CIS 435, Spring 2002, Joseph Leung
Notes on comparing functions.

When comparing the order of growth of functions, there are a few tricks that may come in handy.

Suppose that you want to show that $f(n) = \Omega(g(n))$; that is, a positive multiple of g is asymptotically a lower bound for f. If you show that $f(n)/g(n) \to \infty$ as $n \to \infty$, or even that $f(n)/g(n) \to a$ as $n \to \infty$, where $a > 0$, then $f(n) = \Omega(g(n))$. To see this, note that if $f(n)/g(n) \to a$ as $n \to \infty$, where $a > 0$, then by definition for any $\epsilon > 0$, there exists an integer n_0 such that $n \geq n_0$ implies that $f(n)/g(n) \geq a - \epsilon$. If we take $\epsilon = a/2$, we obtain $f(n)/g(n) \geq a - a/2 = a/2$, so that $n \geq n_0$ implies that $f(n) \geq (a/2)g(n)$, and so $f(n) = \Omega(g(n)$ by the definition on page 27 of the texbook (with $c = a/2$).

Applying this to the problem on the first homework, let's compare $n2^n$ and $(3/2)^n$. Taking the ratio, we have

$$\frac{n2^n}{(3/2)^n} = n\left(\frac{4}{3}\right)^n \to \infty,$$

and so we conclude that $n2^n = \Omega((3/2)^n)$. (Exercise: if$f(n)/g(n) \to a$, where $0 < a < \infty$, then $f(n) = \Theta(g(n))$. What if $f(n)/g(n) \to 0$?)

Another trick is to apply an increasing function to the ratio; often the logarithm function is useful. Since the logarithm is positive and $lg(x) \to \infty$ as $x \to \infty$, we know that if $lg(f(n)/g(n))$ goes to ∞, then so does $f(n)/g(n)$ (and if the lg of the ratio converges to a finite constant, then so does the ratio).

For example, to show that $lg^2(n) = \Omega(ln\, ln(n))$, observe that

$$lg\left(\frac{lg^2(n)}{ln\, ln(n)}\right) = lg\left(lg^2(n)\right) - lg\left(ln\, ln(n)\right) \geq 2lg\, lg(n) - lg\, lg\, lg(n) \to \infty.$$

As another example, $n^{lg\, lg(n)} = \Omega(n^2)$, since

$$lg\frac{n^{lg\, lg(n)}}{n^2} = lg\, lg(n) \cdot lg(n) - 2lg(n) = lg(n)\left(lg\, lg(n) - 2\right) \to \infty.$$