CS 341 Practice Final

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1. Short answers:

(a) Define the following terms and concepts:

i. Union, intersection, set concatenation, Kleene-star, set subtraction, complement

Answer:

- Union: $S \cup T = \{ x \mid x \in S$ **or** $x \in T \}$
- Intersection: $S \cap T = \{ x \mid x \in S$ **and** $x \in T \}$
- Concatenation: $S \circ T = \{ xy \mid x \in S, y \in T \}$
- Kleene-star:
  $S^* = \{ w_1 w_2 \cdots w_k \mid k \geq 0, w_i \in S \; \forall \; i = 1, 2, \ldots, k \}$
- Subtraction: $S - T = \{ x \mid x \in S, x \notin T \}$
- Complement: $\overline{S} = \{ x \in \Omega \mid x \notin S \} = \Omega - S$,
  where Ω is the universe of all elements under consideration.

ii. A set S is closed under an operation f

Answer: S is closed under f if applying f to members of S always returns a member of S.

iii. Regular language

Answer: A regular language is defined by a DFA.

iv. Kleene's theorem

Answer: A language is regular if and only if it has a regular expression.

v. Context-free language

Answer: A CFL is defined by a context-free grammar (CFG).

vi. Chomsky normal form

Answer: A CFG is in Chomsky normal form if each of its rules has one of 3 forms:

$$A \rightarrow BC, \quad A \rightarrow x, \quad \text{or} \quad S \rightarrow \varepsilon,$$

where A, B, C are variables, B and C are not the start variable, x is a terminal, and S is the start variable.

vii. Church-Turing Thesis

Answer: The informal notion of algorithm corresponds exactly to a Turing machine that always halts (i.e., a decider).

viii. Turing-decidable language

Answer: A language A that is **decided** by a Turing machine; i.e., there is a Turing machine M such that

- M halts and accepts on any input $w \in A$, and
- M halts and rejects on input input $w \notin A$.

Looping cannot happen.

ix. Turing-recognizable language

Answer: A language A that is **recognized** by a Turing machine; i.e., there is a Turing machine M such that

- M halts and accepts on any input $w \in A$, and
- M rejects **or loops** on any input $w \notin A$.

x. co-Turing-recognizable language

Answer: A language whose **complement** is Turing-recognizable.

xi. Countable and uncountable sets

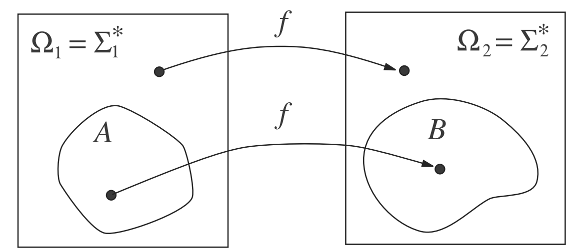
Answer:

- A set S is countable if it is finite or we can define a correspondence between the positive integers and S.
- In other words, can create (possibly infinite) list of all elements in S and each specific element will eventually appear in list.
- An uncountable set is a set that is not countable.
- A common approach to prove a set is uncountable is by using a diagonalization argument.

xii. Language A is mapping reducible to language B, $A \leq_{\mathrm{m}} B$

Answer:

- Suppose A is a language defined over alphabet Σ_1, and B is a language defined over alphabet Σ_2.
- Then $A \leq_{\mathrm{m}} B$ means there is a computable function $f : \Sigma_1^* \to \Sigma_2^*$ such that $w \in A$ iff $f(w) \in B$.



$w \in A \iff f(w) \in B$

YES instance for problem A $\iff$ YES instance for problem B

xiii. Function $f(n)$ is $O(g(n))$

Answer: There exist constants c and n_0 such that $|f(n)| \leq c \cdot g(n)$ for all $n \geq n_0$.

xiv. Classes P and NP

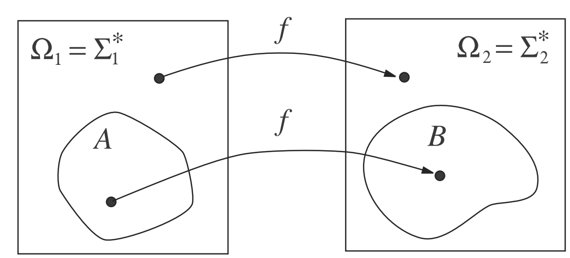
Answer:

- P is the class of languages that can be decided by a **deterministic** Turing machine in polynomial time.
- NP is the class of languages that can be verified in (**deterministic**) polynomial time.
- Equivalently, NP is the class of languages that can be decided by a **nondeterministic** Turing machine in polynomial time.

xv. Language A is polynomial-time mapping reducible to language B, $A \leq_{\mathrm{P}} B$.

Answer:

- Suppose A is a language defined over alphabet Σ_1, and B is a language defined over alphabet Σ_2.
- Then $A \leq_{\mathrm{P}} B$ means $\exists$ polynomial-time computable function $f : \Sigma_1^* \to \Sigma_2^*$ such that $w \in A$ iff $f(w) \in B$.

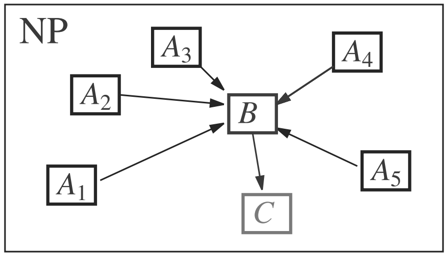


$w \in A \iff f(w) \in B$

YES instance for problem A $\iff$ YES instance for problem B

xvi. NP-complete

Answer: Language B is NP-Complete if $B \in \text{NP}$, and B is NP-Hard ($\forall\ A \in \text{NP}$, we have $A \leq_{\text{P}} B$).



Typical approach for proving language C is NP-Complete:

- first show $C \in \text{NP}$
- then show a known NP-Complete language B satisfies $B \leq_{\text{P}} C$.

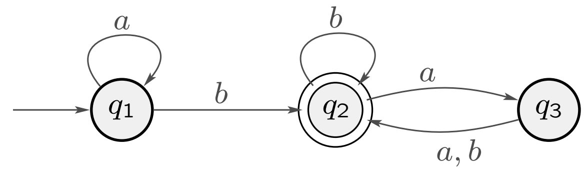
xvii. NP-hard

Answer: Lang B is NP-hard if $A \leq_{\text{P}} B$ for every lang $A \in \text{NP}$.

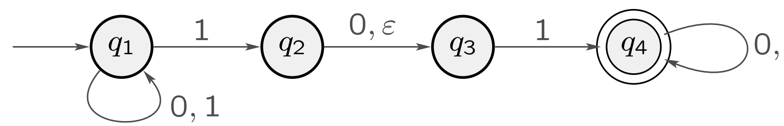
(b) Give the transition functions δ (i.e., give domain and range) of a DFA, NFA, PDA, Turing machine and nondeterministic Turing machine.

Answer:

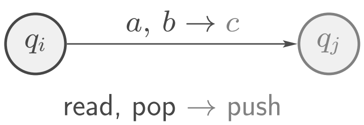
- DFA, $\delta : Q \times \Sigma \to Q$,
  where Q is the set of states and Σ is the alphabet.

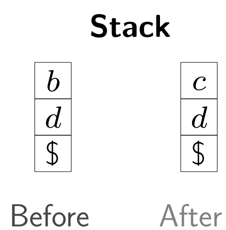


- NFA, $\delta : Q \times \Sigma_\varepsilon \to \mathcal{P}(Q)$,
  where $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$ and $\mathcal{P}(Q)$ is the power set of Q

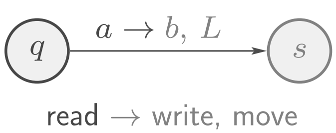


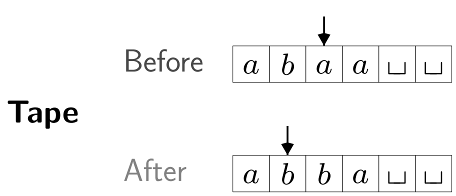
- PDA, $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to \mathcal{P}(Q \times \Gamma_\varepsilon)$,
  where Γ is the stack alphabet and $\Gamma_\varepsilon = \Gamma \cup \{\varepsilon\}$.



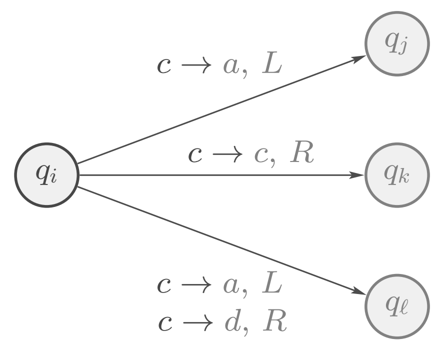


- Turing machine, $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$,
  where Γ is the tape alphabet, L means move tape head one cell left, and R means move tape head one cell right.





- Nondeterministic Turing machine,
  $\delta : Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$



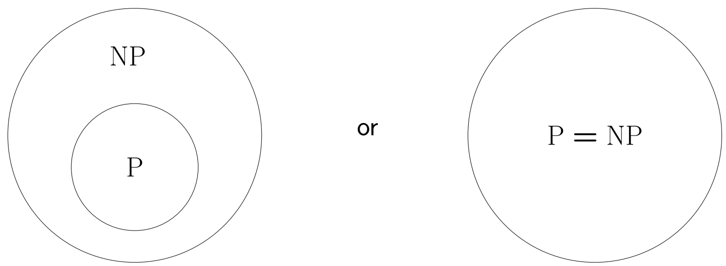
Multiple choices when in state q_i and read c from tape:

$$\delta(q_i, c) = \{\, (q_j, a, L),\ (q_k, c, R),\ (q_\ell, a, L),\ (q_\ell, d, R)\,\}$$

(c) Explain the “P vs. NP” problem.

Answer:

- P is class of languages that can be solved in deterministic poly time.
- NP is class of languages that can be verified in deterministic poly time (equivalently, solved by poly-time NTM).
- We know that $\mathrm{P} \subseteq \mathrm{NP}$.
  - Each poly-time DTM is also a poly-time NTM.
- But it is currently unknown if $\mathrm{P} = \mathrm{NP}$ or $\mathrm{P} \neq \mathrm{NP}$.

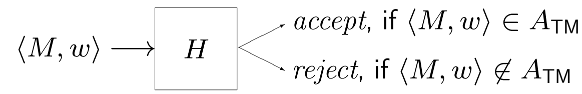


2. Recall that $A_{\mathsf{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM that accepts string } w \}$.

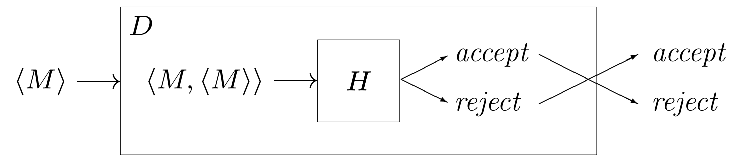
(a) Prove that A_{TM} is undecidable. You may not cite any theorems or corollaries in your proof.

Overview of Proof:

- Suppose A_{TM} is decided by some TM H, taking input $\langle M, w \rangle \in \Omega = \{ \langle M, w \rangle \mid M \text{ is a TM and } w \text{ a string} \}$.



- Define another TM D using H as a subroutine.



- What happens when we run D with input $\langle D \rangle$?
  - D accepts $\langle D \rangle$ iff D doesn't accept $\langle D \rangle$, which is impossible.

Detailed Proof:

- Suppose there exists a TM H that decides A_{TM}.
- Consider language $L = \{ \langle M \rangle \mid M \text{ is a TM that does not accept } \langle M \rangle \}$.
- Now construct a TM D for L using TM H as a subroutine:

  $D =$ “On input $\langle M \rangle$, where M is a TM:
  1. Run H on input $\langle M, \langle M \rangle \rangle$.
  2. If H accepts, *reject*. If H rejects, *accept*.”

- If we run TM D on input $\langle D \rangle$, then D accepts $\langle D \rangle$ if and only if D doesn't accept $\langle D \rangle$.
- Since this is impossible, TM H must not exist.

(b) Show that A_{TM} is Turing-recognizable.

Answer: Universal TM (UTM) U recognizes A_{TM}:

$U =$ “On input $\langle M, w \rangle \in \Omega$, where M is a TM and w is a string:
1. Run M on w.
2. If M accepts w, *accept*; if M rejects w, *reject*.”

U recognizes A_{TM} but does not decide A_{TM}

- When we run M on w, there is the possibility that M neither accepts nor rejects w but rather loops on w.

3. Each of the languages below in parts (a), (b), (c), (d) is of one of the following types:
   Type REG. It is regular.
   Type CFL. It is context-free, but not regular.
   Type DEC. It is Turing-decidable, but not context-free.

   For each of the following languages, specify which type it is. Also, follow these instructions:

   - If a language L is of Type REG, give a regular expression **and** a DFA (5-tuple) for L.
   - If a language L is of Type CFL, give a context-free grammar (4-tuple) **and** a PDA (6-tuple) for L. **Also, prove that L is not regular.**
   - If a language L is of Type DEC, give a description of a Turing machine that decides L. **Also, prove that L is not context-free.**

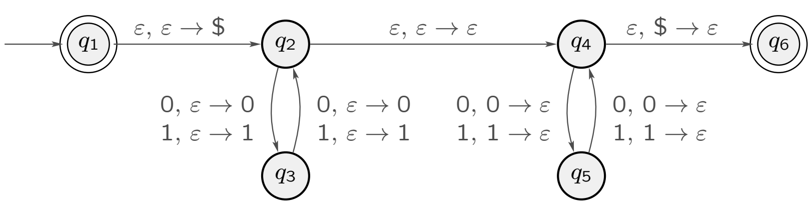
(a) $A = \{ w \in \Sigma^* \mid w = \text{reverse}(w)$ and the length of w is divisible by 4 $\}$, where $\Sigma = \{0, 1\}$.

Answer: A is of type CFL.

A CFG $G = (V, \Sigma, R, S)$ for A has

- $V = \{S\}$,
- $\Sigma = \{0, 1\}$,
- starting variable S,
- rules $R = \{ S \to 00S00 \mid 01S10 \mid 10S01 \mid 11S11 \mid \varepsilon \}$.

PDA for $A = \{ w \in \Sigma^* \mid w = w^{\mathcal{R}}, |w| \text{ divisible by } 4 \}$:



The above PDA has 6-tuple $(Q, \Sigma, \Gamma, \delta, q_1, F)$, with $Q = \{q_1, q_2, \ldots, q_6\}$, $\Sigma = \{0, 1\}$, $\Gamma = \{0, 1, \$\}$, starting state q_1, $F = \{q_1, q_6\}$, and transition function $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to \mathcal{P}(Q \times \Gamma_\varepsilon)$ defined by

Input:	0				1				ε			
Stack:	0	1	\$	ε	0	1	\$	ε	0	1	\$	ε
q_1												$\{(q_2, \$)\}$
q_2				$\{(q_3, 0)\}$				$\{(q_3, 1)\}$				$\{(q_4, \varepsilon)\}$
q_3				$\{(q_2, 0)\}$				$\{(q_2, 1)\}$				
q_4	$\{(q_5, \varepsilon)\}$					$\{(q_5, \varepsilon)\}$					$\{(q_6, \varepsilon)\}$	
q_5	$\{(q_4, \varepsilon)\}$					$\{(q_4, \varepsilon)\}$						
q_6												

Blank entries are $\emptyset$.

Prove $A = \{ w \in \Sigma^* \mid w = w^{\mathcal{R}}, \text{length of } w \text{ is divisible by } 4 \}$ nonregular.

- For a contradiction, suppose that A is regular.
- Pumping Lemma (Theorem 1.I): If L is regular language, then $\exists$ number p where, if $s \in L$ with $|s| \geq p$, then can split $s = xyz$ satisfying conditions (1) $xy^iz \in L$ for each $i \geq 0$, (2) $|y| > 0$, (3) $|xy| \leq p$
- Let $p \geq 1$ be the pumping length of the pumping lemma.
- Consider string $s = 0^p\ 1^{2p}\ 0^p \in A$, and note that $|s| = 4p > p$, so conclusions of pumping lemma must hold.
- Since all of the first p symbols of s are 0s, (3) implies that x and y must only consist of 0s. Also, z must consist of rest of 0s at the beginning, followed by $1^{2p}0^p$.
- Hence, we can write $x = 0^j$, $y = 0^k$, $z = 0^m\ 1^{2p}\ 0^p$, where $j + k + m = p$ since $s = 0^p1^{2p}0^p = xyz = 0^j\ 0^k\ 0^m\ 1^{2p}\ 0^p$.
- Moreover, (2) implies that $k > 0$.
- Finally, (1) states that $xyyz$ must belong to A. However,

$$xyyz = 0^j\ 0^k\ 0^k\ 0^m\ 1^{2p}\ 0^p = 0^{p+k}\ 1^{2p}\ 0^p$$

  since $j + k + m = p$.
- But, $k > 0$ implies reverse$(xyyz) \neq xyyz$, which means $xyyz \notin A$, which contradicts (1).
- Therefore, A is a nonregular language.

(b) $B = \{\, b^n a^n b^n \mid n \geq 0 \,\}$.

Answer: B is of type DEC.

Below is a description of a Turing machine that decides B.

$M =$ "On input string $w \in \{a, b\}^*$:

1. Scan input to check if it's in $b^* a^* b^*$; *reject* if not.
2. Return tape head to left-hand end of tape.
3. Repeat following until no more b's left on tape.
4. Replace the leftmost b with x.
5. Scan right until a occurs. If no a's, *reject*.
6. Replace the leftmost a with x.
7. Scan right until b occurs. If no b's, *reject*.
8. Replace the leftmost b (after the a's) with x.
9. Return tape head to left end of tape; go to stage 3.
10. If tape contains any a's, *reject*. Else, *accept*."

We now prove that B is not context-free by contradiction.

- Suppose that $B = \{\, b^n a^n b^n \mid n \geq 0 \,\}$ is context-free.
- PL for CFL (Thm 2.D): For every CFL L, $\exists$ pumping length p such that $\forall\ s \in L$ with $|s| \geq p$, can split $s = uvxyz$ with (1) $uv^i xy^i z \in L\ \forall i \geq 0$, (2) $|vy| \geq 1$, (3) $|vxy| \leq p$.
- Let p be pumping length of CFL pumping lemma
- Consider string $s = b^p a^p b^p \in B$.
  Note that $|s| = 3p > p$, so the pumping lemma will hold.
- Thus, can split $s = b^p a^p b^p = uvxyz =$ satisfying (1)–(3)
- We now consider all of the possible choices for v and y:
  - Suppose strings v and y are **both uniform** (e.g., $v = b^j$ for some $j \geq 0$, and $y = a^k$ for some $k \geq 0$). Then $|vy| \geq 1$ implies that $v \neq \varepsilon$ or $y \neq \varepsilon$ (or both), so uv^2xy^2z won't have the correct number of b's at the beginning, a's in the middle, and b's at the end. Hence, $uv^2xy^2z \notin B$.
  - Now suppose strings v and y are **not both uniform**. Then uv^2xy^2z won't have form $b \cdots ba \cdots ab \cdots b$, so $uv^2xy^2z \notin B$.
- Every case gives contradiction, so B is not a CFL.

(c) $C = \{\, w \in \Sigma^* \mid n_a(w) \mod 4 = 1 \,\}$, where $\Sigma = \{a, b\}$ and $n_a(w)$ is the number of a's in string w. For example, $n_a(babaabb) = 3$. Also, $3 \mod 4 = 3$, and $9 \mod 4 = 1$.

Answer: C is of type REG.

A regular expression for C is

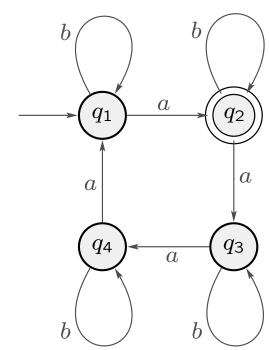
$$(b^*ab^*ab^*ab^*ab^*)^*b^*ab^*$$

$C = \{\, w \in \Sigma^* \mid n_a(w) \mod 4 = 1 \,\}$

DFA 5-tuple $(Q, \Sigma, \delta, q_1, F)$

- $Q = \{q_1, q_2, q_3, q_4\}$
- $\Sigma = \{a, b\}$
- q_1 is start state
- $F = \{q_2\}$
- transition fcn $\delta : Q \times \Sigma \to Q$

	a	b
q_1	q_2	q_1
q_2	q_3	q_2
q_3	q_4	q_3
q_4	q_1	q_4



(d) $D = \{ b^n a^n b^k c^k \mid n \geq 0,\ k \geq 0 \}$.

[Hint: Recall that the class of CFLs is closed under concatenation.]

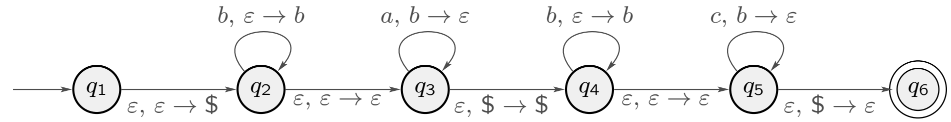
Answer: D is of type CFL.

A CFG $G = (V, \Sigma, R, S)$ for D has

- $V = \{S, X, Y\}$
- $\Sigma = \{a, b, c\}$
- starting variable S
- Rules R:

$$\begin{aligned} S &\rightarrow XY \\ X &\rightarrow bXa \mid \varepsilon \\ Y &\rightarrow bYc \mid \varepsilon \end{aligned}$$

PDA for $D = \{ b^n a^n b^k c^k \mid n \geq 0,\ k \geq 0 \}$:



Important: q_3 to q_4 pops and pushes \$ to make sure stack is empty.

PDA as a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_1, F)$, where

$Q = \{q_1, q_2, \ldots, q_6\}$, $\Sigma = \{a, b, c\}$, $\Gamma = \{b, \$\}$,

q_1 is the start state, $F = \{q_6\}$, and the transition function

$\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow \mathcal{P}(Q \times \Gamma_\varepsilon)$ is defined by

Input:	a			b			c			ε		
Stack:	b	\$	ε	b	\$	ε	b	\$	ε	b	\$	ε
q_1												$\{(q_2, \$)\}$
q_2						$\{(q_2, b)\}$						$\{(q_3, \varepsilon)\}$
q_3	$\{(q_3, \varepsilon)\}$										$\{(q_4, \$)\}$	
q_4						$\{(q_4, b)\}$						$\{(q_5, \varepsilon)\}$
q_5							$\{(q_5, \varepsilon)\}$				$\{(q_6, \varepsilon)\}$	
q_6												

Blank entries are $\emptyset$.

Prove $D = \{ b^n a^n b^k c^k \mid n \geq 0,\ k \geq 0 \}$ not regular.

- Suppose that D is regular. Let $p \geq 1$ be pumping length of pumping lemma (Theorem 1.I).
- Consider string $s = b^p\ a^p\ b^p\ c^p \in D$, and note that $|s| = 4p > p$, so conclusions of pumping lemma must hold.
- Thus, can split $s = xyz$ satisfying
  (1) $xy^iz \in D$ for all $i \geq 0$, (2) $|y| > 0$, (3) $|xy| \leq p$.
- Since all of the first p symbols of s are b's,
  (3) implies that x and y must consist of only b's.
  Also, z is rest of b's at beginning, followed by $a^p\ b^p\ c^p$.
- Hence, we can write $x = b^j$, $y = b^k$, $z = b^m\ a^p\ b^p\ c^p$, where $j + k + m = p$ since
  $s = b^p\ a^p\ b^p\ c^p = xyz = b^j\ b^k\ b^m\ a^p\ b^p\ c^p$.
- Moreover, (2) implies that $k > 0$.
- Finally, (1) states that $xyyz$ must belong to D, but

$$xyyz = b^j\ b^k\ b^k\ b^m\ a^p\ b^p\ c^p = b^{p+k}\ a^p\ b^p\ c^p$$

since $j + k + m = p$. Also $k > 0$, so $xyyz \notin D$, which contradicts (1). Therefore, D is a nonregular language.

4. Each of the languages below in parts (a), (b), (c), (d) is of one of the following types:

   Type DEC. It is Turing-decidable.

   Type TMR. It is Turing-recognizable, but not decidable.

   Type NTR. It is not Turing-recognizable.

   For each of the following languages, specify which type it is. Also, follow these instructions:

   - If a language L is of Type DEC, give a description of a Turing machine that decides L.
   - If a language L is of Type TMR, give a description of a Turing machine that recognizes L. **Also, prove that L is not decidable.**
   - If a language L is of Type NTR, give a proof that it is not Turing-recognizable.

In each part below, if you need to prove that the given language L is decideable, undecidable, or not Turing-recognizable, you must give an explicit proof of this; i.e., do not just cite a theorem that establishes this without a proof. However, if in your proof you need to show another language L' has a particular property for which there is a theorem that establishes this, then you may simply cite the theorem without proof.

(a) $\overline{A_{\text{TM}}}$, where $A_{\text{TM}} = \{ \langle M, w \rangle \mid M$ is a TM that accepts string $w \}$.

Answer: $\overline{A_{\text{TM}}}$ is of type NTR, which is just Theorem 4.M.

Proof:

- If $\overline{A_{\text{TM}}}$ were Turing-recognizable, then A_{TM} would be both Turing-recognizable (see slide 4-25) and co-Turing-recognizable.
- But then Theorem 4.L would imply that A_{TM} is decidable, which we know is not true by Theorem 4.I.
- Hence, $\overline{A_{\text{TM}}}$ is not Turing-recognizable.

(b) $EQ_{\text{TM}} = \{ \langle M_1, M_2 \rangle \mid M_1, M_2$ are TMs with $L(M_1) = L(M_2) \}$.
[Hint: show $\overline{A_{\text{TM}}} \leq_{\text{m}} EQ_{\text{TM}}$.]

Answer: EQ_{TM} is of type NTR (see Theorem 5.K).
Prove by showing $\overline{A_{\text{TM}}} \leq_{\text{m}} EQ_{\text{TM}}$ and applying Corollary 5.I.

- $\overline{A_{\text{TM}}} \subseteq \Omega_1 = \{ \langle M, w \rangle \mid M$ is a TM, w is a string $\}$,
  $EQ_{\text{TM}} \subseteq \Omega_2 = \{ \langle M_1, M_2 \rangle \mid M_1, M_2$ are TMs $\}$.
- Define reducing function $f(\langle M, w \rangle) = \langle M_1, M_2 \rangle$, where
  - $M_1 =$ "*reject* on all inputs."
  - $M_2 =$ "On input x:
    1. Ignore input x, and run M on w.
    2. If M accepts w, *accept*; if M rejects w, *reject*."
- $L(M_1) = \emptyset$.
- If M accepts w (i.e., $\langle M, w \rangle \notin \overline{A_{\text{TM}}}$), then $L(M_2) = \Sigma^*$.
  If M doesn't accept w (i.e., $\langle M, w \rangle \in \overline{A_{\text{TM}}}$), then $L(M_2) = \emptyset$.
- Thus, $\langle M, w \rangle \in \overline{A_{\text{TM}}} \iff f(\langle M, w \rangle) = \langle M_1, M_2 \rangle \in EQ_{\text{TM}}$, so $\overline{A_{\text{TM}}} \leq_{\text{m}} EQ_{\text{TM}}$.
- But $\overline{A_{\text{TM}}}$ is not TM-recognizable (Corollary 4.M), so EQ_{TM} is not TM-recognizable by Corollary 5.I.

(c) $HALT_{\text{TM}} = \{ \langle M, w \rangle \mid M$ is a TM that halts on input $w \}$.
[Hint: modify universal TM to show $HALT_{\text{TM}}$ is TM-recognizable.]

Answer: $HALT_{\text{TM}}$ is of type TMR (see Theorem 5.A).

- **Decision problem:** Given TM M and string w, does M **halt** on input w?
- **Universe:** $\Omega_H = \{ \langle M, w \rangle \mid$ TM M, string $w \}$.
- Consider following Turing machine T:

  $T =$ "On input $\langle M, w \rangle \in \Omega_H$, where M is TM and w is string:
  1. Run M on w.
  2. If M halts (i.e., accepts or rejects) on w, *accept*."
- TM T recognizes $HALT_{\text{TM}}$
  - accepts each $\langle M, w \rangle \in HALT_{\text{TM}}$
  - loops on each $\langle M, w \rangle \notin HALT_{\text{TM}}$

We now prove that $HALT_{\text{TM}}$ is undecidable, which is Theorem 5.A.

- We will show that A_{TM} reduces to $HALT_{\text{TM}}$, where
  - $A_{\text{TM}} \subseteq \Omega_A \equiv \{ \langle M, w \rangle \mid$ TM M, string $w \}$
  - $HALT_{\text{TM}} \subseteq \Omega_H \equiv \{ \langle M, w \rangle \mid$ TM M, string $w \}$.
- Suppose $\exists$ TM R that decides $HALT_{\text{TM}}$.
- Then could use R to build a TM S to decide A_{TM} by modifying UTM to first use R to check if it's safe to run M on w.

  $S =$ "On input $\langle M, w \rangle \in \Omega_A$, where M is TM and w is string:
  1. Run R on input $\langle M, w \rangle$.
  2. If R rejects, *reject*.
  3. If R accepts, simulate M on input w until it halts.
  4. If M accepts, *accept*; otherwise, *reject*."
- Since TM R is a decider, TM S always halts and decides A_{TM}.
- However, A_{TM} is undecidable (Theorem 4.I), so that must mean that $HALT_{\text{TM}}$ is also undecidable.

(d) $EQ_{\text{DFA}} =$
$\{ \langle M_1, M_2 \rangle \mid M_1, M_2$ are DFAs with $L(M_1) = L(M_2) \}$.

Answer: EQ_{DFA} is of type DEC (see Theorem 4.E).

- **Decision problem:** For DFAs M_1, M_2, is $L(M_1) = L(M_2)$?
- **Universe:** $\Omega = \{ \langle M_1, M_2 \rangle \mid M_1, M_2$ are DFAs $\}$.
- The following TM T decides EQ_{DFA}:

$T =$ "On input $\langle A, B \rangle \in \Omega$, where A and B are DFAs:

1. Check if $\langle A, B \rangle$ properly encodes 2 DFAs. If not, *reject*.
2. Construct DFA C such that
$$L(C) = [L(A) \cap \overline{L(B)}] \cup [\overline{L(A)} \cap L(B)]$$
using algorithms for DFA union, intersection and complementation.
3. Run TM that decides E_{DFA} (Theorem 4.D) on $\langle C \rangle$.
4. If $\langle C \rangle \in E_{\text{DFA}}$, *accept*; if $\langle C \rangle \notin E_{\text{DFA}}$, *reject*."

5. - Let $L_1, L_2, L_3, \ldots$ be an infinite sequence of regular languages, each of which is defined over a common input alphabet Σ.
   - Let $L = \cup_{k=1}^{\infty} L_k$ be the infinite union of $L_1, L_2, L_3, \ldots$.
   - Is it always the case that L is a regular language?
   - If your answer is YES, give a proof.
   - If your answer is NO, give a counterexample.
   - Explain your answer.
   - Hint: Consider, for each $k \geq 1$, the language $L_k = \{a^k b^k\}$.

Answer: The answer is NO.

- For each $k \geq 1$, let $L_k = \{a^k b^k\}$, so L_k is a language consisting of just a single string $a^k b^k$.
- Since L_k is finite, it must be a regular language by Theorem 1.F.
- But $L = \cup_{k=1}^{\infty} L_k = \{ a^k b^k \mid k \geq 1 \}$, which we know is not regular (see end of Chapter 1).

6. Let L_1, L_2, and L_3 be languages defined over the alphabet $\Sigma = \{a, b\}$, where
   - L_1 consists of all possible strings over Σ except the strings $w_1, w_2, \ldots, w_{100}$; i.e.,
     - start with all possible strings over the alphabet
     - take out 100 particular strings
     - the remaining strings form the language L_1;
   - L_2 is recognized by an NFA; and
   - L_3 is recognized by a PDA.

   Prove that $(L_1 \cap L_2)L_3$ is a context-free language.

   [Hint: First show that L_1 and L_2 are regular.
   Also, consider $\overline{L_1}$.]

Answer:

- $\overline{L_1} = \{w_1, w_2, \ldots, w_{100}\}$, so $|\overline{L_1}| = 100$. Thus, $\overline{L_1}$ is a regular language since it is finite by Theorem 1.F.
- Then Theorem 1.H implies that the complement of $\overline{L_1}$ must be regular, but the complement of $\overline{L_1}$ is L_1. Thus, L_1 is regular.
- Language L_2 has an NFA, so it also has a DFA by Theorem 1.C. Therefore, L_2 is regular.
- Since L_1 and L_2 are regular, $L_1 \cap L_2$ must be regular by Theorem 1.G. Theorem 2.B then implies that $L_1 \cap L_2$ is CFL.
- Since L_3 has a PDA, L_3 is CFL by Theorem 2.C.
- Hence, since $L_1 \cap L_2$ and L_3 are both CFLs, their concatenation is CFL by Theorem 2.F.

7. Write Y or N in the entries of the table below to indicate which classes of languages are closed under which operations.

Operation	Regular languages	CFLs	Decidable languages	Turing-recognizable languages
Union				
Intersection				
Complementation				

Answer:

Op	Regular languages	CFLs	Decidable languages	Turing-recog languages
$\cup$	Y (Thm 1.A)	Y (Thm 2.E)	Y (HW 7, prob 2a)	Y (HW 7, prob 2b)
$\cap$	Y (Thm 1.G)	N (HW 6, prob 2a)	Y	Y
Compl.	Y (Thm 1.H)	N (HW 6, prob 2b)	Y (swap acc/rej)	N (e.g., A_{TM})

8. Consider the following CFG G in Chomsky normal form:

$$S \to a \mid YZ$$
$$Z \to ZY \mid a$$
$$Y \to b \mid ZZ \mid YY$$

Use CYK (dynamic programming) algorithm to fill in following table to determine if G generates string $babba$. Does G generate $babba$?

$$S \to a \mid YZ$$
$$Z \to ZY \mid a$$
$$Y \to b \mid ZZ \mid YY$$

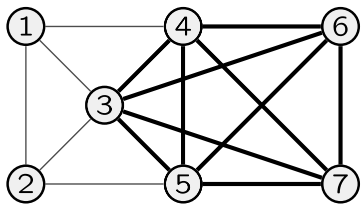
	1	2	3	4	5
1	Y	S	S	S	Y
2		S, Z	Z	Z	Y
3			Y	Y	S
4				Y	S
5					S, Z
	b	a	b	b	a

G does not generate $babba$ because S is not in $(1, 5)$ entry

9. Recall that

$$\begin{aligned} \textit{CLIQUE} &= \{ \langle G, k \rangle \mid G \text{ is undirected graph with } k\text{-clique} \}, \\ &\subseteq \{ \langle G, k \rangle \mid G \text{ is undirected graph, integer } k \} \equiv \Omega_C, \\ \textit{3SAT} &= \{ \langle \phi \rangle \mid \phi \text{ is } \textbf{satisfiable} \text{ 3cnf-function} \} \\ &\subseteq \{ \langle \phi \rangle \mid \phi \text{ is 3cnf-function } \} \equiv \Omega_3. \end{aligned}$$

- Show that *CLIQUE* is NP-Complete by showing that *CLIQUE* $\in$ NP and *3SAT* $\leq_P$ *CLIQUE*.
- Be sure to prove your reduction works and that it takes polynomial time.
- Also, be sure to provide proofs of these results, and don't just cite a theorem.



Answer:

Prove *CLIQUE* $\in$ NP

- The clique is the certificate c.
- Here is a verifier for *CLIQUE*:

  $V =$ "On input $\langle \langle G, k \rangle, c \rangle$:
  1. Test whether c is a set of k different nodes in G.
  2. Test whether G contains all edges connecting nodes in c.
  3. If both tests pass, *accept*; otherwise, *reject*."
- If graph G has m nodes, then (when G is encoded as list of nodes followed by list of edges)
  - Stage 1 takes $O(k)O(m) = O(km)$ time.
  - Stage 2 takes $O(k^2)O(m^2) = O(k^2m^2)$ time.

Prove *3SAT* $\leq_m$ *CLIQUE*

Proof Idea: Convert instance ϕ of *3SAT* problem with k clauses into instance $\langle G, k \rangle$ of clique problem.

- Reducing fcn $f : \Omega_3 \to \Omega_C$
  - $\langle \phi \rangle \in$ *3SAT* iff $f(\langle \phi \rangle) = \langle G, k \rangle \in$ *CLIQUE*
- Suppose ϕ is a 3cnf-function with k clauses, e.g.,

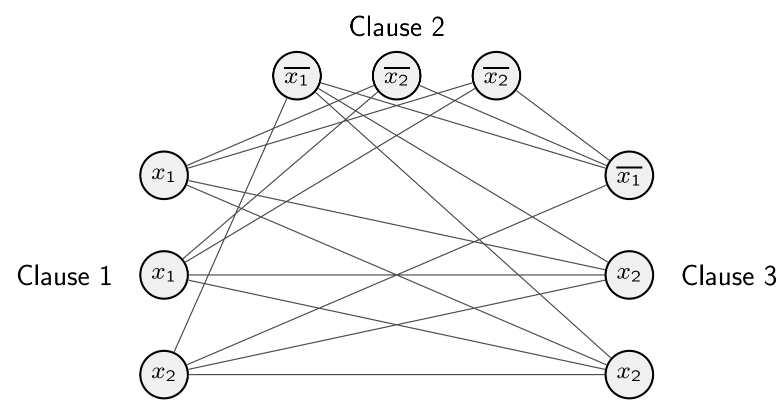
  $$\phi = (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_3 \vee \overline{x_5} \vee x_6) \wedge (x_3 \vee \overline{x_6} \vee x_4) \wedge (x_2 \vee x_1 \vee x_5)$$
- Convert ϕ into a graph G as follows:
  - Nodes in G are organized into k triples $t_1, t_2, \ldots, t_k$.
  - Triple t_i corresponds to the ith clause in ϕ.
  - Each node in a triple corresponds to a literal within the clause.
  - Add edges between each pair of nodes, except
    - within same triple
    - between contradictory literals, e.g., x_1 and $\overline{x_1}$
- Prove $\langle \phi \rangle \in$ *3SAT* iff $\langle G, k \rangle \in$ *CLIQUE*.

3SAT $\leq_m$ CLIQUE

Example: 3cnf-function with $k = 3$ clauses and $m = 2$ variables:

$$\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_2)$$

Corresponding Graph:

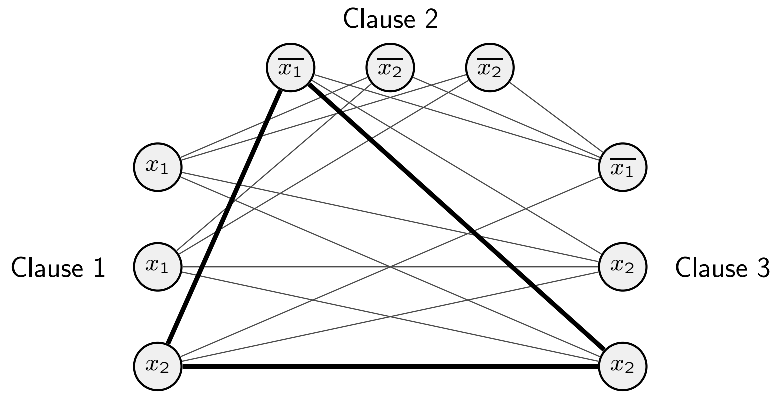


3SAT $\leq_m$ CLIQUE

- 3cnf-formula with $k = 3$ clauses and $m = 2$ variables

$$\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_2)$$

  is satisfiable by assignment $x_1 = 0$, $x_2 = 1$.
- Corresponding graph has k-clique:



Claim: $\langle \phi \rangle \in$ *3SAT* iff $\langle G, k \rangle \in$ *CLIQUE*.

Proof. Use that G has edges between every pair of nodes except for

- pairs in same triple
- contradictory literals.

Also, ϕ satisfiable iff each clause has ≥ 1 true literal.

Claim: The mapping $\phi \to \langle G, k \rangle$ is polynomial-time computable.

Proof.

- Given 3cnf-function ϕ with
  - k clauses
  - m variables.
- Constructing graph G
  - G has $3k$ nodes
  - Adding edges entails considering each pair of nodes in G:

$$\binom{3k}{2} = \frac{3k(3k-1)}{2} = O(k^2)$$

  - Time to construct G is polynomial in size of 3cnf-function ϕ.

10. Recall that

$$\begin{aligned} ILP &= \{ \langle A, b \rangle \mid \text{matrix } A \text{ and vector } b \text{ satisfy } Ay \leq b \\ &\qquad \text{with } y \text{ an } \textbf{integer} \text{ vector} \} \\ &\subseteq \{ \langle A, b \rangle \mid \text{matrix } A \text{ and vector } b \} \equiv \Omega_I \end{aligned}$$

- Show that *ILP* is NP-Complete by showing that *ILP* $\in$ NP and *3SAT* $\leq_P$ *ILP*.
- Be sure to prove your reduction works and that it takes polynomial time.
- Also, be sure to provide proofs of these results, and don't just cite a theorem.

$$\begin{array}{ccccccccc} a_{11}\, y_1 & + & a_{12}\, y_2 & + & \cdots & + & a_{1n}\, y_n & \leq & b_1 \\ a_{21}\, y_1 & + & a_{22}\, y_2 & + & \cdots & + & a_{2n}\, y_n & \leq & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}\, y_1 & + & a_{m2}\, y_2 & + & \cdots & + & a_{mn}\, y_n & \leq & b_m \end{array}$$

ILP $\in$ NP

Proof.

- The certificate c is an integer vector satisfying $Ac \leq b$.
- Here is a verifier for *ILP*:

  V = "On input $\langle \langle A, b \rangle, c \rangle$:
  1. Test whether c is a vector of all integers.
  2. Test whether $Ac \leq b$.
  3. If both tests pass, *accept*; otherwise, *reject*."
- If $Ay \leq b$ has m inequalities and n variables, then
  - Stage 1 takes $O(n)$ time
  - Stage 2 takes $O(mn)$ time
  - So verifier V runs in $O(mn)$, which is polynomial in size of problem instance.

Now prove *ILP* is NP-Hard by showing *3SAT* $\leq_P$ *ILP*.

3SAT $\leq_m$ ILP

- Reductn $f : \Omega_3 \to \Omega_I$, $\langle\phi\rangle \in$ *3SAT* iff $f(\langle\phi\rangle) = \langle A, b\rangle \in$ *ILP*.
- Consider 3cnf-formula with $m = 4$ variables and $k = 3$ clauses:

$$\phi = (x_1 \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_4) \wedge (\overline{x_2} \vee \overline{x_4} \vee \overline{x_3})$$

- Define integer linear program with
  - $2m = 8$ variables $y_1, y'_1, y_2, y'_2, y_3, y'_3, y_4, y'_4$
    - y_i corresponds to x_i
    - y'_i corresponds to $\overline{x_i}$
  - 3 sets of inequalities for each of pair y_i, y'_i:

$$\begin{array}{lll} 0 \leq y_1 \leq 1, & 0 \leq y'_1 \leq 1, & y_1 + y'_1 = 1 \\ 0 \leq y_2 \leq 1, & 0 \leq y'_2 \leq 1, & y_2 + y'_2 = 1 \\ 0 \leq y_3 \leq 1, & 0 \leq y'_3 \leq 1, & y_3 + y'_3 = 1 \\ 0 \leq y_4 \leq 1, & 0 \leq y'_4 \leq 1, & y_4 + y'_4 = 1 \end{array}$$

  which guarantee that exactly one of y_i and y'_i is 1, and other is 0.
  - $0 \leq y_i \leq 1 \iff -y_i \leq 0 \ \& \ y_i \leq 1$
  - $y_i + y'_i = 1 \iff y_i + y'_i \leq 1 \ \& \ y_i + y'_i \geq 1$

3SAT $\leq_m$ ILP

- Recall 3cnf-formula with $m = 4$ variables and $k = 3$ clauses:

$$\phi = (x_1 \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_4) \wedge (\overline{x_2} \vee \overline{x_4} \vee \overline{x_3})$$

  - ϕ satisfiable iff each clause evaluates to 1.
  - A clause evaluates to 1 iff at least one literal in the clause equals 1.
  - For each clause $(x_i \vee \overline{x_j} \vee x_\ell)$, create inequality $y_i + y'_j + y_\ell \geq 1$.
  - For our example, ILP has inequalities

$$\begin{array}{c} y_1 + y_2 + y'_3 \geq 1 \\ y'_1 + y'_2 + y_4 \geq 1 \\ y'_2 + y'_4 + y'_3 \geq 1 \end{array}$$

  which guarantee that each clause evaluates to 1.

3SAT $\leq_m$ ILP

- Given 3cnf-formula:

$$\phi = (x_1 \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_4) \wedge (\overline{x_2} \vee \overline{x_4} \vee \overline{x_3})$$

- Constructed ILP:

$$\begin{array}{lll} 0 \leq y_1 \leq 1, & 0 \leq y'_1 \leq 1, & y_1 + y'_1 = 1 \\ 0 \leq y_2 \leq 1, & 0 \leq y'_2 \leq 1, & y_2 + y'_2 = 1 \\ 0 \leq y_3 \leq 1, & 0 \leq y'_3 \leq 1, & y_3 + y'_3 = 1 \\ 0 \leq y_4 \leq 1, & 0 \leq y'_4 \leq 1, & y_4 + y'_4 = 1 \end{array}$$

$$\begin{array}{c} y_1 + y_2 + y'_3 \geq 1 \\ y'_1 + y'_2 + y_4 \geq 1 \\ y'_2 + y'_4 + y'_3 \geq 1 \end{array}$$

- Note that:

  ϕ satisfiable $\iff$ constructed ILP has solution (with values of variables $\in \{0, 1\}$)

Reducing 3SAT to ILP Takes Polynomial Time

- Given 3cnf-formula ϕ with
  - m variables: $x_1, x_2, \ldots, x_m$
  - k clauses
- Constructed ILP has
  - $2m$ variables: $y_1, y'_1, y_2, y'_2, \ldots, y_m, y'_m$
  - $6m + k$ inequalities:
    - 3 sets of inequalities for each pair y_i, y'_i:

$$0 \leq y_i \leq 1, \quad 0 \leq y'_i \leq 1, \quad y_i + y'_i = 1,$$

      so total of $6m$ inequalities of this type.
    - For each clause in ϕ, ILP has corresponding inequality, e.g.,

$$(x_1 \vee x_2 \vee \overline{x_3}) \quad \longleftrightarrow \quad y_1 + y_2 + y'_3 \geq 1,$$

      so total of k inequalities of this type.
- Thus, size of ILP is polynomial in m and k.