Plasma Physics II:
Kinetic Theory of Warm Plasmas

- The Cold Plasma Approach
- Generalization of the Cold Plasma Theory
- Vlasov Equation
- Landau Damping
- ~~Quasi-Linear Theory~~
- ~~Instabilities and Nonlinear Effects~~

⇨ **Introduction to solar radiations**

Credit to: pma.caltech.edu/Courses/ph136 (chapters 20-21)

The previous lecture: kinetics of individual particles

- The field particles : a mean temperature and density
- The test particle is mono-energetic.

Remaining Issues

- The m.f.p. $\gg$ macroscopic inhomogeneity scale.
- Collective excitation of internal EM field
- A variety of different kinds of plasmons
- Presence of magnetic fields

This lecture

- Start with the cold plasma approach
- Some Generalization
- The Vlasov theory (a linear theory)
- Nonlinear Phenomena

Contents

1. The Cold Plasma Approach

- Deals with linear waves propagating in a plasma ignoring the effects of the thermal motions of the particles.
- can guide us to the correct dispersion relation for two simple modes in unmagnetized plasma: *Langmuir waves* & *ion acoustic waves.*

Maxwell's equation

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\varepsilon_0}, \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = -\mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

From the last two equations,

$$\nabla \times (\nabla \times E) + \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = -\mu_0 \frac{\partial j}{\partial t}$$

Fourier transforming

$$\mathbf{k} \times [\mathbf{k} \times \mathbf{E}(\omega, \mathbf{k})] + \frac{\omega^2}{c^2}\mathbf{E}(\omega, \mathbf{k}) = i\mu_0 \omega \mathbf{J}(\omega, \mathbf{k})$$

or

$$\Big[\frac{c^2}{\omega^2} k_i k_j - \frac{c^2}{\omega^2} k^2 \delta_{ij} + \delta_{ij} + \frac{i}{\varepsilon_0 \omega}\sigma_{ij}\Big] E_j = 0$$

where we used $J_i = \sigma_{ij} E_j$ and $\mu_0 \varepsilon_0 = 1/c^2$.

Alternatively we write

$$\begin{aligned} L_{ij}E_j &= 0, \\ L_{ij} &= \frac{c^2}{\omega^2}(k_i k_j - k^2\delta_{ij}) + \epsilon_{ij} \\ \epsilon_{ij} &= \delta_{ij} + \frac{i}{\varepsilon_0\omega}\sigma_{ij}(\omega, \mathbf{k}) \end{aligned} \tag{1}$$

- The *dielectric tensor*, ϵ_{ij}, specifies the EM property of the medium.
- For (1) to have a solution, $||L_{ij}|| = 0$, which gives a particular wave mode.
- We must give a prescription for calculating the conductivity tensor σ_{ij}.

A simple prescription involves treating electrons and ions collectively as fluids, and apply the equation of continuity and an equation of motion

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s) = 0$$

$$n_s m_s \left[\frac{\partial \mathbf{u}_s}{\partial t} + (\mathbf{u}_s \cdot \nabla)\mathbf{u}_s\right] = -\nabla \mathbf{P}_s + n_s q_s (\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

where $\mathbf{P}_s = n_s m_s \langle(\mathbf{v} - \mathbf{u}) \times (\mathbf{v} - \mathbf{u})\rangle$, the pressure tensor.

By a "cold" plasma we mean $P = 0$, and further simplify considering an unmagnetized plasma ($B_0 = 0$) and only keep the linear wave modes,

$$-i\omega n_s m_s u_s = q_s n_s E$$

We then obtain the linearized current density and the conductivity tensor:

$$\mathbf{j} = \sum_s n_s q_s \mathbf{u}_s = \sum_s \frac{i n_s q_s^2}{m_s \omega} \mathbf{E}, \qquad \sigma = \sum_s \frac{i n_s q_s^2}{m_s \omega}.$$

Note that σ is pure imaginary to guarantee $\langle \mathbf{j} \cdot \mathbf{E} \rangle = 0$. Since σ is known we can express the dielectric tensor as

$$\epsilon_{ij} = \delta_{ij} + \frac{i}{\varepsilon_0 \omega} \sigma_{ij} = \left(1 - \frac{\omega_p^2}{\omega^2}\right) \delta_{ij}$$

with $\omega_p^2 = \sum_s \frac{n_s q_s^2}{m_s \varepsilon_0}$.

If we let $\mathbf{k} = k\hat{e}_3$

$$\mathbf{L} = \begin{pmatrix} 1 - \frac{c^2k^2}{\omega^2} - \frac{\omega_p^2}{\omega^2} & 0 & 0 \\ 0 & 1 - \frac{c^2k^2}{\omega^2} - \frac{\omega_p^2}{\omega^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\omega^2} \end{pmatrix},$$

and use $||L_{ij}|| = 0$ to give the dispersion relation

$$\left(1 - \frac{c^2k^2}{\omega^2} - \frac{\omega_p^2}{\omega^2}\right)^2 \left(1 - \frac{\omega_p^2}{\omega^2}\right) = 0$$

Formally there are six solutions corresponding to three pairs of waves.

- Two pairs of waves are degenerate with frequency

$$\omega^2 = \omega_p^2 + c^2 k^2$$

The associated eigenvectors have electric vectors lying along $\hat{e}_1$ and $\hat{e}_2$, i.e. perpendicular to $\hat{k}$. They are known as *electromagnetic modes*. We can use the 3rd Maxwell equation to solve for the magnetic perturbation

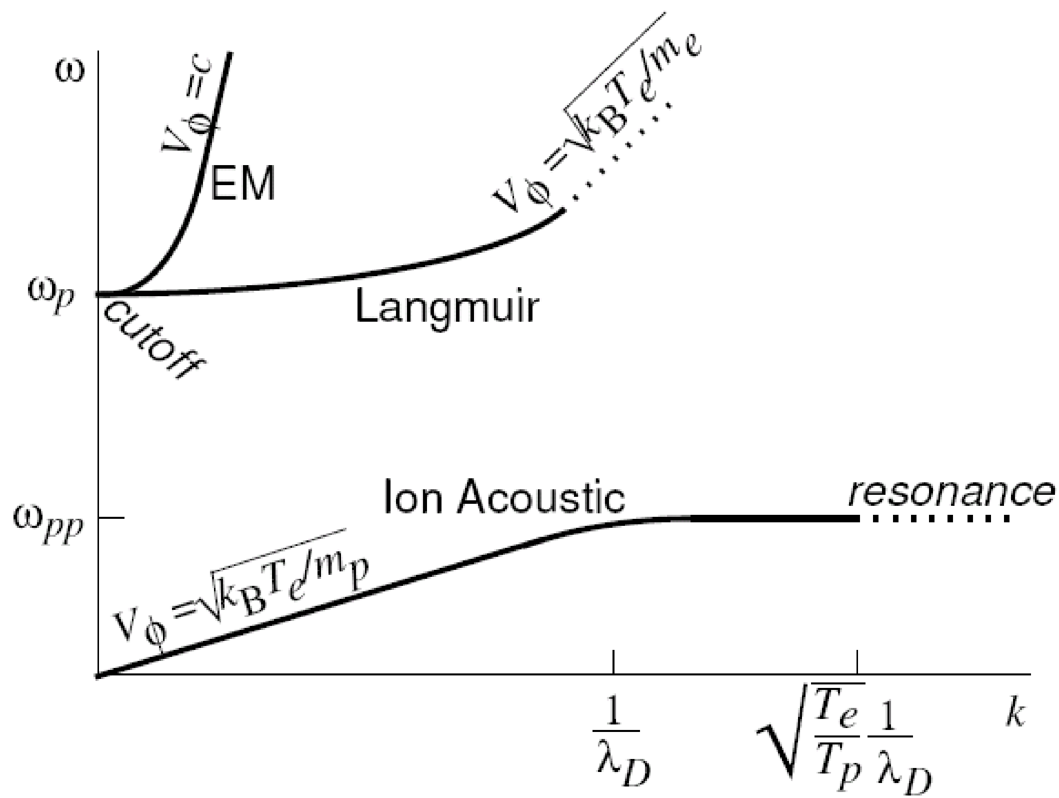
$$\mathbf{B} = \frac{\mathbf{k} \times \mathbf{E}}{\omega}$$

Note $\mathbf{B} \to 0$ as $\omega \to \omega_p$.

- The other pair of modes only propagates with a single frequency, the plasma frequency in this limit.

$$\omega^2 = \omega_p^2$$

This is just the plasma oscillation (Langmuir waves) in which the whole plasma oscillates in phase. This is an *electrostatic mode* in which $\mathbf{E} \parallel \mathbf{k}$.



Wave modes in magnetized plasma

$$-i\omega \mathbf{u}_s = \frac{q_s \mathbf{E}}{m_s} + \frac{q_s}{m_s} \mathbf{u} \times \mathbf{B}_0$$

gives

$$\epsilon = \begin{pmatrix} \epsilon_1 & -i\epsilon_2 & 0 \\ i\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix},$$

where

$$\epsilon_1 = 1 - \sum_s \frac{\omega_{p,s}^2}{\omega^2 - \omega_{c,s}^2},$$

$$\epsilon_2 = \sum_s \frac{\omega_{p,s}^2 \omega_{c,s}}{\omega(\omega^2 - \omega_{c,s}^2)}, \quad \epsilon_3 = 1 - \sum_s \frac{\omega_{p,s}^2}{\omega^2}$$

Let wave propagate in the x-z plane at θ to z-axis, i.e. $\mathbf{B}_0$, then the corresponding L-matrix is

$$\left(L_{ij}\right)=\frac{\omega^2}{c^2}\begin{pmatrix}\varepsilon_1-n^2\cos^2\theta & -i\varepsilon_2 & n^2\sin\theta\cos\theta\\ i\varepsilon_2 & \varepsilon_1-n^2 & 0\\ n^2\sin\theta\cos\theta & 0 & \varepsilon_3-n^2\sin^2\theta\end{pmatrix}$$

where $\quad n=\dfrac{ck}{\omega}$

$\det\left(L_{ij}\right)=0 \quad$ gives

$$\tan^2\theta=-\frac{\varepsilon_3\left(n^2-\varepsilon_R\right)\left(n^2-\varepsilon_L\right)}{\varepsilon_1\left(n^2-\varepsilon_3\right)\left(n^2-\dfrac{\varepsilon_R\varepsilon_L}{\varepsilon_1}\right)}$$

Consider parallel propagation $\theta=0$

& perpendicular propagation $\theta=\pi/2$

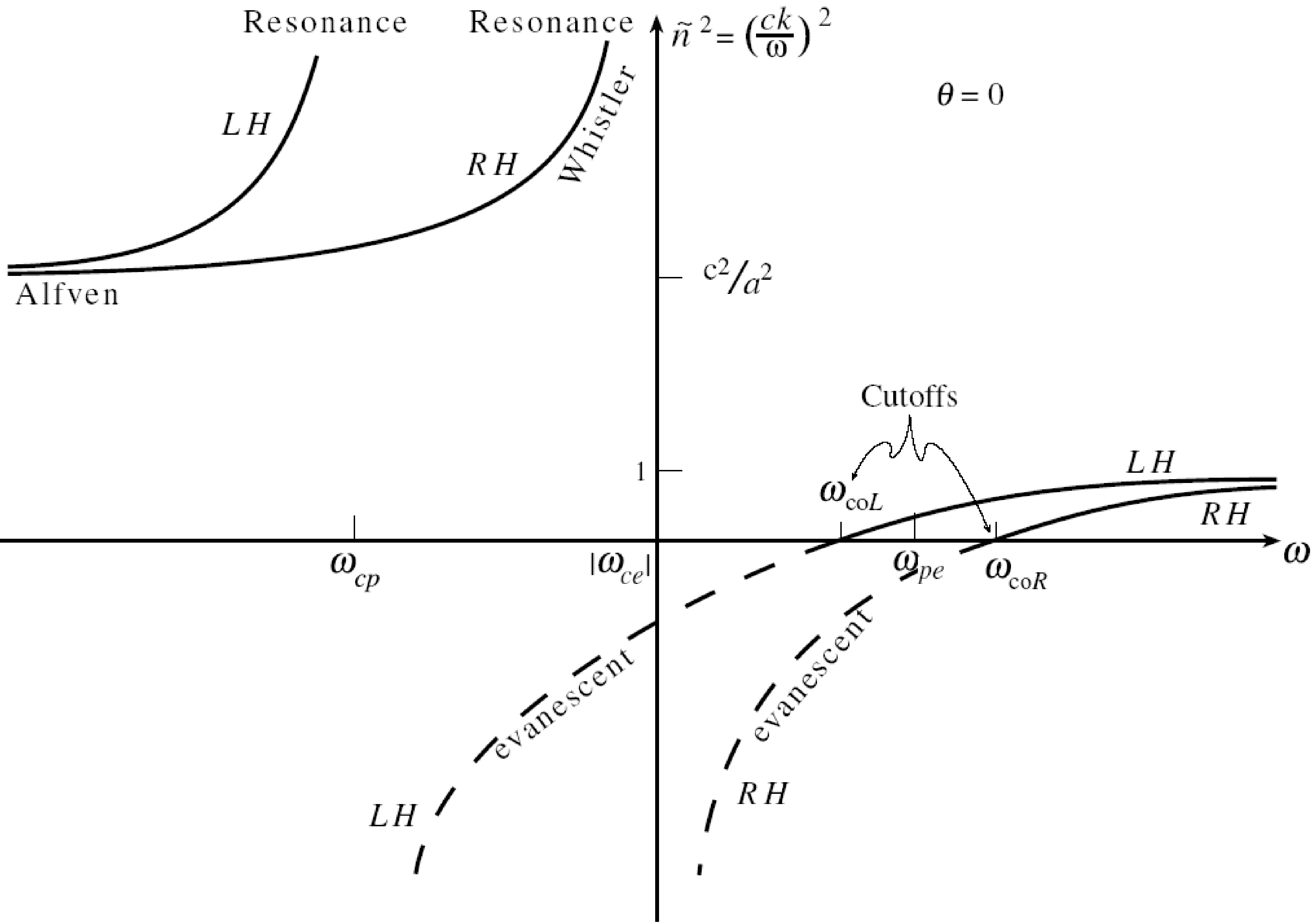
Parallel propagation ($\mathbf{k} = k\hat{e}_3$)

$$\frac{C^2 k^2}{\omega^2} = \epsilon_R = \epsilon_1 - \epsilon_2 \quad \text{(RH)}$$

$$\frac{C^2 k^2}{\omega^2} = \epsilon_L = \epsilon_1 + \epsilon_2 \quad \text{(LH)}$$

$$0 = \epsilon_3 \qquad \text{(electronstatic plasma osc.)}$$

- Plots of square of the index of refrarction
- Cut-off : $n = 0$: wave modes becomes evanescent.
- Resonance: $n = \infty$, $v_\phi = 0$, assoc. with strong absorp.
- Faraday rotation: $n_R - n_L = \frac{\omega_p^2 |\omega_c|}{\omega^3}$
- Whistler mode: LH, only $\omega_{c,i} < \omega < \omega_{c,e}$



Resonance
Resonance
$\tilde{n}^2 = \left(\frac{ck}{\omega}\right)^2$
$\theta = 0$
LH
RH
Whistler
Alfven
c^2/a^2
Cutoffs
1
ω_{coL}
LH
RH
ω
ω_{cp}
$|\omega_{ce}|$
ω_{pe}
ω_{coR}
evanescent
evanescent
RH
LH

Perpendicular propagation ($\mathbf{k} = k\hat{e}_1$)

$$(n^2 - \epsilon_3)([n^2 - \epsilon_1]\epsilon_1 + \epsilon_2^2) = 0$$

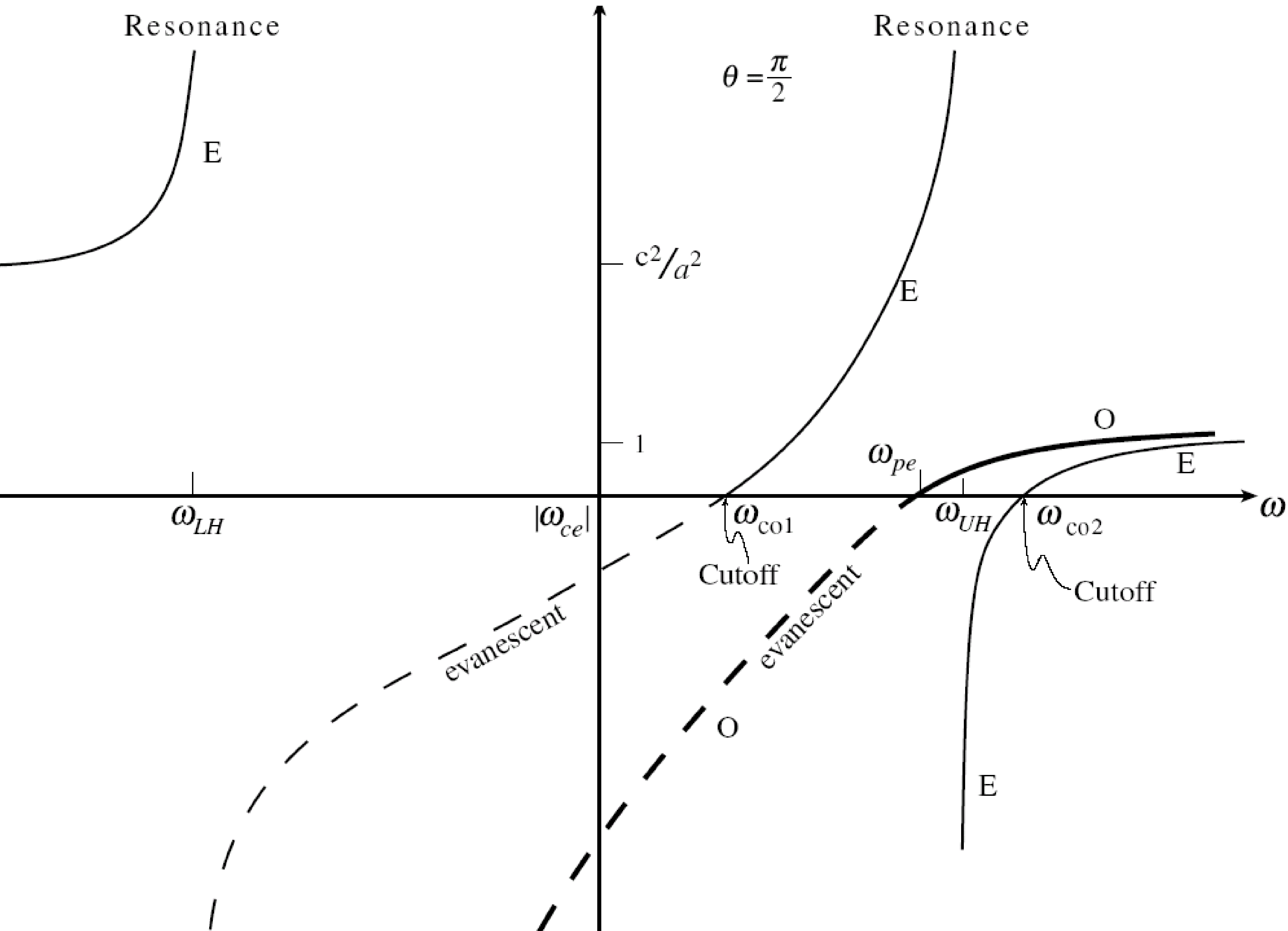
- O-mode : $n = (1 - \omega_{pe}^2/\omega^2)^{1/2}$ - unmagnetized.
- X-mode : $n^2 = (\epsilon_1^2 - \epsilon_2^2)/\epsilon_1$
- Two cutoffs:

$$\omega_{C1,2} \simeq \left(\omega_{pe}^2 + \frac{1}{4}\omega_{ce}^2\right)^{1/2} \pm \frac{1}{2}\omega_{ce}$$

- Two resonances:

$$\omega_{UH} \simeq (\omega_{pe}^2 + \omega_{ce}^2)^{1/2} \sim \omega_{ce} \quad \text{if} \quad \omega_{ce} \gg \omega_{pe}$$

$$\omega_{LH} \simeq (.....)^{1/2} \sim \omega_{ci} \quad \text{if} \quad \omega_{ce} \gg \omega_{pe}$$



Resonance
Resonance
$\theta = \frac{\pi}{2}$
E
E
c^2/a^2
O
1
ω_{pe}
E
ω
ω_{LH}
$|\omega_{ce}|$
ω_{co1}
ω_{UH}
ω_{co2}
Cutoff
Cutoff
evanescent
evanescent
O
E

Magnetoionic Theory

- The discovery that radio waves could be reflected off the ionosphere revolutionized research on communications and radio wave propagation in a magnetoactive plasma.
- Ion motions are ignored. Cold Plasma.
- Appleton-Hartree dispersion relation

$$X = \frac{\omega_p^2}{\omega^2}, \qquad Y = \frac{|\omega_c|}{\omega}$$

$$n^2 = 1 - \frac{X}{1 - \frac{Y^2 \sin^2\theta}{2(1-X)} \pm \left[\frac{Y^4 \sin^4\theta}{2(1-X)^2} + Y^2 \cos^2\theta\right]^{1/2}}$$

- Quasi-transverse approximation (large θ)

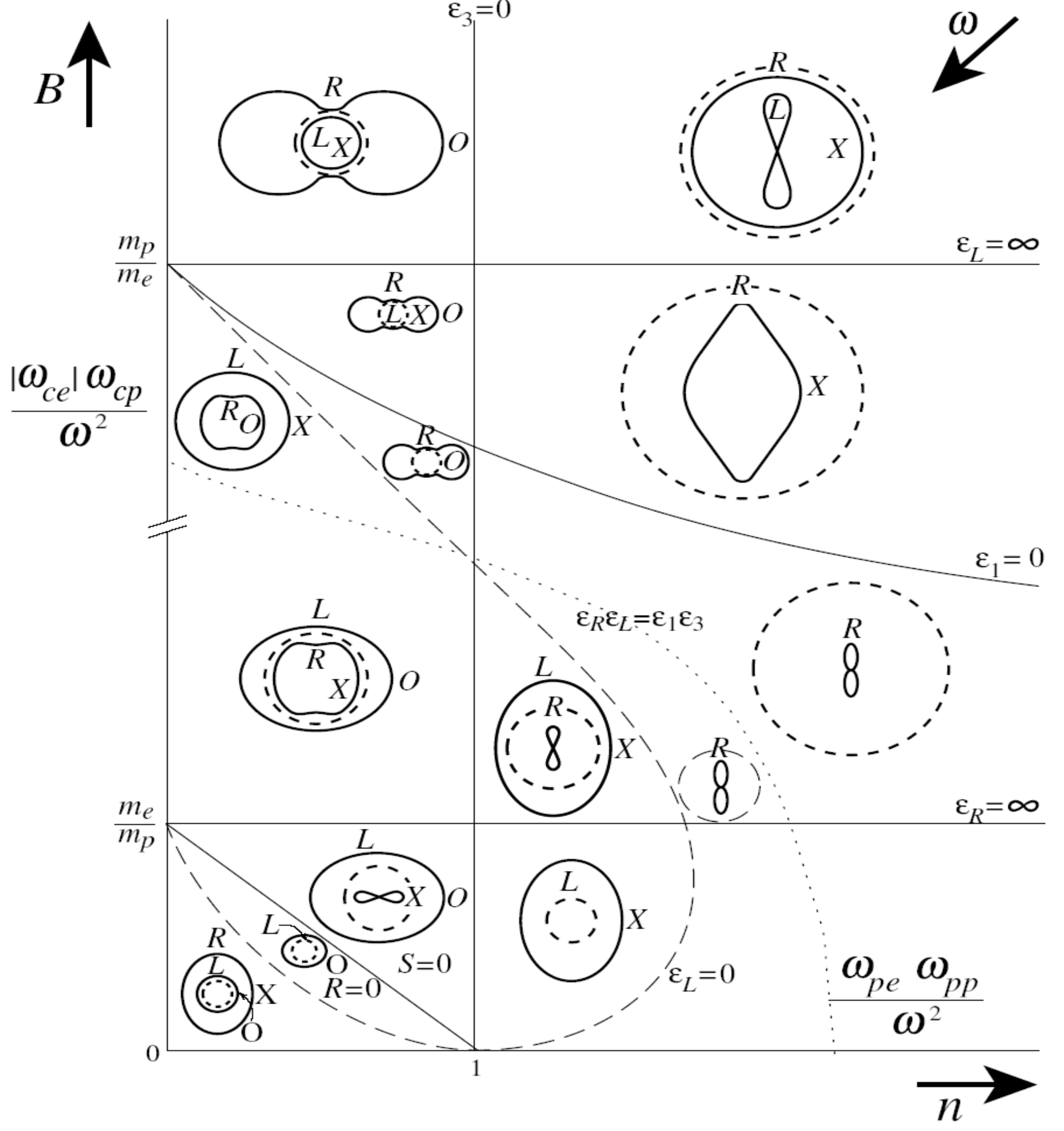
$$\text{when} \qquad Y^2 \sin^2\theta \ll 2(1-X)\cos\theta$$

$$n_O = 1 - X, \qquad n_X^2 = 1 - \frac{X(1-X)}{1 - X - Y^2\sin^2\theta}$$

- Quasi-longitudinal approximation (small θ)

$$n^2 \simeq 1 - \frac{X}{1 \pm Y\cos\theta} \quad \text{when} \quad Y^2\sin^2\theta \gg 2(1-X)\cos\theta.$$

used to simplify the ray-tracing thru the ionosphere.



Clemmow-
Mullally-Allis
(CMA)
diagram

2. Some Generalization of Cold Plasma Theory

A. Electron Thermal Pressure:

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s) = 0$$

$$n_s m_s \left[\frac{\partial \mathbf{u}_s}{\partial t} + (\mathbf{u}_s \cdot \nabla)\mathbf{u}_s \right] = -\nabla \mathbf{P}_s + n_s q_s (\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

Use $\delta n_e = n_e \frac{\mathbf{k}\cdot\mathbf{u}}{\omega}, \quad \delta p_e = m_e s^2 \delta n_e, \quad s = \sqrt{3kT_e/m_e}$. This leads to Böhm-Gross dispersion relation:

$$\omega^2 = \omega_{P_e}^2 + \frac{3kT_e k^2}{m_e} = \omega_{P_e}^2 (1 + 3\lambda_D^2 k^2)$$

B. Collisional Damping:

- Consider a transverse e.m. mode propagating in an unmagnetized partially ionized gas in which the electron-neutral collision frequency is ν_c. and introduce a term $-n_e m_e \nu_c \mathbf{u}$ to the eq. of motion.

$$-i\omega n_e m_e u_e = q_e n_e E - n_e m_e \nu_c u_e$$

- Put $u = ...$ to J to get ϵ:

$$J = n_e q_e u_e = \frac{n_e e^2}{m_e(\nu_c - i\omega)} E = \frac{\epsilon_0 \omega_p^2}{\nu_c - i\omega} E$$

$$\epsilon_{ij} = \delta_{ij} + \frac{i}{\epsilon_0 \omega} \sigma_{ij} = \Big[1 - \frac{\omega_p^2}{\omega(\omega + i\nu_c)}\Big] \delta_{ij}$$

- The L_{ij} in the coordinates where $\mathbf{k} = k\hat{e}_3$

$$L_{ij} = \begin{pmatrix} -k^2 + (\omega^2/c^2)A & 0 & 0 \\ 0 & -k^2 + (\omega^2/c^2)A & 0 \\ 0 & 0 & (\omega^2/c^2)A \end{pmatrix}$$

where $A = 1 - \omega_p^2/\omega(\omega + i\nu_c)$.

- The two degenerate transverse modes

$$c^2k^2 = \omega^2 - \omega_p^2 + i\frac{\omega_p^2\nu_c}{\omega} + O\left(\frac{\nu_c}{\omega}\right)^2$$

$$\omega^2 \approx \omega_p^2 + c^2k^2 - i\frac{\omega_p^2\nu_c}{\sqrt{\omega_p^2 + c^2k^2}}$$

where $\nu_c \ll \omega$.

NB: the wave energy lost by damping should be balanced by the energy gain of the plasma via Ohmic heating. This can be shown explicitly by calculating

- the rate of energy loss per unit volume $= -\nabla \cdot \mathbf{S}$ and
- the ohmic heating of the plasma $= \mathbf{j} \cdot \mathbf{E}$.

C. Beam instability in Cold Plasma: Recall the dispersion relation for longitudinal plasma oscillation.

$$1 - \frac{\omega_p^2}{\omega^2} = 0$$

If we consider a simple cold plasma of electrons and ions at rest, this corresponds to the dispersion relation for Langmuir waves with $\omega_p^2 = \omega_{pi}^2 + \omega_{pe}^2$.

In some other frames in which electrons and ions are moving with speed u, the observed frequency is Doppler-shifted and so

$$\frac{\omega_p^2}{(\omega - ku)^2} = 1$$

where ω is now the measured in this frame.

To generalize to several cold plasma streams, each moving with speed u_i,

$$1 - \frac{\omega_{p1}^2}{(\omega - ku_1)^2} - \frac{\omega_{p2}^2}{(\omega - ku_2)^2} = 0$$

The maximum value for the growth rate is

$$\omega_i = \frac{3^{1/2}\alpha^{1/3}\omega_p}{2^{4/3}}$$

Use $\omega_p \sim 10^5\ \mathrm{s}^{-1}$, $\alpha \sim 10^{-3}$, $V \sim 10^4\ \mathrm{km\ s}^{-1}$. The wave should grow in a length of 30 km.

3. Vlasov equation

- Ask how many particles are found in a fixed volume of one particle phase space.
- In a volume that is small compared with a Debye sphere, but still large enough to contain many particles, the continuity of particles belonging to species s in one particle phase space as:

$$\frac{\partial f_s}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f_s) + \nabla_{\mathbf{v}} \cdot (\mathbf{a} f_s) = 0$$

where

$$\mathbf{a} = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Now $\mathbf{x}$, $\mathbf{v}$ are independent variables; $\mathbf{E}$, $\mathbf{B}$ are functions of $\mathbf{x}$, t but not of $\mathbf{v}$ and the term $\mathbf{v} \times \mathbf{B}$ is perpendicular to $\mathbf{v}$, therefore

$$\nabla \cdot \mathbf{v} = 0, \qquad \nabla_{\mathbf{v}} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = 0$$

Thus we rewrite it

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0$$

This is the Vlasov equation for a *collisionless* particles species in a plasma.

Relationship of the Vlasov Eqn with the fluid approach

$$
\begin{aligned}
n_s &= \int d\mathbf{v} f_s \\
\mathbf{u}_s &= \frac{1}{n_s} \int d\mathbf{v}\mathbf{v} f_s \\
\mathbf{P} &= m_s \int d\mathbf{v}(\mathbf{v}-\mathbf{u}) \times (\mathbf{v}-\mathbf{u}) f_s \\
&\vdots
\end{aligned}
$$

Integrating the Vlasov equation over velocity space,

$$
\frac{\partial n_s}{\partial t} + \frac{\partial (n_s \mathbf{u}_s)}{\partial \mathbf{x}} = 0.
$$

- To solve for n_s, we need to know $\mathbf{v}$. We multiply the Vlasov equation by $\mathbf{v}$ and integrate over $\int \cdot d\mathbf{v}$:

$$\frac{\partial \mathbf{u}_s}{\partial t} + (\mathbf{u}_s \cdot \nabla)\mathbf{u}_s = \frac{1}{n_s m_s}[-\nabla \cdot \mathbf{P}_s + (\rho_e \mathbf{E} + \mathbf{j} \times \mathbf{B})].$$

- To solve for n_s and $\mathbf{u}_s$ we need to know $\mathbf{P}_s$. To know $\mathbf{P}_s$, we multiply the Vlasov equation by $\mathbf{v} \times \mathbf{v}$ and integrate over the third moment $\int d\mathbf{v}\ \mathbf{v} \times \mathbf{v} \times \mathbf{v}$, related to the heat flux.

. . . .

- This procedure will never terminate unless we introduce a closure relation. Typically either the heat flux to vanish or $P_s = n_s k T_s$. Our fluid theory is certainly no more accurate than this closure relation.

4. Landau Damping

Try to solve the Vlasov equation directly (re-derive the dispersion relation for Langmuir waves in an unmag. plasma). Let

$$f(x,v,t) = f_0(v) + f_1(x,v,t)$$
$$E(x,t) = E_0(x) + E_1(x,t)$$

Assume $E_0 = 0$. $f_0(v)$: the unperturbed electrons in the absence of waves, $f_1(v)$: the induced electron distribution due to the perturb. Linearizing the Vlasov equation,

$$\frac{\partial}{\partial t}(f_0 + f_1) + v\frac{\partial}{\partial z}(f_0 + f_1) - \frac{e}{m}E_1\frac{\partial}{\partial v}(f_0 + f_1) = 0$$

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} = \frac{e}{m}E_1\frac{\partial f_0}{\partial v}$$

$$\frac{\partial E_1}{\partial x} = -\frac{e}{\varepsilon_0}\int f_1 dv$$

Seek a wave solution with f_1, $E_1 \propto \exp[i(kx - \omega t)]$:

$$(-i\omega + ikv)f_1 = \frac{e}{m_e}E_1\frac{df_0}{dv}$$

$$ikE_1 = -\frac{e}{\varepsilon_0}\int f_1 dv$$

Solve for E_1

$$ikE_1 = -\frac{e^2}{m_e\varepsilon_0}E_1\int\frac{f_0'}{-i\omega + ikv}dv$$

Thus our dispersion relation is

$$\epsilon(k,\omega) = 1 + \frac{e^2}{m_e \varepsilon_0 k} \int \frac{f_0'}{\omega - kv} dv = 0$$

---

* Connection to the wave dispersion relation

$$\epsilon(k,\omega) = 1 + \frac{e^2}{m_e \varepsilon_0 k} \int \frac{f_0'}{\omega - kv} dv = 0$$

Integrating by part and rewrite this relation as

$$\frac{e^2}{m_e \varepsilon_0} \int \frac{f}{(\omega - kv)^2} dv = 1.$$

This is generalization of the two stream equation.

How do we perform the integral? - Case A

Suppose $f_0(v)$ is an even function of v and vanishes for sufficiently high $v > u_m$ (and for $|\omega/k| > u_m$)

$$1 = -\frac{e^2}{m\varepsilon_0 k\omega}\int_{-u_m}^{+u_m} \frac{f_0'}{1 - kv/\omega} dv$$

$$= -\frac{e^2}{m\varepsilon_0 k\omega}\int_{-u_m}^{+u_m} f_0' \left[1 + \frac{kv}{\omega} + \frac{k^2v^2}{\omega^2} + ...\right] dv$$

Finally we get

$$1 = \frac{\omega_p^2}{\omega^2}\left[1 + 3\frac{k^2\langle v^2\rangle}{\omega^2} + 5\frac{k^4\langle v^4\rangle}{\omega^4} + ...\right]$$

So

$$\omega^2 \simeq \omega_p^2 + 3(kT/m)k^2$$

NB 1: This is an approximate result.

NB 2: What happens when $v = \omega/k$? (*i.e., warm*)

How do we perform the integral? - Case B

$$\epsilon(\omega, k) = 1 + \frac{e^2}{m_e \varepsilon_0 k} \int \frac{f_0'}{\omega - kv} dv = 0$$

If ω, k real, then $\exists$ a singularity at $v = \omega/k$.
Possible Remedy: include collision in a crude way

$$\frac{\partial f_1}{\partial t} + v \frac{\partial f_1}{\partial z} - \frac{eE}{m} \frac{\partial f_0}{\partial v} = -\nu f_1$$

This will cause $f_1 \sim e^{-\nu t}$ (thus, in fact, $\sim e^{-i(\omega - i\nu)t}$) and

$$\begin{aligned} \epsilon &= 1 - \frac{e^2}{m_e \varepsilon_0 k^2} \int \frac{f_0'}{v - \omega/k + i\nu/k} dv \\ &= 1 - \frac{e^2}{m_e \varepsilon_0 k^2} \Big[\int_{PV} \frac{f_0'}{v - \omega/k + i\nu/k} dv - i\pi \frac{df_0}{dv}\Big|_{v=\omega/k} \Big] \end{aligned}$$

in the limit of of $\nu \to 0$.

Suppose

$$f_0 = \frac{n_0}{\sqrt{\pi} v_e} e^{-v^2/v_e^2}$$

$$\left.\frac{df_0}{dv}\right|_{\omega/k} = -2\frac{n_0 v}{\sqrt{\pi} v_e^3} e^{-v^2/v_e^2}\Big|_{v=\omega/k}$$

$$\epsilon \simeq 1 - \frac{\omega_p^2}{\omega^2}\left(1 + \frac{3k^2\langle v^2\rangle}{\omega^2} + \ldots\right) - 2i\sqrt{\pi}\frac{\omega_p^2}{k^2 v_e^2}\frac{\omega}{k v_e} e^{-(\omega/k v_e)^2} = 0$$

So

$$\omega^2 \simeq \omega_p^2 + \frac{3kT}{m}k^2 + 2i\alpha\omega_p^2$$

With $\omega = \omega_r - i\nu, \quad \omega^2 \approx \omega_r^2 - 2i\omega_r\nu$ we obtain

$$\omega_r^2 \approx \omega_p^2 + \frac{3kT_e}{m_e}k^2,$$

$$\nu \approx -\alpha\omega_p \quad \leftarrow \textbf{sign reversed}???$$

How to perform the integral? - Case C

$$\epsilon(\omega, k) = 1 + \frac{e^2}{m_e \varepsilon_0 k} \int \frac{f_0'}{\omega - kv} dv$$

- In 1946, Landau recognized that the difficulty could be resolved by treating the problem as an I.V.P. and take a Laplace transform in time rather than using the Fourier Transform of Vlasov.
- However, if we're not concerned with the transient we can just persist with Fourier analysis supplemented by a fairly subtle though important physical argument.
- When $\omega_i > 0$ the wave will grow exponentially and the initial disturbance is small in the distant past. The calculation of ϵ does not include the effect of the transients and the above can be just used in its form. (*analytic*)

- If $\omega_i < 0$ the amplitude was large in the past, and we cannot assume that the transients are ignorable (*not analytic*). At this point introduce causality into our time-symmetric equations by allowing the past (but not the future) to influence the present.
- Mathematically we use the *analytic continuation* in the range for which our physical argument fails. $\epsilon(k, \omega)$ should be analytic in ω. When $\omega_i > 0$ the integral is along real v axis. However, if $\omega_i \leq 0$ the only the function can remain analytic is by deforming the contour so as to pass below the pole. This rule is known as the *Landau prescription*:

$$\int_L \frac{dv f'(v)}{v - \frac{\omega}{k}} = \int_{PV} \frac{dv f'(v)}{v - \frac{\omega}{k}} + i\pi f'(v = \omega/k).$$

Summary of Landau Damping

We have demonstrated that Langmuir waves will be damped by a Maxwellian distribution of electrons:

$$\omega_r = \omega_p(1 + 3k^2\lambda_D^2)^{1/2}$$

$$\omega_i = -\left(\frac{\pi}{8}\right)^{1/2} \frac{\omega_p}{k^3\lambda_D^3} e^{-(1/2k^2\lambda_D^2 + 3/2)}$$

- The Böhm-Gross dispersion relation recovered.
- $\lambda \uparrow$ $(k \to 0)$: the wave $v_\phi \uparrow$: # of electrons resonant with the wave $\downarrow$: the damping rate $\downarrow$
- The sign of ω_i = the sign of $f'(v)$ at resonance.
- When $\omega/k \sim V_{\mathrm{th}}$, the damping rate will be so severe that this calculation is invalid. (Our assumptions: linearized Vlasov eq, $|\omega_i| \ll \omega_r$, $k\lambda_D \ll 1$)

This result is *not apparent (and could not have been)* from the *cold plasma theory.*