Math 340 * Review for Final Exam * Victor Matveev

Problem 1: Assuming $|x|<<1$, for what values of x does the function evaluation below **completely fail** (i.e. relative error $\geq 100\%$) in double precision? What result would MATLAB produce for those values of x? Using Taylor polynomials, find **leading two terms** in the limiting behavior as $x \to 0$:

$$f(x) = \frac{x^5}{\sin(x^2) - \log(1+x^2)} = \frac{x^5}{\left(\cancel{x^2} - \frac{x^6}{3!} + O(x^{10})\right) - \left(\cancel{x^2} - \frac{x^4}{2} + \frac{x^6}{3} + O(x^8)\right)} = \frac{x^5}{\frac{x^4}{2} - x^6\left(\frac{1}{6}+\frac{1}{3}\right) + O(x^8)}$$

$$= \frac{x}{\frac{1}{2} - \frac{1}{2}x^2 + O(x^4)} = \frac{2x}{1 - x^2 + O(x^4)} \approx \boxed{2x\left(1+x^2\right)} + O(x^5)$$

This is the correct limiting behavior for $x \to 0$ (and also very accurate given that the error $\sim x^5$)

Denominator will yield 0 in double precision when $x^4/2$ is rounded off when added to x^2: $\frac{x^4/2}{x^2} = \frac{x^2}{2} \leq 2^{-53} \doteq 10^{-16}$

(under this condition both $x^4/2$ and $x^6/3!$ will be rounded off to zero when subtracted from x^2)

$\Rightarrow$ In MATLAB $f(x)$ will evaluate to **Inf** when $|x| \leq 10^{-8}$, although the true limit for $x \to 0$ is **zero**!

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Problem 2: Find values w_1, w_2 and x_2 so that the following integration rule has the highest degree of precision:

$$\int_0^1 \ln(x)\, f(x)\, dx = w_1 f(0) + w_2 f(x_2) \qquad \left(\text{Use the result } \int_0^1 x^m \ln(x) dx = -\frac{1}{(m+1)^2}\right)$$

$$f = 1\ (m=0): \int_0^1 \ln(x)\, dx = -1 = w_1 + w_2$$

$$f = x\ (m=1): \int_0^1 \ln(x)\, dx = -\frac{1}{4} = x_2\, w_2$$

$$f = x^2\ (m=2): \int_0^1 \ln(x)\, dx = -\frac{1}{9} = x_2^2\, w_2$$

$$\boxed{\div} \Rightarrow \boxed{\begin{matrix} x_2 = \frac{4}{9} \\ w_2 = -\frac{9}{16} \end{matrix}} \qquad \boxed{w_1 = -\frac{7}{16}} \Rightarrow \boxed{\int_0^1 \ln(x)\, f(x)\, dx \approx -\frac{7f(0) + 9f\left(\frac{4}{9}\right)}{16}}$$

Use the resulting integration rule to estimate $\int_0^1 \frac{\ln(x)}{1+x} dx$, comparing with the exact value $\int_0^1 \frac{\ln(x)}{1+x} dx = -\frac{\pi^2}{12} \doteq -0.8225$

$$\int_0^1 \frac{\ln(x)}{1+x} dx \approx -\frac{7f(0) + 9f\left(\frac{4}{9}\right)}{16} = -\frac{7 + \frac{9}{1+4/9}}{16} = -\frac{7 + \frac{81}{13}}{16} = -\frac{172}{13 \cdot 16} = -\frac{43}{13 \cdot 4} = -\frac{43}{52} \doteq 0.827$$

Very good given singular integrand and 2-point rule

Compare the accuracy of this estimate with the midpoint rule with $n=1$, noting that $\ln(2) \approx 0.69315$:

This is much better than midpoint with $n=1$: $M_1 = h\, f\left(\frac{1}{2}\right) = \frac{\ln(1/2)}{1+1/2} = -\frac{2\ln 2}{3} \doteq -\frac{2 \cdot 0.6931}{3} \doteq -0.462$

Problem 3:

Find values of constants A, B and C so that the following finite difference approximates the 2nd derivative of $f(x)$ at x_0. What is the error of this approximation? To check your answer, apply this formula to $f(x)=x^2$

$$Df(x_0) = A\,f(x_0-2h) + B\,f(x_0) + C\,f(x_0+3h)$$

Expand the 1st and 3rd terms in Taylor series up to 3rd order, and sum the right-hand side:

$$\begin{array}{r|l|} A\times & f(x_0-2h) \approx f(x_0) - 2h\,f'(x_0) + \frac{(2h)^2}{2}\,f''(x_0) - \frac{(2h)^3}{6}\,f'''(x_0) \\ \boxed{+}\quad B\times & \qquad\qquad\quad f(x_0) \\ C\times & f(x+3h) \approx f(x_0) + 3h\,f'(x_0) + \frac{(3h)^2}{2}\,f''(x_0) + \frac{(3h)^3}{6}\,f'''(x_0) \end{array}$$

$$f''(x) \quad \boxed{=} \quad \underbrace{(A+B+C)}_{=0}\,f(x) + \underbrace{(3C-2A)}_{=0}\,h\,f'(x) + \underbrace{(4A+9C)\frac{h^2}{2}}_{=1}\,f''(x) + \underbrace{(27C-8A)\frac{h^3}{6}\,f'''(x)}_{\text{ERROR}}$$

$$\begin{cases} A+B+C=0 \\ 2A=3C \\ 4A+9C=15C=\dfrac{2}{h^2} \Rightarrow C=\dfrac{2}{15h^2} \end{cases} \Rightarrow \begin{cases} B=-A-C=-\dfrac{5}{15h^2} \\ A=\dfrac{3C}{2}=\dfrac{3}{15h^2} \\ C=\dfrac{2}{15h^2} \end{cases} \Rightarrow \boxed{f''(x_0)\approx\frac{3f(x_0-2h)-5f(x_0)+2f(x_0+3h)}{15h^2}}$$

The error of this numerical second derivative is given by the cubic term:

$$E \approx \underbrace{(27C-8A)}_{2/h^2}\frac{h^3}{6}\,f'''(x_0) = \boxed{\frac{h}{3}\,f'''(x_0)}$$

More accurate error formula is $\frac{h}{3}\,f'''(c)$, where $c\in[x_0-2h,\ x_0+3h]$

Therefore, the derivative calculated with non-equidistant points is less accurate than the usual symmetric $D_h^{(2)}f(x_0)$

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Problem 4:

Find the value of constants γ_1 and γ_2 so that the following method of integrating a differential equation $dY/dx = f(x, Y)$ has second order of accuracy, by comparing Taylor expansions up to order h^3

$$y_{n+1} = y_n + h\left[\gamma_1 f(x_n, y_n) + \gamma_2 f\left(x_n + \frac{2}{3}h,\ y_n + \frac{2}{3}h\, f(x_n, y_n)\right)\right]$$

$$= y_n + h\left[\gamma_1 f(x_n\ y_n) + \gamma_2\left(f(x_n, y_n) + \frac{2h}{3}\frac{\partial f}{\partial x}(x_n\ y_n) + \frac{2h}{3} f(x_n, y_n)\frac{\partial f}{\partial Y}(x_n, y_n) + O(h^2)\right)\right]$$

$$= y_n + h(\gamma_1 + \gamma_2)\, f(x_n, y_n) + \frac{2h^2\gamma_2}{3}\left[\frac{\partial f}{\partial x}(x_n, y_n) + f(x_n, y_n)\frac{\partial f}{\partial Y}(x_n, y_n)\right] + O(h^3)$$

Compare this with the Taylor expansion of exact solution:

$$Y_{n+1} = Y_n + hY_n' + \frac{h^2}{2}Y_n'' + O(h^3) = Y_n + h\, f(x_n, Y_n) + \frac{h^2}{2}\frac{df}{dx}(x_n, Y_n) + O(h^3)$$

Now, use the chain rule to find the "full" x-derivative of $f(x_n, Y(x_n))$: $\frac{df}{dx}(x, Y(x)) = \frac{\partial f}{\partial x}(x, Y) + \frac{\partial f}{\partial Y}(x, Y)\underbrace{\frac{dY(x)}{dx}}_{f(x,Y)}$

$$\Rightarrow\ Y_{n+1} = Y_n + h\, f(x_n, Y_n) + \frac{h^2}{2}\left[\frac{\partial f}{\partial x}(x_n, Y_n) + \frac{\partial f}{\partial Y}(x_n, Y_n)\, f(x_n, Y_n)\right] + O(h^3)$$

The equations for y_n and Y_n agree up to terms of order h^3 if $\begin{cases} \gamma_1 + \gamma_2 = 1 \quad \Rightarrow \gamma_1 = 1 - \gamma_2 = \dfrac{1}{4} \\ \dfrac{2\cancel{h^2}\gamma_2}{3} = \dfrac{\cancel{h^2}}{2} \quad \Rightarrow \gamma_2 = \dfrac{3}{4} \end{cases}$

Thus, the 2nd order accurate method is:

$$\boxed{y_{n+1} = y_n + \frac{h}{4}\left[f(x_n, y_n) + 3f\left(x_n + \frac{2}{3}h,\ y_n + \frac{2}{3}h\, f(x_n, y_n)\right)\right]}$$

Problem 4 (continued): To examine stability, consider special case $f(x, Y) = \lambda Y$:

$$y_{n+1} = y_n + \frac{h}{4}\left[f(x_n, y_n) + 3f\left(x_n + \frac{2h}{3},\ y_n + \frac{2h}{3} f(x_n, y_n) \right) \right]$$

$$= y_n + \frac{h}{4}\left[\lambda y_n + 3\lambda\left(y_n + \frac{2h}{3}\lambda y_n \right) \right] = y_n + h\lambda y_n + \frac{h^2\lambda^2}{2} y_n = \left(1 + h\lambda + \frac{(h\lambda)^2}{2} \right) y_n$$

Since this agrees with exact solution up to order h^2, the propagated error directly follows:

$$\varepsilon_{n+1} = \left(1 + h\lambda + \frac{(h\lambda)^2}{2} \right) \varepsilon_n$$

Stability requires $\left| 1 + h\lambda + \frac{(h\lambda)^2}{2} \right| < 1$

Denoting $z = h\lambda$, stability is satisfied when $\left| 1 + z + \frac{z^2}{2} \right| = \left| \frac{(z+1)^2 + 1}{2} \right| < 1$

$$\Rightarrow \left| (z+1)^2 + 1 \right| < 2 \Rightarrow (z+1)^2 < 1 \Rightarrow -2 < z < 0 \Rightarrow \begin{cases} \lambda < 0 \\ h < \dfrac{2}{|\lambda|} \end{cases} \text{Conditionally stable}$$

Problem 5:

Find the value of constant k so that the following function is a cubic spline, and sketch this spline:

$$s(x) = \begin{cases} 1 + x + (x-1)^2 + (x-1)^3 & \text{if } 1 \le x \le 2 \\ 5 + 6(x-2) + k(x-2)^2 & \text{if } 2 \le x \le 3 \end{cases}$$

Note that $s(x)$ and $s'(x)$ are continuous at $x = 2$ for any k: $s(2) = 5$; $s'(2) = 6$

Need to make second derivative continuous also:

$$s''(x) = \begin{cases} 2 + 6(x-1) & \text{if } 1 \le x \le 2 \\ 2k & \text{if } 2 \le x \le 3 \end{cases} \Rightarrow s''(2) = \begin{cases} 8 \\ 2k \end{cases} \Rightarrow \boxed{k = 4}$$

Problem 6:

Use the least sum of squares method to fit a curve $y(x) = a\cos x + b$ to the data points $(0,\ 1)$, $(\pi/3,\ 0)$, $(\pi/2,\ -2)$

Solution:

1) Find the sum of squares of residuals:

$$G(a,b) = \sum_{i=1}^{3}(y(x_i) - y_i)^2 = \sum_{i=1}^{3}(\cos(x_i) + b - y_i)^2 = (a+b-1)^2 + \left(\frac{a}{2} + b\right)^2 + (b+2)^2$$

2) Minimize the sum of squares of residuals:

$$\begin{cases} \dfrac{\partial G}{\partial a} = 2\left(a+b-1\right) + \left(\dfrac{a}{2} + b\right) = \dfrac{5}{2}a + 3b - 2 = 0 \quad \Big\| \times 2 \\ \dfrac{\partial G}{\partial b} = 2\left(a+b-1\right) + \left(a+2b\right) + 2(b+2) = 3a + 6b = -2 \end{cases} \Rightarrow \begin{cases} 5a + 6b = 4 \\ 3a + 6b = -2 \end{cases} \Rightarrow \boxed{\begin{matrix} a = 3 \\ b = -\dfrac{11}{6} \end{matrix}}$$

Thus, the curve of given form that fits data the best is $\boxed{y(x) = 3\cos x - \dfrac{11}{6}}$