Math 335-002
Homework #8

Problems 1-5 are due April 2; problems 6-7 are due April 4

<u>Please show all work in detail to receive full credit. Late homework is not accepted</u>

1. Problems 5.8 and 5.10 on page 98.

2. Verify the Stokes theorem $\left(\iint_S \nabla\times\vec{\mathbf{u}}\cdot\vec{\mathbf{n}}\,dS = \oint_C \vec{\mathbf{u}}\cdot\mathbf{d}\vec{\mathbf{r}}\right)$ by calculating both the surface integral and the closed curve integral for vector field $\vec{\mathbf{u}}=(0, 2x, z)$, with surface S given by $z = 2 - 2x - y$, with constraints $x>0$, $y>0$, $z>0$. Note that the curve integral breaks down into three separate pieces.

3. Sketch and use cylindrical coordinates to calculate the mass of a paraboloid cut in half, defined by $x^2 + y^2 \le 1 - z$, $0 \le z \le 1$, $\boldsymbol{x\geq 0}$, with the mass density function given by $\rho = z\,x\sqrt{x^2+y^2}$. Make sure you use correct limits for the angle φ.

4. According to the Stokes theorem, the integral of $\nabla\times\vec{\mathbf{u}}$ of any field $\vec{\mathbf{u}}$ over any **closed** surface is zero. *Use cylindrical coordinates* to verify this fact for the vector field $\vec{\mathbf{u}}=(xz, xz, 0)$, for the surface of the cylinder $x^2 + y^2 = 1$, $0 \le z \le 1$. Recall that the normal to the side surface is the unit vector $\vec{e}_R$, and the normal to the top and bottom surfaces is the same as in Cartesian coordinates: $\vec{\mathbf{n}}_{t,b} = \pm\vec{e}_z$

5. Problem 6.3 on page 107 – but instead of calculating the volume, simply find the unit vectors, the stretch factors, and the volume element, as we did in class for the cylindrical coordinate system (differentiate the position vector). Note that x_1, x_2 and x_3 denote the Cartesian coordinates x, y, and z.

6. Consider a part of the sphere $x^2 + y^2 + z^2 \le 1$ satisfying $0<\theta<\pi/6$, $0<\varphi<\pi/2$. Sketch (roughly) this object and use spherical coordinates for the following calculations:

   a) Find the volume of this object.
   b) Find its mass, given the mass density function $\rho = y$.
   c) Verify the divergence theorem for the vector field $\vec{\mathbf{u}}=(0, 0, z^2)$ (two of the four surface integrals are zero)
   d) Find the surface area, including both the flat and the curved boundaries of this object.

7. Convert the Cartesian vector field $\vec{\mathbf{v}} = (y, 0, z^2)_{x\,y\,z}$ into the cylindrical coordinate system, $\vec{\mathbf{v}}=(v_R, v_\varphi, v_z)_{R\,\varphi\,z}$, and the spherical system, $\mathbf{v}=(v_r, v_\theta, v_\varphi)_{r\,\theta\,\varphi}$ (see note on the next page if in doubt)

Note on converting vectors between different coordinate systems

A vector should not depend on a coordinate system we choose to use, so

$$\mathbf{v} = \mathrm{v}_x\, \mathbf{e}_x + \mathrm{v}_y\, \mathbf{e}_y + \mathrm{v}_z\, \mathbf{e}_z = \mathrm{v}_1\, \mathbf{e}_1 + \mathrm{v}_2\, \mathbf{e}_2 + \mathrm{v}_3\, \mathbf{e}_3$$

where $\mathbf{e}_{1,\,2,\,3}$ are the unit vectors of any curvilinear orthogonal coordinate system. We may re-write the above equation in component form as

$$\mathbf{v} = (\mathrm{v}_x, \mathrm{v}_y, \mathrm{v}_z)_{x\,y\,z} = (\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3)_{\mathrm{u1\;u2\;u3}}$$

where the subscripts indicate the coordinate system of the components. The vector components in brackets are found by projecting the vector onto each of the unit vectors:

$$\mathrm{v}_{x,\,y,\,z} = \mathbf{v} \cdot \mathbf{e}_{x,\,y,\,z} \quad \text{and} \quad \mathrm{v}_{1,\,2,\,3} = \mathbf{v} \cdot \mathbf{e}_{1,\,2,\,3}$$

where the relationship between the curvilinear basis vectors $\mathbf{e}_{1,\,2,\,3}$ and the cartesian basis vectors $\mathbf{e}_{x,\,y,\,z}$ is given by

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u_i} \Big/ \left| \frac{\partial \mathbf{r}}{\partial u_i} \right| = \frac{1}{h_i} \frac{\partial \mathbf{r}}{\partial u_i} = \frac{1}{h_i} \frac{\partial}{\partial u_i} (x, y, z)_{x\,y\,z}, \quad i = 1, 2, 3$$

For cylindrical coordinates, we have (see page 108)

$$\mathbf{e}_R = (1, 0, 0)_{R\,\varphi\,z} = (\cos\varphi, \sin\varphi, 0)_{x\,y\,z} \qquad \mathrm{v}_R = \mathbf{v} \cdot \mathbf{e}_R$$
$$\mathbf{e}_\varphi = (0, 1, 0)_{R\,\varphi\,z} = (-\sin\varphi, \cos\varphi, 0)_{x\,y\,z} \qquad \mathrm{v}_\varphi = \mathbf{v} \cdot \mathbf{e}_\varphi$$
$$\mathbf{e}_z = (0, 0, 1)_{R\,\varphi\,z} = (0, 0, 1)_{x\,y\,z} \qquad \mathrm{v}_z = \mathbf{v} \cdot \mathbf{e}_z$$

For spherical coordinates, we have (see page 111)

$$\mathbf{e}_r = (1, 0, 0)_{r\,\theta\,\varphi} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)_{x\,y\,z} \qquad \mathrm{v}_r = \mathbf{v} \cdot \mathbf{e}_r$$
$$\mathbf{e}_\theta = (0, 1, 0)_{r\,\theta\,\varphi} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)_{x\,y\,z} \qquad \mathrm{v}_\theta = \mathbf{v} \cdot \mathbf{e}_\theta$$
$$\mathbf{e}_\varphi = (0, 0, 1)_{r\,\theta\,\varphi} = (-\sin\varphi, \cos\varphi, 0)_{x\,y\,z} \qquad \mathrm{v}_\varphi = \mathbf{v} \cdot \mathbf{e}_\varphi$$