Math 335-002
Homework #9
Due date: Monday, April 9, 2007

<u>Please show all work in detail to receive full credit. Late homework is not accepted.</u>

1. Derive the expressions for the gradient, the divergence and the Laplacian in cylindrical coordinates, starting with the general expressions
$$\vec{\nabla} f = \left(\frac{1}{h_1} \frac{\partial f}{\partial u_1}, \frac{1}{h_2} \frac{\partial f}{\partial u_2}, \frac{1}{h_3} \frac{\partial f}{\partial u_3} \right)_{u_1 u_2 u_3}$$
$$\vec{\nabla} \cdot \vec{v} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial (h_2 h_3 v_1)}{\partial u_1} + \frac{\partial (h_1 h_3 v_2)}{\partial u_2} + \frac{\partial (h_1 h_2 v_3)}{\partial u_3} \right], \ \nabla^2 f = \vec{\nabla} \cdot \vec{\nabla} f$$
where u_1=R, u_2=φ, u_3=z. Make sure to simplify whenever possible.

2. Convert the Cartesian components of the vector field $\mathbf{v} = (x^2, 0, 0)_{x\,y\,z}$ into spherical components, $\mathbf{v} = (v_r, v_\theta, v_\varphi)_{r\,\theta\,\varphi}$. Then, find the divergence of this vector field using spherical coordinates (Eq. 6.23 on p. 111 we derived in class), and show that the result agrees with the simple Cartesian calculation, $\vec{\nabla} \cdot \vec{\mathbf{v}} = 2x$

3. Find the gradient and the Laplacian of the scalar field $f = z^2$ using **spherical** coordinate expressions derived in class (Eqs. 6.22-6.24 on p. 111), and show that the results agree with the calculations carried out in Cartesian coordinates (to verify the agreement for the gradient, you have to convert the Cartesian gradient calculation result into the spherical coordinate system, or *vice versa*)

4. In Cartesian coordinates, the basis vectors ($\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$) have constant direction everywhere, and therefore they have zero divergence (and curl). This is not always the case for the curvilinear coordinate basis vectors. Calculate the divergence of unit vectors $\mathbf{e}_R$, $\mathbf{e}_\varphi$ and $\mathbf{e}_z$ in cylindrical coordinates, using the expression obtained in problem 1. Note that the cylindrical components of these unit vectors are (1,0,0), (0,1,0), and (0,0,1), respectively. Explain why one of the results is non-zero (one sentence or a rough sketch is sufficient).

5. Repeat problem 4 for the **spherical** coordinate system, that is, calculate the divergence of unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$, using the divergence expression derived in class for the spherical coordinate system. Explain why two of the vectors have a non-zero divergence, using a simple sketch.

Note on converting vectors between different coordinate systems

A vector should not depend on a coordinate system we choose to use, so

$$\mathbf{v} = v_x\,\mathbf{e}_x + v_y\,\mathbf{e}_y + v_z\,\mathbf{e}_z = v_1\,\mathbf{e}_1 + v_2\,\mathbf{e}_2 + v_3\,\mathbf{e}_3$$

where $\mathbf{e}_{1,\,2,\,3}$ are the unit vectors of any curvilinear orthogonal coordinate system. We may re-write the above equation in component form as

$$\mathbf{v} = (v_x,\ v_y,\ v_z)_{x\,y\,z} = (v_1,\ v_2,\ v_3)_{\mathrm{u1\ u2\ u3}}$$

where the subscripts indicate the coordinate system of the components. The vector components in brackets are found by projecting the vector onto each of the unit vectors:

$$v_{x,\,y,\,z} = \mathbf{v}\cdot\mathbf{e}_{x,\,y,\,z} \quad\text{and}\quad v_{1,\,2,\,3} = \mathbf{v}\cdot\mathbf{e}_{1,\,2,\,3}$$

where the relationship between the curvilinear basis vectors $\mathbf{e}_{1,\,2,\,3}$ and the cartesian basis vectors $\mathbf{e}_{x,\,y,\,z}$ is given by

$$\mathbf{e}_i = \frac{\partial\mathbf{r}}{\partial u_i}\Bigg/\left|\frac{\partial\mathbf{r}}{\partial u_i}\right| = \frac{1}{h_i}\frac{\partial\mathbf{r}}{\partial u_i} = \frac{1}{h_i}\frac{\partial}{\partial u_i}(x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z), \quad i = 1,\ 2,\ 3$$

For cylindrical coordinates, we have (see page 108)

$\mathbf{e}_R = (1, 0, 0)_{R\,\varphi\,z} = (\cos\varphi, \sin\varphi, 0)_{x\,y\,z}$ $\qquad v_R = \mathbf{v}\cdot\mathbf{e}_R$
$\mathbf{e}_\varphi = (0, 1, 0)_{R\,\varphi\,z} = (-\sin\varphi, \cos\varphi, 0)_{x\,y\,z}$ $\qquad v_\varphi = \mathbf{v}\cdot\mathbf{e}_\varphi$
$\mathbf{e}_z = (0, 0, 1)_{R\,\varphi\,z} = (0, 0, 1)_{x\,y\,z}$ $\qquad v_z = \mathbf{v}\cdot\mathbf{e}_z$

For spherical coordinates, we have (see page 111)

$\mathbf{e}_r = (1, 0, 0)_{r\,\theta\,\varphi} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)_{x\,y\,z}$ $\qquad v_r = \mathbf{v}\cdot\mathbf{e}_r$
$\mathbf{e}_\theta = (0, 1, 0)_{r\,\theta\,\varphi} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)_{x\,y\,z}$ $\qquad v_\theta = \mathbf{v}\cdot\mathbf{e}_\theta$
$\mathbf{e}_\varphi = (0, 0, 1)_{r\,\theta\,\varphi} = (-\sin\varphi, \cos\varphi, 0)_{x\,y\,z}$ $\qquad v_\varphi = \mathbf{v}\cdot\mathbf{e}_\varphi$