Chapter 10 Quantum Statistics (1)

In chapter 4, using grand canonical ensemble, we derived:

$$\Xi(V,T,\lambda) = \prod_{k} (1 \pm \lambda e^{-\beta \varepsilon_k})^{\pm 1}$$

$$N = \sum_k \frac{\lambda e^{-\beta \varepsilon_k}}{1 \pm \lambda e^{-\beta \varepsilon_k}} \qquad \beta = \frac{1}{kT}$$

$$\bar{n}_k = \frac{\lambda e^{-\beta \varepsilon_k}}{1 \pm \lambda e^{-\beta \varepsilon_k}} \qquad \lambda = e^{\beta \mu}$$

$$E = \sum_k \frac{\lambda \varepsilon_k e^{-\beta \varepsilon_k}}{1 \pm \lambda e^{-\beta \varepsilon_k}}$$

$+$: Fermi - Dirac Statistics

$-$: Bose - Einstein Statistics

if λ is very small, both become Boltzman distribution

This chapter consider weak & strong degeneracy.

Weakly Degenerate Fermi-Dirac Gas

$$N = \sum_k \frac{\lambda e^{-\beta \varepsilon_k}}{1 + \lambda e^{-\beta \varepsilon_k}}$$

$$PV = kT \sum_k \ln (1 + \lambda e^{-\beta \varepsilon_k})$$

$$\varepsilon_{n_x, n_y n_z} = \frac{h^2}{8m V^{2/3}} (n_x^2 + n_y^2 + n_z^2) \qquad a^2 = V^{2/3}$$

at room temperature

$$\varepsilon_{k+1} \sim \varepsilon_k \qquad \Sigma \rightarrow \int$$

$$(10\text{-}9) \quad N = 2\pi \left(\frac{2m}{h^2}\right)^{3/2} V \int_0^\infty \frac{\lambda \varepsilon^{\frac{1}{2}} e^{-\beta\varepsilon} d\varepsilon}{1+\lambda e^{-\beta\varepsilon}}$$

(10-10)

$$PV = 2\pi kT \left(\frac{2m}{h^2}\right)^{3/2} V \int_0^\infty \varepsilon^{\frac{1}{2}} \ln(1+\lambda e^{-\beta\varepsilon}) d\varepsilon$$

$$\left(1\text{-}35 \quad w(\varepsilon, \Delta\varepsilon) = \text{const.}\ \varepsilon^{\frac{1}{2}} \Delta\varepsilon \right)$$

10-9 & 10-10 can be expanded

$$\frac{N}{V} = \rho = \frac{1}{\Lambda^3} \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1} \lambda^\ell}{\ell^{3/2}} \qquad (10\text{-}11)$$

$$\frac{P}{kT} = \frac{1}{\Lambda^3} \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1} \lambda^\ell}{\ell^{5/2}} \qquad (10\text{-}12)$$

$$\Lambda = \left(\frac{h^2}{2\pi m kT}\right)^{1/2}$$ Thermal De Broglie λ

idea: find λ as a function of ρ and plug into 10-12 to derive state function

Assume $\lambda = a_0 + a_1\rho + a_2\rho^2 + \cdots$

compare with (10-11)

$$a_0 = 0, \quad a_1 = \Lambda^3, \quad a_2 - \frac{a_1^2}{2^{3/2}} = 0$$

$$a_3 - \frac{a_1 a_2}{2^{1/2}} + \frac{a_1^3}{3^{3/2}} = 0$$

$$\lambda = \rho\Lambda^3 + 2^{-3/2}(\rho\Lambda^3)^2 + \left(\frac{1}{4} - \frac{1}{3^{3/2}}\right)(\rho\Lambda^3)^3 + \cdots$$

$$\frac{P}{KT} = \rho + \frac{\Lambda^3}{2^{5/2}}\rho^2 + \left(\frac{1}{8} - \frac{2}{3^{5/2}}\right)\Lambda^6 \rho^3 + \cdots$$

$$= \rho + B_2(T)\rho^2 + B_3(T)\rho^3 + \cdots$$

$B_j(T)$ jth virial coefficient

which desaibes interaction among particles.

although we do not have real interaction, but quantum effect (symmetry requirement in ψ) caused some effective interaction.

Λ is small. quantum effect is small

Further:

$$E = \frac{3}{2} VKT \frac{1}{\Lambda^3} \sum_{l=1}^{\infty} \frac{(-1)^{l+1}\lambda^l}{l^{5/2}}$$

$$= \frac{3}{2} NKT \left(1 + \frac{\Lambda^3}{2^{5/2}}\rho + \cdots\cdots\right)$$

Strongly Degenerate Ideal Fermi-Dirac Gas

Degeneracy $\uparrow$ as $T \downarrow$, $\rho \uparrow$, quantum effect $\uparrow$

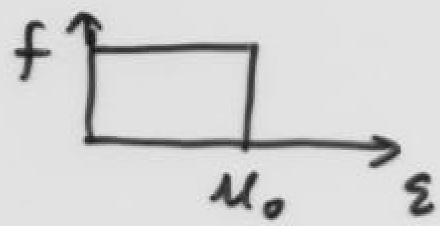
number of particles in ε_K is

$$\bar{n}_K = \frac{\lambda e^{-\beta \varepsilon_K}}{1+\lambda e^{-\beta \varepsilon_K}} = \frac{1}{1+e^{\beta(\varepsilon_K - \mu)}}$$

again we can treat it as a continous function

$$f(\varepsilon) = \frac{1}{1+e^{\beta(\varepsilon-\mu)}}$$

$T > 0$ $\qquad\qquad$ $T = 0$



μ_0 : Fermi energy

let's consider $\omega(\varepsilon)$ again, except $\times 2$ (for electrons) (2 spin states)

$$\omega(\varepsilon)\, d\varepsilon = 4\pi \left(\frac{2m}{h^2}\right)^{3/2} V \varepsilon^{\frac{1}{2}} d\varepsilon$$

Total number of electrons

$$N = 4\pi \left(\frac{2m}{h^2}\right)^{3/2} V \int_0^{\mu_0} \varepsilon^{\frac{1}{2}} d\varepsilon$$

$$= \frac{8\pi}{3} \left(\frac{2m}{h^2}\right)^{3/2} V (\mu_0)^{3/2}$$

$$\boxed{\mu_0 = \frac{h^2}{2m} \left(\frac{3}{8\pi}\right)^{2/3} \left(\frac{N}{V}\right)^{2/3}}$$

e.g. $\mu_0(Na) = 3.1\,eV$

μ_0/k — Fermi temperature

$$E_0 = 4\pi\left(\frac{2m}{h^2}\right)^{3/2} V \int_0^{\mu_0} \varepsilon^{3/2}\, d\varepsilon$$

$$= \frac{3}{5} N \mu_0$$

$$P_0 = 4\pi kT \left(\frac{2m}{h^2}\right)^{3/2} \int_0^{\mu_0} \varepsilon^{\frac{1}{2}} \ln\left(1+e^{-\beta(\varepsilon-\mu_0)}\right) d\varepsilon$$

$$\simeq 4\pi \left(\frac{2m}{h^2}\right)^{3/2} \int_0^{\mu_0} \varepsilon^{\frac{1}{2}} (\mu_0 - \varepsilon)\, d\varepsilon$$

$$= \frac{2}{5} N \mu_0 / V \qquad \simeq O(10^6) \text{ atm}$$

we $T \neq 0$.

$$\eta = (\beta\mu_0)^{-1} < 1$$

$$\mu = \mu_0 \left[1 - \frac{\pi^2}{12}\eta^2 + \cdots\right]$$

$$E = E_0 \left[1 + \frac{5\pi^2}{12}\eta^2 + \cdots\right]$$

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{\pi^2 NkT}{2(\mu_0/k)} = \frac{\pi^2}{2} Nk \left(\frac{T}{T_F}\right)$$

Weakly Degenerate Ideal Bose-Einstein Gas

$$N = \sum_k \frac{\lambda e^{-\beta\varepsilon_k}}{(1-\lambda e^{-\beta\varepsilon_k})}$$

$$PV = -kT \sum_k \ln\left(1 - \lambda e^{-\beta\varepsilon_k}\right)$$

special care: ground state must be excluded to convert Σ to $\int$

$$N = \frac{\lambda e^{-\beta\varepsilon_0}}{1-\lambda e^{-\beta\varepsilon_0}} + 2\pi\left(\frac{2m}{h^2}\right)^{3/2} V \int_{\varepsilon>\varepsilon_0}^{\infty} \frac{\lambda \varepsilon^{\frac{1}{2}} e^{-\beta\varepsilon} d\varepsilon}{1-\lambda e^{-\beta\varepsilon}}$$

$$\varepsilon_0 = \frac{3h^2}{8mV^{2/3}} \approx 0$$

$$\rho = \frac{N}{V} = 2\pi\left(\frac{2m}{h^2}\right)^{3/2} \int_{\varepsilon>0}^{\infty} \frac{\lambda \varepsilon^{\frac{1}{2}} e^{-\beta\varepsilon} d\varepsilon}{1-\lambda e^{-\beta\varepsilon}} + \frac{\lambda}{V(1-\lambda)}$$

$$\frac{P}{KT} = -2\pi\left(\frac{2m}{h^2}\right)^{3/2} \int_{\varepsilon>0}^{\infty} \varepsilon^{\frac{1}{2}} \ln(1-\lambda e^{-\beta\varepsilon}) d\varepsilon - \frac{1}{V}\ln(1-\lambda)$$

$$0 \le \lambda < 1$$

very similar to F-D gas, assume

$$\rho = \frac{1}{\Lambda^3} g_{3/2}(\lambda)$$

$$\frac{P}{KT} = \frac{1}{\Lambda^3} g_{\frac{5}{2}}(\lambda) \qquad g_n(\lambda) = \sum_{\ell=1}^{\infty} \frac{\lambda^\ell}{\ell^n}$$

$$\frac{P}{\rho KT} = 1 - \frac{\Lambda^3}{2^{5/2}}\rho + \cdots$$

effective interaction: attractive

$$E = \frac{3}{2} VKT \frac{1}{\Lambda^3} g_{\frac{5}{2}}(\lambda)$$

$$= \frac{3}{2} NKT \left(1 - \frac{\Lambda^3}{2^{5/2}}\rho + \cdots\right)$$