



Some new results on Yang–Lee zeros of the Ising model partition function

Victor Matveev¹, Robert Shrock²

Institute for Theoretical Physics, State University of New York, Stony Brook, NY 11794-3840, USA

Received 22 December 1995; accepted for publication 18 March 1996

Communicated by A.R. Bishop

Abstract

We prove that for the Ising model on a lattice of dimensionality $d \geq 2$, the zeros of the partition function Z in the complex μ plane (where $\mu = e^{-2\beta H}$) lie on the unit circle $|\mu| = 1$ for a wider range of $K_{nn'} = \beta J_{nn'}$ than the range $K_{nn'} \geq 0$ assumed in the premise of the Yang–Lee circle theorem. This range includes complex temperatures, and we show that it is lattice-dependent. Our results thus complement the Yang–Lee theorem, which applies for any d and any lattice if $J_{nn'} \geq 0$. For the case of uniform couplings $K_{nn'} = K$, we show that these zeros lie on the unit circle $|\mu| = 1$ not just for the Yang–Lee range $0 \leq u \leq 1$, but also for (i) $-u_{c,\text{sq}} \leq u \leq 0$ on the square lattice, and (ii) $-u_{c,t} \leq u \leq 0$ on the triangular lattice, where $u = z^2 = e^{-4K}$, $u_{c,\text{sq}} = 3 - 2^{3/2}$, and $u_{c,t} = 1/3$. For the honeycomb, 3×12^2 , and 4×8^2 lattices we prove an exact symmetry of the reduced partition functions, $Z_r(z, -\mu) = Z_r(-z, \mu)$. This proves that the zeros of Z for these lattices lie on $|\mu| = 1$ for $-1 \leq z \leq 0$ as well as the Yang–Lee range $0 \leq z \leq 1$. Finally, we report some new results on the patterns of zeros for values of u or z outside these ranges.

The Ising model has long served as a simple prototype of a statistical mechanical system which (for d greater than the lower critical dimensionality $d_{l.c.d.} = 1$) undergoes a second-order phase transition with associated spontaneous symmetry breaking and long range magnetic order. The general zero-field (spin 1/2) Ising model on a lattice A at temperature T and external magnetic field H is defined by the partition function

$$Z = \sum_{\{\sigma_n\}} e^{-\beta \mathcal{H}}, \quad (1)$$

with the Hamiltonian

$$\mathcal{H} = - \sum_{\langle nn' \rangle} \sigma_n J_{nn'} \sigma_{n'} - H \sum_n \sigma_n, \quad (2)$$

where $\sigma_n = \pm 1$ are the spin variables on each site n of the lattice, $J_{nn'}$ is the spin–spin exchange constant, and the units are defined such that the magnetic moment which would multiply $H \sum_n \sigma_n$ is unity. We shall prove a general result for the case of arbitrary $J_{nn'}$ (connecting any two sites n and n') and then concentrate on the usual nearest-neighbor model with uniform coupling $J_{nn'} = J \delta_{n,n' \pm e_j}$, where e_j is a lattice vector. We use the standard notation $\beta = (k_B T)^{-1}$, $K = \beta J$, $h = \beta H$, $z = e^{-2K}$, $u = z^2 = e^{-4K}$, and $\mu = e^{-2h}$. The reduced free energy (per site) is $f = -\beta F = \lim_{N \rightarrow \infty} N^{-1} \ln Z$ in the thermodynamic limit, where N denotes the number of sites on the lattice. For the uniform case, we can write

¹ E-mail: vmatveev@insti.physics.sunysb.edu.

² E-mail: shrock@insti.physics.sunysb.edu.

$$Z = e^{l(q/2)K + h|N} Z_T, \quad (3)$$

where q denotes the coordination number of the lattice. For fixed N , Z_T is then a polynomial in μ and either u (if q is even) or z (if q is odd). The 2D Ising model has the appealing feature of exact solvability; the zero-field free energy f and spontaneous magnetization M were first derived by Onsager and Yang, respectively [1,2] (both for the square lattice; these solutions were later generalized to other 2D lattices).

Yang and Lee pioneered a very interesting line of research in which one studies the model with the external magnetic field generalized from real to complex values [3,4]. For an arbitrary (finite as well as infinite) lattice with ferromagnetic spin-spin couplings $J_{nn'} \geq 0$ (and physical temperature $0 \leq \beta \leq \infty$), Yang and Lee proved a classic theorem stating that the zeros of the partition function lie on the unit circle $|\mu| = 1$ in the μ plane and pinch the positive real μ axis as the temperature T decreases through the critical temperature T_c [3,4] (see also Ref. [5]). In the thermodynamic limit, these zeros become dense and determine the continuous locus of points in the μ plane where the free energy, for fixed T , is non-analytic. Henceforth, we shall concentrate on the thermodynamic limit, and on results from finite graphs which can give insight into this limit³. An elementary property of (1) is its invariance under the simultaneous transformations $\sigma_n \rightarrow -\sigma_n$, $h \rightarrow -h$. This implies that the phase diagram in the μ plane is invariant under the inversion

$$\mu \rightarrow 1/\mu, \quad (4)$$

with the obvious sign flip $M(\mu) = -M(1/\mu)$. With no loss of generality, one may therefore restrict one's considerations just to the interior and rim of the unit circle $|\mu| = 1$ in the μ plane.

³ Several further questions concerning these zeros have never been answered exactly. These include, for the case of lattices of dimensionality $d \geq 2$ (i) the density of zeros, and (ii) the location of the complex-conjugate endpoints of the distribution for $T > T_c$. Although we shall not need this here, we mention that there has been intensive study of these zeros and generalizations since the original works [3,4]; some work (on regular lattices without quenched disorder) is listed in Ref. [7]. For $J_{nn'} < 0$, the zeros do not lie on the unit circle $|\mu| = 1$ [6]. For the 1D uniform- J case, an exact solution by Yang and Lee [4,6] shows that these lie on the negative real μ axis; for $d \geq 2$, there are no exact results on the location of the zeros for $J_{nn'} < 0$. A numerical study was carried out in Ref. [8].

A complementary complexification is to consider the model in zero field with the temperature generalized to complex values. In this case, one is interested in the phase diagram in the complex u plane (complex z plane for odd q). A number of results in this area, both exact and from series analyses, have been obtained for the Ising model with spin 1/2 [9–13] (and with higher spin [14,11,15]). The next logical step is to consider the model with both temperature and magnetic field generalized to complex values. Perhaps the earliest step in this direction was the exact solution by Lee and Yang [4], of the square-lattice Ising model for a particular manifold of pure imaginary values of external field $H = i(\pi/2)k_B T$, i.e. $h = i\pi/2$ ($\mu = -1$). Recently, we have studied this area further [16,17].

In the present note we report some new results on complex-field (Yang–Lee) zeros of the Ising model partition function. We first prove a general result. Consider the Ising model with uniform ferromagnetic spin couplings. For $d > d_{l.c.d.} = 1$, the model has a low-temperature (i.e., small- u for even q , small- z for odd q) series expansion with a finite radius of convergence. This fact allows one to carry out an analytic continuation from $u \sim 0^+$ through $u = 0$ to negative real values of u in the vicinity of the origin (for odd q , a continuation from $z \sim 0^+$ to negative real values of z near $z = 0$). The properties of the model are continuous under this analytic continuation; in particular, the zeros of Z remain on the unit circle⁴ $|\mu| = 1$. As we shall now show, the actual interval in negative u or z in which the zeros remain on $|\mu| = 1$ depends on the lattice. We specialize to (i) uniform nearest-neighbor couplings $J_{nn'} = J\delta_{n\pm e_i, n'}$ and (ii) the thermodynamic limit (which, as usual, can be probed by using sufficiently large finite lattices).

We consider the square lattice first. We found previously [16] that for $\mu = -1$, the locus of points in a complex temperature (the u plane) across which f is non-analytic is the union of the unit circle and a finite line-segment,

$$\{u = e^{i\theta}\} \cup \{1/u_c \leq u \leq u_c\}, \quad (5)$$

⁴ We report results for real u here. We have also calculated zeros for complex u ; in this case, the zeros in the μ plane are not symmetric under complex conjugation. For example, for the square lattice, we find zeros which lie along (i) a spiral curve for $u = i$, and (ii) two linked, translated spiral-like curves for $u = e^{i\pi/4}$.

where $0 \leq \theta < 2\pi$ and the inner endpoint, u_e , of the line segment is given by

$$u_e = -(3 - 2^{3/2}) = -0.171573 \dots \quad (6)$$

Note that $u_e(\mu = -1) = -u_{c,sq}(\mu = 1)$, where $u_{c,sq}$ is the critical point separating the Z_2 -symmetric, paramagnetic (PM) phase and the broken-symmetry, ferromagnetic (FM) phase of the $H = 0$ model on the square lattice. Now we switch our view back from the non-analyticities of f in the u plane as a function of μ to the non-analyticities in the μ plane as a function of u . From our previous result (5), we know that as u moves leftward from the origin along the negative real u axis, there will be a point of non-analyticity when it reaches the value u_e , and this point will occur, in the μ plane, at $\mu = -1$.

To investigate the situation in the interval $u_e < u < 0$, we use two specific methods: (i) exact calculation of the partition function and numerical evaluation of the corresponding zeros in μ on finite lattices; and (ii) analysis of low-temperature, high-field series expansions. All of our finite-lattice calculations for $u_e < u < 0$ yield zeros in μ which, to within numerical accuracy ($\sim O(10^{-9})$) lie on the unit circle $|\mu| = 1$. In Fig. 1a we show a plot of zeros of Z in μ for an 8×8 lattice with $u = -0.1$. For this and the other figures, periodic boundary conditions are used. We have also made calculations with helical and open boundary conditions, and obtain the same conclusions.

As u decreases through the value u_e , the situation becomes more complicated. We find that the zeros no longer all lie on the circle $|\mu| = 1$. For u slightly more negative than u_e , we find that some zeros lie on the negative real μ axis (in a manner symmetric under $\mu \rightarrow 1/\mu$, as implied by Eq. (4)). This is illustrated by Fig. 1b, for $u = -0.25$. In the thermodynamic limit, these presumably merge to form a line segment which originates at the point $\mu = -1$ when $u = u_e$ and spreads outward from this point as u moves to larger negative values. Further structure may develop around $\mu = -1$ (see Fig. 1c for $u = -0.5$). As u becomes more negative, zeros appear elsewhere, e.g. on the arcs evident in Fig. 1c.

We have confirmed the locations of these singularities by analyzing low-temperature, high-field series expansions. With the definitions

$$f = \frac{1}{2}qK + h + f_r(u, \mu) \quad (7)$$

and $f_r = \lim_{N \rightarrow \infty} N^{-1} \ln Z_r$, these expansions can be expressed as a small- μ series,

$$f_r = \sum_{n=1}^{\infty} L_n(u) \mu^n, \quad (8)$$

or a small- u series,

$$f_r = \sum_{n=1}^{\infty} \psi_n(\mu) u^n, \quad (9)$$

where L and ψ are polynomials in u and μ , respectively. For lattices with odd q , similar series hold with u replaced by z . Comparisons of series analyses with exact results, e.g. for the 2D Ising model with $H = 0$ ($\mu = 1$) and $h = i\pi/2$ ($\mu = -1$; see Ref. [16]) show that series of reasonable lengths yield very accurate determinations of the positions of singular points. The series (8) has been computed to order μ^{15} for the square lattice [18] and to order μ^{12} for the triangular lattice [18,19], to be used below. The series (9) has been calculated to $O(u^{23})$ for the square lattice [20] and to $O(u^{21})$ for the triangular lattice [21]. For a given value of u , we use the small- μ series to compute the magnetization M and fit this to a leading singularity

$$M_{\text{sing}} \sim (1 - \mu/\mu_s)^{1/\delta_s}, \quad (10)$$

where μ_s denotes a generic singular point (depending on u). The symmetry (4) automatically implies that M then has the singularity

$$M_{\text{sing}} \sim [(1 - \mu/\mu_s)(1 - \mu_s\mu)]^{1/\delta_s}. \quad (11)$$

As in our earlier papers [12,16,17], we use dlog Padé and (first-order, unbiased) differential approximants for our study. Details of our methods are discussed in these papers. We have checked our results by analyzing the small- u series (9) to compute M , fitting it to a leading singularity of the form

$$M_{\text{sing}} \sim (1 - u/u_s)^\beta, \quad (12)$$

where u_s is a generic singular point (depending on μ). That is, we obtain a series of pairs of singular points $(\mu, u)_s$; these are singular points in the μ plane for a given u or, equivalently, singular points in the u plane for a given μ . The results for the singular point $u_s(\mu)$ are in excellent agreement with the values

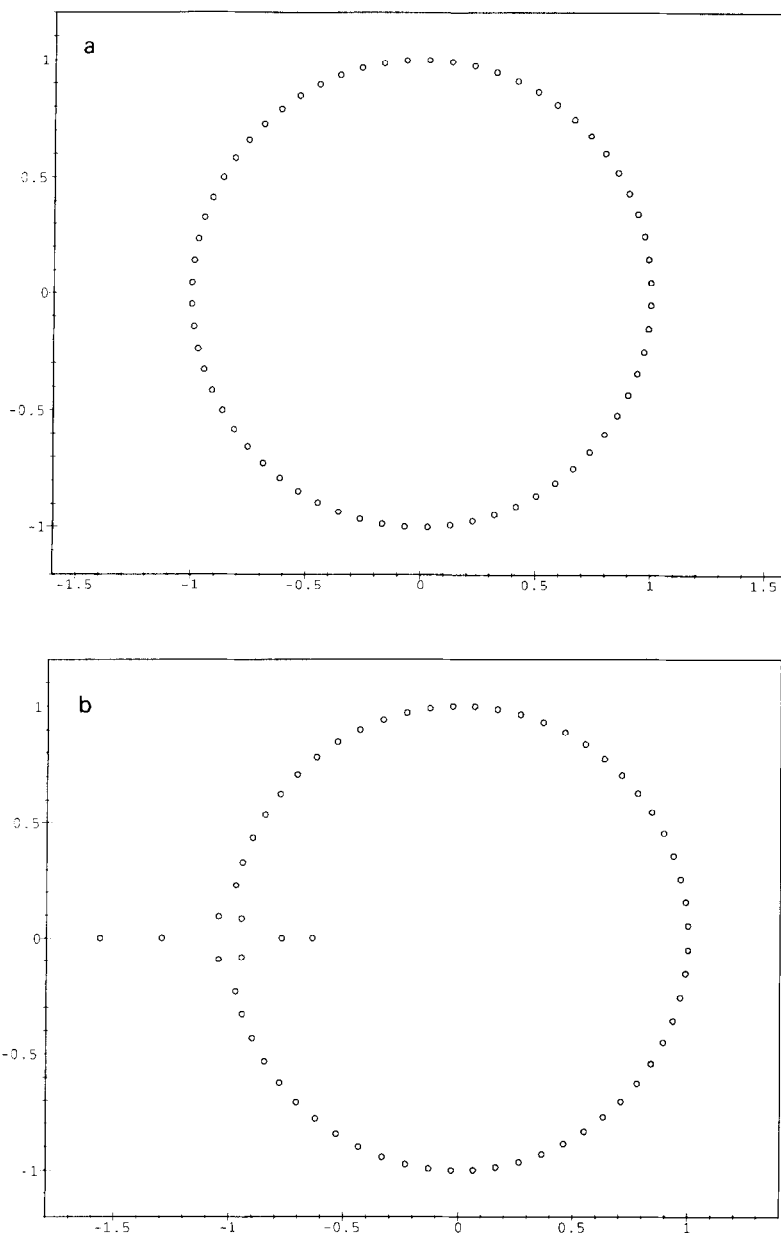


Fig. 1. Zeros of Z in the μ plane for the Ising model on a square lattice of size 8×8 for $u = -0.1$ (a), -0.25 (b), -0.5 (c).

of $\mu_s(u)$ obtained from the small- μ series. As we decrease u from 0 through real values, we first find a firm indication of a singularity when u passes through u_e ; this occurs at $\mu = -1$. This singularity moves inward toward the origin in the μ plane as u moves to the left of u_e . Some typical values are shown in

Table 1.

These results are in very good agreement with our findings from the calculation of zeros of Z , and suggest that the singular points $\mu_s(u)$ and $1/\mu_s(u)$ are the inner and outer endpoints of (what in the thermodynamic limit becomes a dense) line segment of zeros

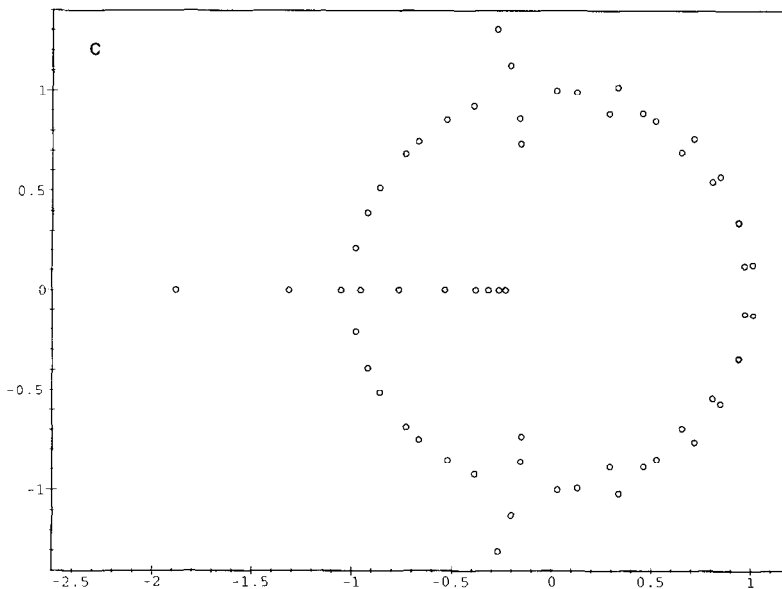


Fig. 1. (continued)

Table 1

Values of μ_s and $1/\delta_s$ for various u from our analysis of the small- μ series for M from (8). For $u = u_e$, the location of the singular point is known exactly as $\mu_s = -1$.

u	μ_s	$1/\delta_s$
-1/4	-0.615(10)	-0.20(2)
-1/3	-0.400(5)	-0.18(3)
-1/2	-0.219(2)	-0.19(1)
-2/3	-0.1407(2)	-0.20(1)
-1	-0.0735(1)	-0.20(1)
-3/2	-0.0372(5)	-0.20(1)

Table 2

Values of u_s and β_s for various μ from our analysis of the small- u series for M from (9). The entries on the last line are exact (u_e is given in Eq. (6)).

μ	u_s	β_s
-0.15	-0.6400(5)	-0.185(15)
-0.2	-0.5314(5)	-0.19(2)
-0.4	-0.3345(4)	-0.19(2)
-0.6	-0.2529(3)	-0.20(3)
-0.8	-0.2058(4)	-0.21(2)
-1	$u_e = -0.171573\dots$	-1/8

of Z (i.e., non-analyticities of f).

As Table 1 shows, we find that M has a divergent singularity at μ_s (and hence, by (4), also at $1/\mu_s$). This is an analogue in the μ plane of the general result in the u plane which we have found in our previous work on complex- T singularities [12,15,16] that M diverges, as a function of u , at the endpoints of arcs or line segments of singularities protruding into the FM phase. In particular, for $\mu = -1$, the exact result [4] for $M(u)$ exhibits divergences at $u = u_e$ and $u = 1/u_e$, the endpoints of the line segment in (5), with $\beta_e = -1/8$, while from the exact result [4] for f one can extract [16] the specific heat exponent $\alpha'_e = 1$. Our series analysis [16] strongly suggested the exact value $\gamma'_e = 5/4$ at this point for the susceptibility exponent (so that $\alpha'_e + 2\beta_e + \gamma'_e = 2$). Although we have found violations of scaling relations for complex- T singularities, we note that, provided the usual relations $\alpha' + \beta(\delta + 1) = 2$ and $\gamma' = \beta(\delta - 1)$ hold at $u = u_e$, it follows that $1/\delta_e = -1/9$ at $(\mu, u) = (-1, u_e)$. The fact that this differs from the value $1/\delta \sim -0.2$ for the (μ, u) entries in Table 1 is not surprising, because the point $\mu = -1$ is quite special, being related by a simple transformation [22,16] to the zero-field model, i.e. to the point $\mu = 1$. Thus, just as a small nonzero

value of the H fundamentally changes the properties of the zero field model (H is a relevant parameter), i.e., the singular properties change abruptly when μ moves slightly away from $\mu = 1$, so also, one expects that a small change in μ away from $\mu = -1$ will have a similarly abrupt effect on the singular properties.

As the results in Table 2 show, M is also divergent at the singular values of u_s corresponding to the μ values listed there. The corresponding exponent $\beta \sim -0.2$, and, again, it is not surprising that this differs from the exactly known value $\beta_c = -1/8$ at $u = u_c$, $\mu = -1$, for the same reason as given above.

For large negative u , we can observe that in the polynomial L_n at n th order in μ in (8), the dominant contribution is given by the term of highest power in u , $\propto u^{q/2}$. Now consider the limit $|u| \rightarrow \infty$, $\mu \rightarrow 0$ with

$$x \equiv u^{q/2} \mu \quad (13)$$

fixed. Then the double series (8) for f_r reduces to a series in the single variable x . We have analyzed this series to determine the singular point x_s . For a given large negative value of u , we can then extract the asymptotic value of the singularity μ_s . Clearly, as $u \rightarrow -\infty$, μ_s approaches the origin like

$$\mu_s \sim a u^{-q/2}, \quad (14)$$

where a is a constant (and $q = 4$ and 6 for the square and triangular lattices).

We also observe from our calculations of zeros that for small negative u (e.g. Fig. 1), the density $g(\theta)$ of zeros is consistent with being constant on the circle. Although $g(\theta, u)$ is not known exactly even in the Yang–Lee region $0 \leq u \leq 1$ or the subinterval of this region lying in the FM phase for the square lattice, viz. $0 \leq u \leq u_c$ (in which the zeros cover the entire circle), we have found by explicit calculation on finite lattices that for u in this latter interval, that as $u \rightarrow 0^+$, the density $g(\theta)$ is again consistent with being constant.

We have also calculated zeros for real $u < 1$. These fall in more complicated patterns. Here our emphasis is on determining the boundaries of the intervals in negative u where the zeros still lie on the unit circle $|\mu| = 1$ and establishing the nature of the singularities (which we show are line segments on the real μ axis)

which first appear when these zeros start moving off the circle.

Next, we carry out an analogous study for the triangular lattice of the size of the interval along the negative u axis for which the zeros stay exactly on the circle $|\mu| = 1$. We bring to bear our knowledge of the complex-temperature phase diagram: for $h = 0$ ($\mu = 1$), the continuous locus of points where f is non-analytic is the union of the circle and semi-infinite line [12]

$$\{u = -\frac{1}{3} + \frac{2}{3}e^{i\theta}\} \cup \{-\infty \leq u \leq -\frac{1}{3}\}, \quad (15)$$

where $0 \leq \theta < 2\pi$. Recall that the physical critical point is $u_{c,t} = 1/3$. For $\mu = -1$, the non-analytic points in the u plane consist of a circular arc and semi-infinite line segment [17]

$$\begin{aligned} \{u = \frac{1}{2}(-1 + e^{i\theta}); \theta_{cc} \leq |\theta| \leq \pi\} \\ \cup \{-\infty \leq u \leq -\frac{1}{2}\}, \end{aligned} \quad (16)$$

where $u_{cc} = e^{i\theta_{cc}} = \frac{1}{9}(-1 + 2^{3/2}i)$. From explicit calculations of zeros in μ and series analysis, we find that for $-1/3 \leq u < 0$, the zeros of Z continue to lie on the circle $|\mu| = 1$. (The positions are exactly on this circle to within the numerical accuracy of $\sim O(10^{-9})$.) Fig. 2a is a plot of zeros for $u = -0.1$ on a triangular lattice⁵. As u decreases through $-1/3$ moving to larger negative values, some zeros appear on the positive real μ axis, starting at $\mu = 1$ and spreading outward along the positive real axis from this point. Fig. 2b shows the zeros for $u = -0.4$. It is plausible that in the thermodynamic limit, these merge to form line segments. We have used the small- μ and small- u series to determine the position of the inner endpoint, $\mu_{e,rhs}$, of this line segment (and hence, by the $\mu \rightarrow 1/\mu$ symmetry, also the outer endpoint). We find results in agreement with the zeros calculated on the finite lattice. Here, lhs and rhs denote “left- and right-hand side”. As u reaches $-1/2$, our exact results in Ref. [17] show that there is a new set of singularities in the μ plane first appearing at $\mu = -1$. This singularity $\mu_{e,lhs}$, moves inward toward the origin along the negative real μ axis (and its inverse moves outward) as u decreases past $-1/2$

⁵ The triangular lattice can be represented by a square lattice with an interaction between, e.g., the spins on the lower-left and upper-right corners of each square. We use a 6×6 lattice of this type with periodic boundary conditions.

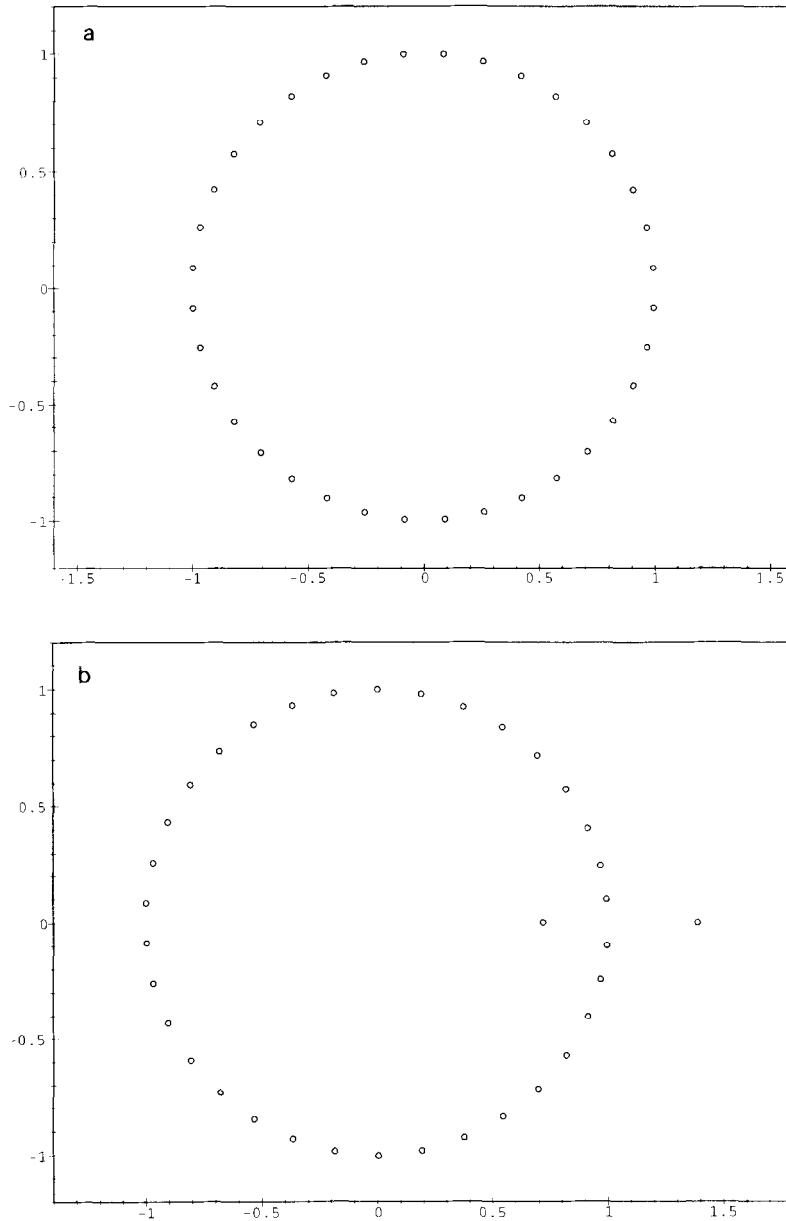


Fig. 2. Zeros of Z in the μ plane for the Ising model on a triangular lattice of size 6×6 for (a) $u = -0.1$, (b) $u = -0.4$.

toward larger negative values. We find that the small- μ series (8), to the order calculated, is not sensitive to $\mu_{e, \text{rhs}}$; however, the small- u series (9) does allow a rough determination of its value.

Finally, we prove a theorem concerning the reduced free energy for the honeycomb (hexagonal) lattice

and for two heteropolygonal lattices: the 3×12^2 and 4×8^2 lattices. (For notation, we refer the reader to our previous paper on heteropolygonal lattices, Ref. [13].) These lattices all have odd coordination number ($q = 3$).

Theorem. For lattices with odd coordination number q ,

$$Z_r(z, -\mu) = Z_r(-z, \mu). \quad (17)$$

This is proved as follows. Now $\mu \rightarrow -\mu$ corresponds to $h \rightarrow h + i\pi/2$. Using the identity $e^{i(\pi/2)\sigma_n} = i\sigma_n$, we have

$$Z(K, h + i\pi/2) = \sum_{\{\sigma\}} \left(\prod_{\langle nn' \rangle} e^{K\sigma_n\sigma_{n'}} \right) \left(\prod_n i\sigma_n e^{h\sigma_n} \right). \quad (18)$$

Since q is odd, we may replace $\sigma_n = \sigma_n^q$ on each site. We can then associate each factor in σ_n^q with one of the q bonds adjacent to this site. The product of σ_n^q over sites is thus re-expressed as a product of pairs $\sigma_n\sigma_{n'}$ over bonds. We next use the identity $e^{i(\pi/2)\sigma_n\sigma_{n'}} = i\sigma_n\sigma_{n'}$ to re-express this product over bonds in terms of an exponential with shifted coupling \tilde{K} ,

$$\begin{aligned} Z(K, h + i\pi/2) &= i^{(2-q)N/2} \sum_{\{\sigma\}} \exp \left(\tilde{K} \sum_{nn'} \sigma_n\sigma_{n'} + h \sum_n \sigma_n \right) \\ &= i^{(2-q)N/2} Z(\tilde{K}, h), \end{aligned} \quad (19)$$

where

$$\tilde{K} = K + \frac{1}{2}i\pi. \quad (20)$$

The shift (20) takes $z = e^{-2K}$ to $-z$; using this together with (3) then yields the theorem.

A consequence of this theorem is that for a lattice with odd q , the zeros of the partition function in the μ plane lie on the unit circle $|\mu| = 1$ not just for the Yang–Lee range $0 \leq z \leq 1$, but for the larger range

$$-1 \leq z \leq 1. \quad (21)$$

In the thermodynamic limit, for $z_c < z \leq 1$ or $-1 \leq z < -z_c$ (where $z_c = e^{-2K_c}$ is the respective critical point in the z variable for each such lattice), these zeros form a circular arc which does not completely enclose the origin in the μ plane, allowing an analytic continuation from the interior of the unit circle to its exterior. As z decreases through z_c or $-z$ increases through $-z_c$ along the real z axis, the arc closes to form a complete circle which prevents an analytic continuation from the interior to the exterior of the unit

circle $|\mu| = 1$. In passing, we recall that for the honeycomb lattice, $z_c = 2 - \sqrt{3}$ (see Ref. [13] for the complex-temperature phase diagrams and the z_c values for the 3×12^2 and 4×8^2 lattices).

This research was supported in part by the NSF grant PHY-93-09888.

References

- [1] L. Onsager, Phys. Rev. 65 (1944) 117.
- [2] C.N. Yang, Phys. Rev. 85 (1952) 808.
- [3] C.N. Yang and T.D. Lee, Phys. Rev. 87 (1952) 404.
- [4] T.D. Lee and C.N. Yang, Phys. Rev. 87 (1952) 410.
- [5] C.N. Yang, Selected papers 1945–1980 with commentary (Freeman, New York, 1983) pp. 14–16.
- [6] C.N. Yang, Special problems of statistical mechanics, Lectures at the University of Washington (1952) pp. 172–173. We thank Professor Yang for kindly lending a copy of these unpublished lectures to us.
- [7] M. Suzuki, Prog. Theor. Phys. 38 (1967) 1225; 40 (1968) 1246; 41 (1969) 1438; T. Asano, Prog. Theor. Phys. 40 (1968) 1328; Phys. Rev. Lett. 24 (1970) 1409; R.B. Griffiths, J. Math. Phys. 10 (1969) 1559; M. Suzuki, C. Kawabata, S. Ono, Y. Karaki and M. Ikeda, J. Phys. Soc. Japan 29 (1970) 837; M. Suzuki and M.E. Fisher, J. Math. Phys. 12 (1971) 235; P. Kortman and R.B. Griffiths, Phys. Rev. Lett. 27 (1971) 1439; S. Katsura and M. Ohnami, J. Phys. A 5 (1972) 95; G.A. Baker Jr. and J. Moussa, J. Appl. Phys. 49 (1978) 1360; M.E. Fisher, Phys. Rev. Lett. 40 (1978) 1610; D. Kurtze and M.E. Fisher, J. Stat. Phys. 19 (1978) 203; G.A. Baker Jr., M.E. Fisher and P. Moussa, Phys. Rev. Lett. 42 (1979) 615; E.H. Lieb and A. Sokal, Commun. Math. Phys. 80 (1981) 153; J. Cardy, Phys. Rev. Lett. 54 (1985) 1354.
- [8] S. Katsura, Y. Abe and M. Yamamoto, J. Phys. Soc. Japan 30 (1971) 347.
- [9] M.E. Fisher, Lectures in theoretical physics, Vol. 7C (Univ. of Colorado Press, 1965) p. 1.
- [10] S. Katsura, Prog. Theor. Phys. 38 (1967) 1415; S. Ono, Y. Karaki, M. Suzuki and C. Kawabata, J. Phys. Soc. Japan 25 (1968) 54; A.J. Guttmann, J. Phys. C 2 (1969) 1900; C. Domb and A.J. Guttmann, J. Phys. C 3 (1970) 1652; C. Itzykson, R. Pearson and J.B. Zuber, Nucl. Phys. B 220 (1983) 415; G. Marchesini and R. Shrock, Nucl. Phys. B 318 (1989) 541.
- [11] I.G. Enting, A.J. Guttmann and I. Jensen, J. Phys. A 27 (1994) 6987.

- [12] V. Matveev and R. Shrock, *J. Phys. A* 28 (1995) 1557; *J. Phys. A*, in press.
- [13] V. Matveev and R. Shrock, *J. Phys. A* 28 (1995) 5235.
- [14] C. Kawabata and M. Suzuki, *J. Phys. Soc. Japan* 27 (1969) 1105;
P.F. Fox and A.J. Guttmann, *J. Phys. C* 6 (1973) 913.
- [15] V. Matveev and R. Shrock, *J. Phys. A* 28 (1995) L533; *Phys. Lett. A* 204 (1995) 353.
- [16] V. Matveev and R. Shrock, *J. Phys. A* 28 (1995) 4859.
- [17] V. Matveev and R. Shrock, *Phys. Rev. E*, in press [cond-mat/9507120].
- [18] M.F. Sykes, J.W. Essam and D.S. Gaunt, *J. Math. Phys.* 6 (1965) 283;
M.F. Sykes, D.S. Gaunt, S.R. Mattingly, J.W. Essam and C.J. Elliott, *J. Math. Phys.* 14 (1973) 1066.
- [19] M.F. Sykes, S. McKenzie, M.G. Watts and D.S. Gaunt, *J. Phys. A* 8 (1975) 1461.
- [20] R.J. Baxter and I.G. Enting, *J. Stat. Phys.* 21 (1979) 103.
- [21] M.F. Sykes, D.S. Gaunt, J.L. Martin, S.R. Mattingly and J.W. Essam, *J. Math. Phys.* 14 (1973) 1071;
M.F. Sykes, M.G. Watts and D.S. Gaunt, *J. Phys. A* 8 (1975) 1448.
- [22] D. Merlini, *Lett. Nuovo Cim.* 9 (1974) 100;
K.Y. Lin and F.Y. Wu, *Int. J. Mod. Phys. B* 4 (1988) 471.