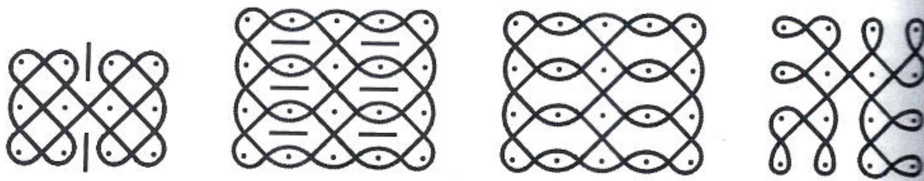

Figure 3: Three examples of traditional Chokwe sona, and the walls that would reflect their construction. The middle one is shown both with the walls and without the walls, as it would be drawn in the sand. The first two sona have D_2 symmetry; the last one has approximate C_4 symmetry, although one additional wall had to be added to achieve monolineality.

The walls shown above are placed perpendicularly between two dots. A similar, but different type of wall can be used to describe many Celtic knot designs, where the walls are placed extending *from* one dot *to* an adjacent one, as in figure 4. While this type of wall is also used occasionally by the Chokwe, we refer to the perpendicular placement of a wall as a "Chokwe wall" to distinguish it from the "Celtic walls". To incorporate the idea of these walls with a sona grid, we say a *sona configuration* is a sona grid together with a set (possibly empty) of internal Chokwe walls. A sona configuration defines a unique lusona that can be drawn on that grid while respecting the bounce action. If a sona configuration $\mathcal{S}$ requires k circuits to draw it, using the Chokwe techniques, then we say $\mathbf{C}_{\mathcal{S}} = k$ (where $\mathbf{C}$ stands for "circuits"). We are thus interested primarily in sona configurations in which $\mathbf{C}_{\mathcal{S}} = 1$ and that are symmetrical in some way.

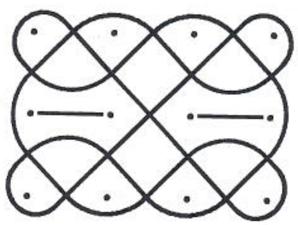

Figure 4: An example of a one-line drawing using "Celtic walls". These walls are used by the Chokwe in some of their sona, but less often, and are not used in this paper.

2. MATHEMATICAL TOOLS TO CONSTRUCT SYMMETRIC SONA

The effect on $\mathbf{C}$ of adding or removing a wall to a sona configuration depends on the exact location of the wall. Chavey & Straffin (2010) show that the number of circuits required for a sona configuration will always change by 1 whenever a Chokwe wall is added or removed. The portion of this theorem relevant here is:

Theorem 1 *(Chavey & Straffin)*: If a Chokwe wall is added to, or removed from, a sona configuration $\mathcal{S}$ that has $\mathbf{C}_{\mathcal{S}}$ circuits, then the new sona $\mathcal{S}'$ will have $\mathbf{C}_{\mathcal{S}'} = \mathbf{C}_{\mathcal{S}} \pm 1$ circuits. If a wall separating two different circuits of $\mathcal{S}$ is removed, then the new sona $\mathcal{S}'$ will have $\mathbf{C}_{\mathcal{S}'} = \mathbf{C}_{\mathcal{S}} - 1$ circuits. If a wall separating two different portions of the same circuit is removed, then the new sona will have 1 additional circuits, i.e. $\mathbf{C}_{\mathcal{S}'} = \mathbf{C}_{\mathcal{S}} + 1$.

This is the primary tool we use to show that certain grid layouts *cannot* give rise to a monolineal sona. This theorem will allow us to calculate the parity of $\mathbf{C}_{\mathcal{S}}$ for a sona configuration. If that parity is even, then we know we cannot have a monolineal sona; if that parity is odd, we can generally find a set of walls that will make $\mathbf{C}_{\mathcal{S}} = 1$.

A second tool useful for the construction of symmetric sona is the following result (Chavey & Straffin, 2010):

Theorem 2 *(Chavey & Straffin)*: Any sona grid will have some arrangement of Chokwe walls so that the resulting sona configuration generates a monolineal drawing.

This theorem is, essentially, a corollary to Euler's theorem on Eulerian circuits. While the sona from this theorem will not, generally, be symmetric, we will be able to use this result consistently on multiple regions of a symmetric sona grid to construct a combined sona that *will* be symmetric.

3. SYMMETRIC SONA

In Chavey (2010), we consider the existence of sona configurations with strip symmetry or wallpaper symmetry groups. In that paper, we limit ourselves to symmetric sona within rectangular sona grids, and allow boundary irregularities necessary in drawing finite sona from what are, theoretically, infinite patterns. Here, in contrast, we investigate the simpler finite design symmetry groups of C_n and D_n (the cyclic and dihedral groups), but we consider any shape of sona grid. In addition, we are interested in constructing sona that are precisely symmetrical, e.g. without any "modifications" at the boundaries, or "approximate" symmetries, such as the last sona in figure 3. While these symmetric designs may seem simpler than, say, the wallpaper designs, they represent an important group of the authentic Chokwe sona. For example, of the sona designs of the type discussed in this paper that appear in Gerdes (1994), we counted the following number of symmetry types:

C_2	C_4	D_1	D_2	D_4	Strip Groups	Wallpaper Groups
6	10	26	28	0	14	5

Thus if we wish to use mathematical ideas to model the sona designs of the Chokwe, this chart clearly shows that it is important to model the construction of these finite design symmetry groups. (These numbers are approximate, because there are subjective issues regarding which designs are of "the type" discussed in this paper, deciding whether a sona is more like a strip design or a finite design, etc.) As a consequence, for the rest of this paper, we consider the question:

Main Problem: Given a sona grid that itself has C_n or D_n symmetry, when will it be possible to find a collection of Chokwe walls so that the resulting sona configuration will have the same symmetries?

Note that our definition of "sona grid" assumes that this grid is connected, with respect to adjacencies in the unit lattice. A disconnected layout cannot generate a monolineal sona, although we may generate disconnected dot layouts temporarily in our proofs. We call a sona grid "*simply connected*" if there are no "holes" in the set of dots, such as occur in the sona of figure 5. Many of our results can be worded more concisely when limited to simply connected layouts. When a theorem is followed by a corollary for simply connected layouts, we suggest reading the corollary first.

To determine when we can convert symmetric sona grids into symmetric, monolineal sona, we will analyze the C_n or D_n symmetry groups individually. An approach we use repeatedly is: Use walls to break the sona into *fundamental regions*, i.e. portions of the design which are moved by the symmetries of Cn or Dn so as to cover all (or most) of the design; construct a lusona on the dots of one fundamental region and replicate that on each copy of that region; then remove some of the original walls so as to merge the result into a monolineal sona. We start with a simple case:

Theorem 3: If S is any sona grid that has bilateral symmetry, i.e. has symmetry group D_1, and *either* (1) S has even width, or (2) the layout dots on the reflection line are connected with respect to adjacencies in the unit lattice, then Chokwe walls can always be added maintaining that symmetry and creating a monolineal sona.

Corollary 3a: If S is a simply connected sona grid with symmetry group D_1, then Chokwe walls can always be added maintaining that symmetry and creating a monolineal sona.

Proof

We orient the layout so the reflection line is vertical. If the layout now has even width, we place walls on the reflection to separate the layout into two fundamental regions (see figure 5). If a fundamental region were to be disconnected, its mirror image would be disconnected in the same places, and hence the combination would be disconnected, which violates our assumptions. Since the left half is connected, theorem 2 tells us that there is a way to place walls so as to create a monolineal sona on that half, and we mirror that on the right half. Each wall on the reflection line now separates the two circuits of this sona, so theorem 1 tells us that we can remove any wall on that reflection line to create a monolineal sona on the full sona grid.

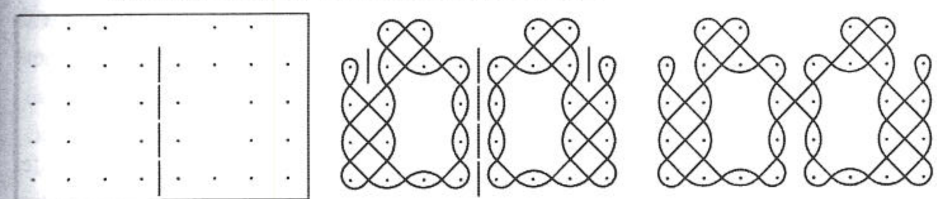

Figure 5: The construction of a monolineal, bilaterally symmetric sona from an even-width bilaterally symmetric sona grid.

If the layout has odd width, we add walls to separate all of those dots on the center reflecting line from the rest of the sona grid (see figure 6). Because of the assumption that the center column is connected, we can draw a sona on those dots with a single line. We construct a monolineal sona on one side of the walls, and *mirror it on the other side, creating a three-line drawing. We then* remove a matching pair of walls separating each of those two main regions from the center column. By theorem 1, removing those two walls combines the three lines into one. •

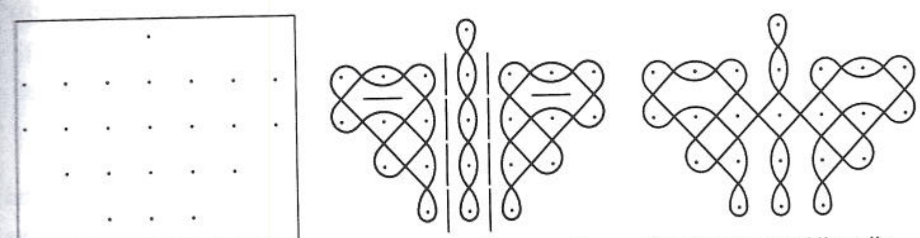

Figure 6: The construction of a monolineal, bilaterally symmetric sona from an odd-width bilaterally symmetric sona grid.

The proof shows that there is at least one way to construct a bilaterally symmetric monolineal sona, but in general there are many ways to do so. For example, there are usually many ways to construct a monolineal sona on one of the fundamental regions of the layout. We can, generally, connect the two halves via several choices of wall(s). Furthermore, if we consider bilateral even-width sona, removing a second center wall will (theorem 1), split the one circuit into two. Because those two circuits will be symmetrical with each other, the location of that wall will be one place where the two circuits cross, but there will usually be other places where another one of the center walls separates branches of those two circuits, hence removing that wall will fuse the two circuits back into one (theorem 1), giving us another monolineal sona on that layout. Similarly, if in figure 6 we were to remove another pair of walls around the central column, theorem 1 tells us that we will end up with either 1 circuit or else with 3. In fact, two of the choices will give us other monolineal sona, and only one of them gives a 3-line sona.

The converse of theorem 3 is also true: If the sona layout has odd width, and the center column is *not* connected, then it will *not* be possible to create a monolineal, symmetric sona on the layout. The proof of this appears to be more difficult than other similar results in this paper, and is not included here. It can be found in a somewhat extended version of this paper at <http://math.beloit.edu/chavey/Sona/>

To continue this process with larger symmetry groups, it is useful to establish a general lemma that applies to all of these cases. In lemma 4, we adopt group theory terminology and we say that an *orbit* of a dot d (in a sona layout) is the set of all dots in the layout to which d is mapped by the symmetries of that layout. The size of such an orbit is always 1, 2, 4, or 8.

Lemma 4: If the orbit of a dot d has size ≥ 2, then all dots of that orbit may be removed from the sona grid without changing the parity of c_S for the resulting sona, *except* for an orbit of size 2 whose dots are adjacent to each other.

Proof:

If d is disconnected from the rest of the sona grid, the sona line at that dot, and each dot in its orbit, will be a single circle around the dot. We can delete these dots and lines with no change in parity. If d is connected to other dots in the layout, it can be disconnected by the insertion of w walls, where $1 \leq w \leq 4$. If d is not adjacent to other dots in its orbit (of size $s \geq 2$), then there is no overlap (i.e. double counting) in these $s{\cdot}w$ walls needed to separate this orbit of dots from the rest of the sona grid, so this separation requires an even number of walls. By theorem 1, adding those walls will not change the parity of lines in the drawing. Deleting the resulting circles around those dots (and the now useless walls, if we wish) also leaves the parity unchanged.

A wall will be double counted in $s{\cdot}w$ only if it is a wall separating two adjacent dots within an orbit. If d is in an orbit of size 4 or 8, then the number of walls that are double counted is even, regardless of the dot adjacencies in that orbit, hence the number of walls needed to separate these dots is still even, and the lemma holds for them as well. •

This lemma will allow us to show that certain sona grids will *not* be able to support monolineal sona that have the same symmetries as the sona grids themselves, because they will not be able to support *any* symmetric sona with an odd number of lines. To describe when this happens, it is useful to have the concept of the *center dots* of a bounding rectangle. These are the locations where sona dots *could* be placed which are closest to (less than one unit away from) the rotational center of the sona grid. If the bounding rectangle has both dimensions with odd lengths, there will be a single center dot; if both dimensions have even length, there will be four center dots forming a square; and if one dimension is odd and the other is even, there will be two center dots, adjacent to each other.

Theorem 5: If S is a sona grid with symmetry group C_2 or C_4, then Chokwe walls can be added maintaining that symmetry and creating a monolineal sona if and only if at least one of the dimensions of the bounding rectangle of S has odd length *and* the center dot(s) of the bounding rectangle are part of the sona grid.

Corollary 5a: If S is a simply connected sona grid with symmetry group C_2 or C_4, then Chokwe walls can be added maintaining that symmetry and creating a monolineal sona if and only if at least one of the dimensions of the bounding rectangle of S has odd length.

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The simplification of the corollary will follow from the theorem because in a simply connected sona with rotational symmetry, the center dot(s) of the bounding rectangle must be part of the sona grid.

If both dimensions of the bounding rectangle are odd, or both are even, then walls in any sona that maintain the grid symmetry come in orbit pairs (or groups of 4), and hence those sona do not have different parity than the sona that would be constructed if there were no walls. If one dimension is odd and one is even, then the same is true for all walls except a possible wall at the location of the rotational center of the sona grid, which is mapped to itself by C_2. Thus we can remove all walls from the sona, except that one which may be at the rotational center for odd by even rectangles, without changing the parity of c_S. From the original sona grid, we can use lemma 4 to remove all dots of the layout except for the center dots without changing the parity of c_S. If both dimensions of the bounding rectangle have odd length, and the single center dot is missing, then removing these dots gives a "sona" with 0 lines, hence any symmetric sona will have an even number of lines, and cannot be monolineal. The same statement holds if the bounding rectangle has dimensions odd by even and it is missing the two center dots (because of the rotational symmetry, it cannot be missing one center dot without missing both). Thus in either case, the sona grid must include its center dots to be able to construct a monolineal sona on the layout.

If both dimensions of the bounding rectangle are even, the center dots will form a 2 X 2 square which, if all 4 dots are present, will require 2 lines. Thus if all 4 dots are present or all 4 are missing, any sona on this layout will require an even number of lines, hence will not be monolineal. If 2 dots are present, then they must be opposite corners of the center square, and the symmetry group must be C_2. But in this case, the sona on those 2 dots would be 2 disconnected circles, i.e. an even number, so again the sona cannot be monolineal.

This establishes the non-existence of symmetric sona as ruled out by the theorem; it remains only to show the existence of the remaining classes of symmetric sona.

If the dimensions of the bounding rectangle are even length by odd length, we use walls to cut the layout in half along the odd length dimension (see figure 7). Unlike the previous theorem, this can decompose each half into multiple connected components, such as happens in figure 7. We apply theorem 2 to each component, e.g. adding the gray walls shown on the left in figure 7. If we have components that are not adjacent to the center dots, we connect them by deleting one of the center walls for each such component, chosen symmetrically, e.g. the "X"-d walls of figure 7. Finally, we delete the wall separating the two adjacent center dots, resulting in a monolineal sona.

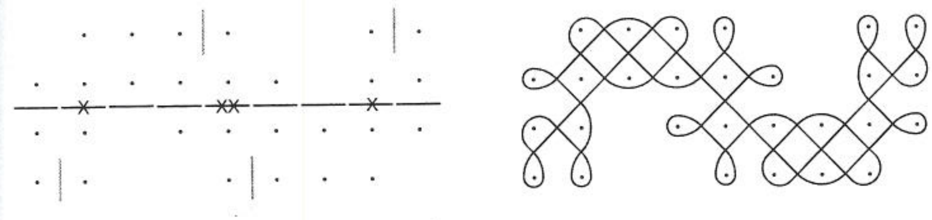


Figure 7: A construction of a monolineal, rotationally symmetric sona from an odd-length, even-height, symmetric sona grid.

If the dimensions of the bounding rectangle are odd length and height, we use walls to cut out a central row or column (see figure 8). For each component, we add walls (shown in gray) to create monolineal components. For each component not adjacent to the center dot, we delete a single wall, chosen symmetrically (shown "X"-d in figure 8, resulting in a 3-line drawing (e.g. the center design in figure 8). We then delete the two walls adjacent to the center dot (shown "XX"-d in figure 8) to combine those three lines into a single line, such as shown on the right in figure 8.

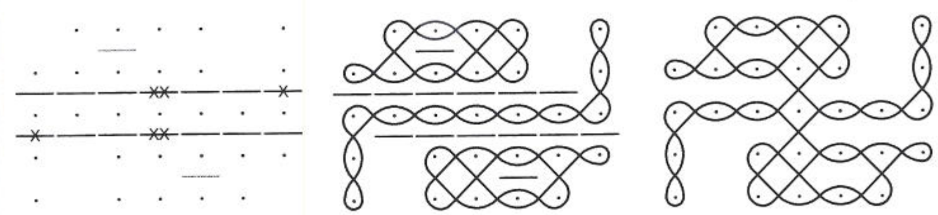


Figure 8: A construction of a monolineal, rotationally symmetric sona from an odd-length, odd-height, symmetric sona grid.

Finally, if the symmetry group is C_4, then the bounding rectangle must be a square, and by the earlier non-existence result in this proof, we may assume it has odd dimension. By the theorem's statement, we may assume that the center dot of this bounding square is an element of the sona grid. We add a wall immediately below this center dot, and add walls to the right of it in a straight line; we then add walls to the other three sides of the center dot to make this symmetrical, e.g. as shown on the left in figure 9. We use theorem 2 to add walls to make each component monolineal. As before, if some components in a quadrant are disconnected from the main component, i.e. the one adjacent to the center dot, we delete a wall along each of the "spiral arms," symmetrically, until we have one connected component for each quadrant. Combined with a single circle around the center dot, this gives us a 5-line drawing. Deleting each of the four walls adjacent to the center dot combines all five lines into one. •

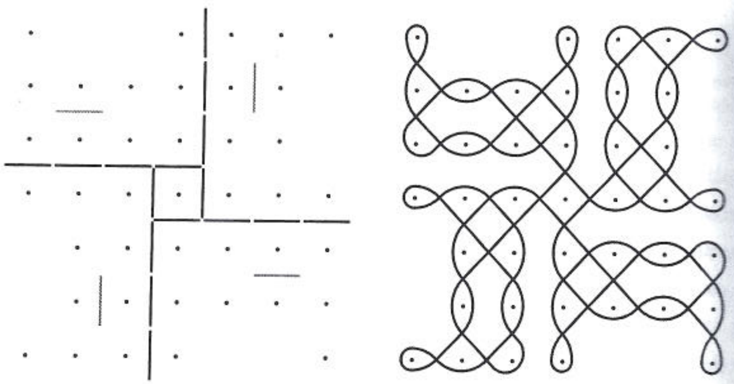


Figure 9: A construction of a monolineal, 4-fold rotationally symmetric sona from an odd-dimension symmetric sona grid.

Note that theorem 5 tells us that there will be no arrangement of walls that will create a monolineal sona on a 4 X 4 square of dots such that the resulting sona has C_4 symmetry. This helps explain why the authentic Chokwe sona on the far right of figure 3 can only have "approximate" C_4 symmetry; no perfect rotation symmetry is possible, and we strongly suspect that the Chokwe artists knew this. See Gerdes (2006), chapter 9 for a discussion of how the Chokwe constructed sona with C_4 symmetry for even sized squares by using the "roving lines" that are a feature of classes of sona not discussed in this paper.

Theorem 6: If S is a sona grid with symmetry group D_2, then Chokwe walls can be added maintaining that symmetry and creating a monolineal sona if and only if at least one of the dimensions of the bounding rectangle of S has odd length *and* the center dot(s) of the bounding rectangle are part of the sona grid.

Corollary 6a: If S is a simply connected sona grid with symmetry group D_2, then Chokwe walls can be added maintaining that symmetry and creating a monolineal sona if and only if at least one of the dimensions of the bounding rectangle of S has odd length.

Proof

The simplification of the corollary will follow from the theorem because in a simply connected sona with rotational symmetry, that center dot(s) of the bounding rectangle must be part of the sona grid. The non-existence aspects of the theorem are inherited from theorem 5, since D_2 designs must also have C_2 symmetry.

To construct a monolineal sona on a grid meeting the conditions of the theorem, we orient the sona grid so the odd dimension is horizontal, use walls to isolate the central column of the layout, and add walls along the horizontal reflection line except at the central column (see figure 10). As noted in the proof of theorem 3, the reflection symmetry guarantees us that each of the quadrants will consist of connected components, so the extra step of theorem 5 to handle multiple components is not necessary. We use theorem 2 to find a monolineal sona for one of the quadrants, and copy that solution, symmetrically, to the other 3 quadrants. This gives us a 5-line drawing. Removing the 4 walls of the central column that are adjacent to the two center dots maintains the D_2 symmetry and merges those 5 lines into one, creating our monolineal sona. Figure 10 shows 2 solutions for a particular sona grid (of the 8 possible solutions that involve only 1 wall inside each quadrant). •

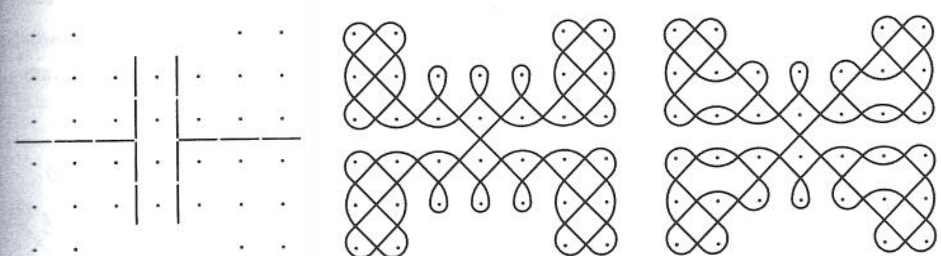


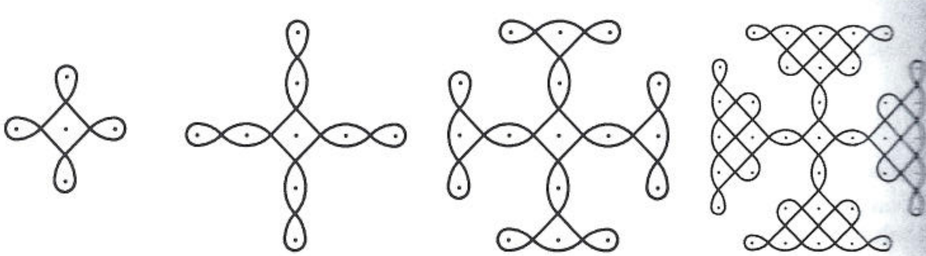
Figure 10: A construction of a monolineal, D_2 symmetric sona from an odd-width, even-height, symmetric sona grid. Two solutions are shown, which differ only by the choice of the monolineal sona chosen for the individual quadrants. There are six other choices involving only one wall per quadrant, and many additional choices involving three walls per quadrant.

Finally, we are faced with the most difficult of these symmetric layout problems, i.e. when the symmetry group is D_4. The table at the beginning of section 3 indicates that none of the authentic sona shown by Gerdes (1994) have D_4 symmetry, although the left-most sona of figure 11 occurs in other works by Gerdes. This agrees with our experience that it is quite difficult to construct interesting sona with D_4 symmetries. We have been unable to satisfactorily answer the question of which D_4 symmetric sona grids have D_4 symmetric sona. A partial solution to this problem is given in theorem 7. This theorem *does* find all D_4 symmetric sona grids of size 5 X 5 or smaller, and we have been unable to find any such layouts that do not fit the conditions of this theorem, but we have not been able to show that these are the only possible solutions.

Theorem 7: If S is a simply connected sona grid with symmetry group D_4, the length of the bounding square of S is odd, and the center 3 X 3 square of the layout consists of exactly those dots of the 3 X 3 sona shown on the left of figure 11 then Chokwe walls can be added maintaining D_4 symmetry and creating a monolineal sona.

Proof

To maintain D_4 symmetry, the center 3 X 3 square of the sona grid must be either a complete square, or else the arrangement shown on the left of figure 11. If the center of the sona grid is the arrangement of figure 11, then the assumption of simply connectedness means that if we place 4 walls around the center dot, the rest of the sona grid breaks down into 4 separate sona grid components, each one of which is connected. Since the full sona grid has D_4 symmetry, these components each have D_1 symmetry. By theorem 3, we can add Chokwe walls to one of these components to create a bilaterally symmetric, monolineal sona, which we replicate on each of these components. Removing the walls we originally placed around the center dot merges each of these component sona into a single, monolineal sona on the full sona grid. •



4. CONCLUSIONS

We have limited ourselves to one type of sona design, and the reader should see the works of Gerdes (e.g. 1994, 2006) for other kinds of sona designs. Most of these other designs share the fascination for symmetry and monolineal drawings that are a cultural value of the Chokwe artistic tradition. However, these other sona use techniques other than just this "billiard ball and walls" model. Nevertheless, within this particular class of Chokwe sona we can find mathematical tools to build additional sona of nearly all shapes and sizes. We can also use these results to explain why the Chokwe found it necessary to "fudge" the symmetry in a sona such as that on the right in figure 3. Through this modeling, it appears that we can build additional sona designs that are consistent enough with the Chokwe artistic aesthetic that we feel the Chokwe would be pleased with the designs. The proofs of the existence of these symmetric sona give guidelines for the construction of such symmetric sona, but leave enough room for flexibility and experimentation that this can make interesting experimental projects for students at the middle school through college level.

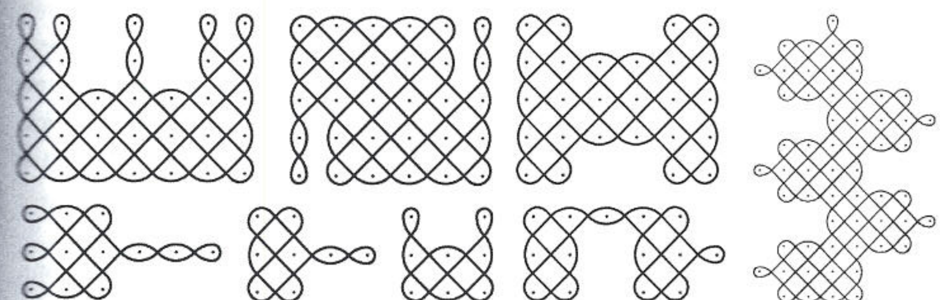


Figure 12: Additional authentic Chokwe sona with C_n and D_n symmetry. Top row: Muyombo trees, Mole-rat and bat, Two men at a table drinking beer, Leopard with 5 Cubs. Bottom row: Scorpion, Mallet, a design to accompany a story that even a slave should be treated humanely (offering at a Muyombo tree), and the torn (or worm-eaten) pelt of a genet (cat). (Fontinha 1983, Gerdes 2007, Kubik 2006)

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