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Exponential Mixing in a Hydrodynamic Pilot–Wave Theory with Singular Potentials

Hung D. Nguyen[1] · **Anand U. Oza**[2]

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Abstract
We conduct an analysis of a stochastic hydrodynamic pilot-wave theory, which is a Langevin equation with a memory kernel that describes the dynamics of a walking droplet (or "walker") subjected to a repulsive singular potential and random perturbations through additive Gaussian noise. Under suitable assumptions on the singularities, we show that the walker dynamics is exponentially attracted toward the unique invariant probability measure. The proof relies on a combination of the Lyapunov technique and an asymptotic coupling specifically tailored to our setting. We also present examples of invariant measures, as obtained from numerical simulations of the walker in two-dimensional Coulomb potentials. Our results extend previous work on the ergodicity of stochastic pilot-wave dynamics established for smooth confining potentials.

Keywords Langevin equation with memory · Hydrdynamic pilot-wave theory · Exponential ergodicity · Singular potentials

Mathematics Subject Classification 60H10 · 34F05

1 Introduction

We are interested in the following stochastic integro-differential equation in $\mathbb{R}^d$:

$$\begin{aligned} \mathrm{d}x(t) &= v\,\mathrm{d}t, \\ m\,\mathrm{d}v(t) &= -v\,\mathrm{d}t - \nabla U(x(t))\,\mathrm{d}t - \nabla G(x(t))\,\mathrm{d}t \\ &\quad + \int_{-\infty}^{t} H(x(t)-x(s))K(t-s)\,\mathrm{d}s\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \end{aligned} \tag{1.1}$$

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✉ Hung D. Nguyen
hnguye53@utk.edu

[1] Department of Mathematics, University of Tennessee, Knoxville, Tennessee, USA

[2] Department of Mathematical Sciences & Center for Applied Mathematics and Statistics, New Jersey Institute of Technology, Newark, New Jersey, USA

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which has been used to model the dynamics of a millimetric droplet (or "walker") moving along the surface of a vibrating fluid bath of the same fluid (Bush and Oza 2021; Couder et al. 2005; Oza et al. 2013). In (1.1), the pair $(x(t), v(t)) \in \mathbb{R}^d \setminus \{0\} \times \mathbb{R}^d$ represents the displacement and velocity of the walker whose mass is denoted by the constant $m > 0$, $U : \mathbb{R}^d \to [0, \infty)$ is a smooth potential satisfying certain polynomial growth, $G : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is a singular potential, $H : \mathbb{R}^d \to \mathbb{R}^d$ is the pilot-wave force with $K : [0, \infty) \to [0, \infty)$ the corresponding memory kernel, and $W(t)$ is a standard d-dimensional Brownian motion with noise strength constant $\sigma > 0$.

1.1 Physical Background

Over the past two decades, a number of experiments have demonstrated that a millimetric oil droplet bouncing on the surface of a vertically vibrating fluid bath exhibits certain phenomena previously thought to be exclusive to the microscopic quantum realm (Bush et al. 2024; Bush and Oza 2021). These include quantized orbital radius (Blitstein et al. 2024; Fort et al. 2010; Harris and Bush 2014) and angular momentum (Durey and Milewski 2017; Perrard et al. 2014), wave-like statistics in cavities (Gilet 2016; Harris et al. 2013; Sáenz et al. 2018), tunneling (Eddi et al. 2009; Hubert et al. 2017; Nachbin et al. 2017; Tadrist et al. 2020), Friedel oscillations (Sáenz et al. 2020) and Anderson localization (Abraham et al. 2024), among others (Eddi et al. 2012; Frumkin and Bush 2023; Frumkin et al. 2023, 2022; Nachbin 2022; Papatryfonos et al. 2022; Valani et al. 2018). The droplet bounces on the surface of the bath and thus generates a guiding or 'pilot' wave field. This wave field imparts a propulsive horizontal force to the droplet, which causes it to move (or "walk") along the surface of the bath.

While the walker's dynamics may be chaotic, its statistics often displays wave-like patterns reminiscent of those exhibited by quantum particles. Specifically, experiments and numerical simulations have suggested that the walker's long-time statistical behavior exhibits a wave-like signature in a rotating frame (Harris and Bush 2014; Oza et al. 2014), harmonic potential (Durey and Milewski 2017; Kurianski et al. 2017; Perrard et al. 2014) and corral geometry (Cristea-Platon et al. 2018; Gilet 2016; Harris et al. 2013; Rahman 2018; Sáenz et al. 2018). There has thus been an interest in understanding the long-time behavior of this hydrodynamic pilot-wave system. To that end, prior studies established a link between the walker's position probability density and the time-averaged pilot-wave field (Durey et al. 2018; Tambasco and Bush 2018).

The discovery of this hydrodynamic pilot-wave system motivated the development of a number of theoretical models that describe the walker dynamics (Bush and Oza 2021; Turton et al. 2018). The so-called stroboscopic model is the focus of this paper, and it was developed by Oza et al. (2013) under the assumptions of no external force ($U = G \equiv 0$) and no stochastic forcing ($\sigma = 0$). The model neglects consideration of the walker's vertical dynamics and instead describes its horizontal position $x(t) \in \mathbb{R}^2$ and velocity $v(t) \in \mathbb{R}^2$ along the bath surface. Moreover, the timescale over which $x(t)$ evolves is assumed to be long relative to the vertical bouncing period. The walker is thus modeled as a moving source that continuously generates standing waves; these waves have a prescribed spatial profile $h(x)$ and decay in time due to the influence of viscosity according to the function $K(t)$ (Moláček and Bush 2013). The walker thus experiences a wave force proportional $\nabla h(x)K(t) = H(x)K(t)$, integrated over its entire past, resulting in the integral shown in Eq. (1.1). A key feature of the system is its "path memory" (Fort et al. 2010), as evidenced by the integral over the walker's past, with the near past having a larger influence than the far owing to the decay of K. The walker also experiences a drag force proportional to its velocity v.

The stroboscopic model (1.1) without stochastic forcing ($\sigma = 0$) and without a singular potential ($G \equiv 0$) has been studied numerically for linear (Valani 2022), quadratic (Budanur and Fleury 2019; Labousse et al. 2016, 2014; Perrard and Labousse 2018), quartic (Montes et al. 2021) or Bessel (Tambasco and Bush 2018) potentials $U(x)$. However, the case of singular potentials has received comparatively little attention: To the authors' knowledge, the only such example is by Tambasco et al. (2016), who used numerical simulations to demonstrate that, for a two-dimensional attractive Coulomb potential, $G(x) = \log(|x|)$, the dynamics undergoes a period-doubling transition to chaos as the pilot-wave force becomes progressively stronger. The Coulomb potential is of particular interest to physicists, as the description of the hydrogen atom as given by Schrödinger's equation is a canonical problem in quantum mechanics textbooks (Cohen-Tannoudji et al. 1977, Chapter VII).

We note that in the absence of the memory term, i.e., setting $K \equiv 0$, equation (1.1) is reduced to the classical Langevin dynamics

$$\begin{aligned} \mathrm{d}x(t) &= v\,\mathrm{d}t, \\ m\,\mathrm{d}v(t) &= -v\,\mathrm{d}t - \nabla U(x(t))\,\mathrm{d}t - \nabla G(x(t))\,\mathrm{d}t + \sigma \mathrm{d}W(t), \end{aligned} \tag{1.2}$$

whose large-time behavior is well understood (Conrad and Grothaus 2010; Cooke et al. 2017; Grothaus and Stilgenbauer 2015; Herzog and Mattingly 2019; Lu and Mattingly 2019). More specifically, under a broad class of polynomial potentials U and singular potentials G, the system (1.2) is exponentially attractive toward a unique invariant probability measure in the phase space $\mathcal{R}^d$ where

$$\mathcal{R}^d = \begin{cases} (0,\infty)\times\mathbb{R}, & d = 1, \\ \mathbb{R}^d \setminus \{0\} \times \mathbb{R}^d, & d \geq 2. \end{cases} \tag{1.3}$$

Here, the difference of the phase space $\mathcal{R}^d$ between dimension $d = 1$ and $d \geq 2$ is due to the fact that $\mathbb{R} \setminus \{0\}$ is not a connected domain. Notably, the ergodicity results established for (1.2) in Conrad and Grothaus (2010); Cooke et al. (2017); Grothaus and Stilgenbauer (2015); Herzog and Mattingly (2019); Lu and Mattingly (2019) cover a wide range of singularities, including the Lennard–Jones function

$$G(x) = \frac{c_1}{|x|^{12}} - \frac{c_2}{|x|^6}, \tag{1.4}$$

and the repulsive Coulomb function

$$G(x) = \begin{cases} -c \log |x|, & d = 2, \\ \frac{c}{|x|^{d-2}}, & d \geq 3. \end{cases} \tag{1.5}$$

Turning back to (1.1), while there is rich literature describing numerical simulations of (1.1), as mentioned above, comparatively less work has been done on proving analytical results about its statistical behavior. To the best of the authors' knowledge, results in this direction were first established in Nguyen and Oza (2024). In particular, for a variety of smooth potentials ($G \equiv 0$), it can be shown that (1.1) admits a unique invariant measure (Nguyen and Oza 2024). However, the argument therein does not cover the setting of singularities, nor does it imply an explicit convergence rate. The main goal of our present paper is to investigate the ergodicity of (1.1) under the impact of singular potentials and, more importantly, to prove the existence of an exponential mixing rate. In what follows, we will provide an overview of the main mathematical results.

1.2 Overview of the Main Results.

Due to the presence of the memory, it is expected that (1.1) is not Markovian. In the previous work of Bakhtin and Mattingly (2005); Itô and Nisio (1964); Nguyen and Oza (2024) without the singularities, this issue was tackled by considering the phase space of trajectories in the negative time interval $C((-\infty, 0]; \mathbb{R}^d \times \mathbb{R}^d)$. Roughly speaking, given an initial path $(x_0, v_0) \in C((-\infty, 0]; \mathbb{R}^d \times \mathbb{R}^d)$ and a finite time window $[0, t]$, we construct a solution evolving according to (1.1), resulting in a trajectory belonging to $C((-\infty, t]; \mathbb{R}^d \times \mathbb{R}^d)$. Then, the whole path is "shifted" back to time 0 so as to remain in $C((-\infty, 0]; \mathbb{R}^d \times \mathbb{R}^d)$. While this approach is useful for establishing strong solutions to and statistical properties of (1.1) for $G \equiv 0$, it is not suitable for proving ergodicity in the presence of singularities, nor for establishing a convergence rate to equilibrium. In a related work Duong and Nguyen (2024a) concerning the generalized Langevin equation, for the case of the memory kernel being a finite sum of exponentials, one can conveniently map the system therein to a finite-dimensional system of stochastic differential equations (SDEs). The approach, however, relies on the so-called fluctuation–dissipation relationship, which is not available in the setting of (1.1).

To circumvent these difficulties, in the present article, we draw upon the framework of Bonaccorsi et al. (2012); Glatt-Holtz et al. (2022); Nguyen (2023a) by augmenting the original process $(x(t), v(t))$ with a "history" variable $\eta(t)$. More precisely, we introduce $\eta(t) = \eta(t; s)$ given by

$$\eta(t; s) = x(t - s), \quad s, t \geq 0.$$

We note that $\eta(t; s)$ satisfies the transport equation

$$\partial_t \eta(t; s) = -\partial_s \eta(t; s), \quad \eta(t; 0) = x(t). \tag{1.6}$$

To see the role of $\eta(t;s)$ in (1.1), by making a simple change of variable $r = t - s$, the integral on the right-hand side of (1.1) is equivalent to

$$\begin{aligned}\int_{-\infty}^{t} H(x(t) - x(s))K(t-s)\,\mathrm{d}s &= \int_0^\infty H(x(t) - x(t-r))K(r)\,\mathrm{d}r\\ &= \int_0^\infty H(x(t) - \eta(t;r))K(r)\,\mathrm{d}r.\end{aligned}$$

So, we may recast the v-equation of (1.1) as

$$\begin{aligned} m\,\mathrm{d}v(t) &= -v\,\mathrm{d}t - \nabla U(x(t))\,\mathrm{d}t - \nabla G(x(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)\\ &\quad + \int_0^\infty H(x(t) - \eta(t;r))K(r)\,\mathrm{d}r\,\mathrm{d}t. \end{aligned} \tag{1.7}$$

This results in a jointly Markov process $(x(t), v(t), \eta(t))$ evolving in an extended phase space, which will be described later in Sect. 2.1. It is worth pointing out that the extended phase space approach in this work as well as in Bonaccorsi et al. (2012); Nguyen (2023a); Nguyen and Oza (2024) is valid thanks to the critical assumption that the memory kernel is well defined on $[0,\infty)$ and decays exponentially fast, cf., Assumption 2.1. While the latter assumption is restrictive from a mathematical standpoint, it is physically well motivated, as fluid viscosity is known to lead to exponential decay in time of the surface waves (Eddi et al. 2011; Moláček and Bush 2013). Moreover, the ergodicity results established in Theorem 1.1 can accommodate a rather wide class of nonlinear functions U, G and H.

We note that while augmenting by η is useful for analyzing ergodicity, we are mainly interested in the long-time statistical behavior of the pair $(x(t), v(t))$, i.e., the displacement and velocity of the walker. Our main result of the article concerning the convergence rate in terms of observables can be summarized as follows:

Theorem 1.1 *Under suitable assumptions on the functions U, G, H and K, and suitable initial data (x_0, v_0), let $(x(t), v(t))$ be the solution of (1.1). Then, there exists a unique probability measure ν on $\mathcal{R}^d$ such that for every suitable observable $f \in C(\mathcal{R}^d;\mathbb{R})$, the following holds:*

$$\Big|\mathbb{E}\big[f(x(t), v(t))\big] - \int_{\mathcal{R}^d} f(x,v)\nu(\mathrm{d}x,\mathrm{d}v)\Big| \le Ce^{-ct}, \quad t \ge 0, \tag{1.8}$$

for some positive constants C and c independent of t.

Here, $\mathbb{E}$ denotes the expectation with respect to the probability measure induced by $W(t)$. We refer the reader to Corollary 2.13 for a more precise version of Theorem 1.1. We note that Theorem 1.1 extends an ergodicity result in Nguyen and Oza (2024) established only for a polynomial potential U and pilot-wave force H. In particular, the result presented in Theorem 1.1 applies to a variety of singular potentials, e.g., the Riesz-type functions (i.e., $G \sim |x|^{-q}$ for $q > 0$ as $x \to 0$), of which the Lennard–Jones function (1.4) and the Coulomb function (1.5) in dimension $d \ge 3$ are members, as well as the log function $G(x) = -\alpha \log |x|$ for some constant $\alpha > 0$ sufficiently large. Also, thanks to the suitable energy estimates performed in Sect. 3, (1.8) is valid for a broad class of observables, including those having exponential growth. See Remark 2.7 and Remark 2.14 for a further discussion of these points.

Historically, SDEs with memory similar to (1.1) were studied as early as in the seminal work of Itô and Nisio (1964). Since then, the stationary solution approach therein has been employed in Bakhtin and Mattingly (2005); Mattingly and Sinai (2001); Liu (2002); Herzog et al. (2023); Nguyen and Oza (2024) to investigate unique ergodicity. On the other hand, the extended phase space approach as in (1.7) dates back to at least the work of Miller (1974) and appeared later in Bonaccorsi et al. (2012); Engel et al. (2000); Nguyen (2023a); Nguyen and Oza (2024). Appending η to (x, v) allows one to rewrite (1.2) as an appropriate Cauchy system while preserving the Markovian framework, which is very convenient for exploring mixing properties. However, the introduction of η unfortunately induces an infinite-dimensional system, which in comparison with (1.2) requires a more careful analysis so as to establish geometric ergodicity.

More specifically, for finite-dimensional SDEs such as (1.2), mixing results are typically a consequence of suitable Lyapunov functions. In other words, the returning time to the center of the phase space is shown to be exponentially fast. Together with the strong Feller property and an irreducibility condition, these three properties are sufficient to establish a convergence rate with respect to a weighted total variation norm in terms of the Lyapunov function (Khasminskii 2011). We remark that while verifying the latter two properties is usually straightforward, the Lyapunov construction itself is much more nontrivial (Duong and Nguyen 2024a; Herzog and Mattingly 2019; Lu and Mattingly 2019), owing to the presence of singularities.

On the other hand, concerning (1.1) and (1.7), since this is an infinite-dimensional system with degenerate noise, it typically precludes access to the strong Feller property, let alone measuring the convergence rate with respect to total variation norm. Nevertheless, in this work, we effectively overcome the challenge by resorting to the framework of *asymptotic coupling* developed in Butkovsky et al. (2020); Hairer et al. (2011) for stochastic PDEs. More precisely, instead of exploring total variation distance between probability measures, we opt for a more suitable Wasserstein metric defined in terms of a carefully chosen distance-like function ϱ, cf. Section 2.3. The convergence argument then relies on two crucial ingredients. The first is the *contracting* property of ϱ for the Markov semigroup associated with (x, v, η). The second is the ϱ-*small* property of bounded sets, to which the dynamics returns to infinitely often. (See Definition 4.5 part 2 . and part 3 ., respectively). The first can be achieved by establishing a so-called asymptotic strong Feller property, cf. Lemma 4.2, which can be regarded as a large-time smoothing effect of the Markov semigroup. In turn, this follows from a delicate estimate on the derivative of the solutions with respect to initial data. The second can be validated by proving an irreducibility condition, cf. Proposition 4.3. In order to do so, we compare the walker dynamics with the classical Langevin equation and show that they behave similarly on any finite time window. It is important to point out that in both auxiliary results, we have to deal with corresponding control problems to ensure legitimate changes of measures via Girsanov's theorem.

We refer the reader to Proposition 4.6 for the precise statements of the *contracting* and ϱ-*small* properties, while their detailed proofs will be supplied at the end of Sect. 4.

1.3 Organization of the Paper

The rest of the paper is organized as follows. In Sect. 2, we introduce the relevant function spaces (Sect. 2.1) and the assumptions (Sect. 2.2). We also state the main result of the paper, Theorem 2.12, which establishes the existence and uniqueness of the invariant probability measure ν for (1.1), as well as the exponential mixing rate toward ν. In Sect. 3, we collect useful moment bounds on the solutions. Theorem 2.12 is proved in Sect. 4. Numerical examples of the invariant measure for the case of a repulsive Coulomb potential in dimension $d = 2$ are provided in Sect. 5.

2 Main Results

2.1 Functional Setting

We start by discussing the functional settings which will be employed for the analysis. We use $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively, to denote the Euclidean norm and inner product in $\mathbb{R}^d$. For $q \in (1,\infty)$, we introduce the memory space $\mathcal{C}_q$ given by

$$\mathcal{C}_q = \Big\{\eta : [0,\infty) \to \mathbb{R}^d, \eta \text{ is measurable}, \|\eta\|_q \overset{\text{def}}{=} \Big|\int_0^\infty |\eta(r)|^q K(r)\mathrm{d}r\Big|^{1/q} < \infty\Big\}, \tag{2.1}$$

which is a Banach space (Rudin 1987, Theorem 3 .13) and whose dual is given by $\mathcal{C}_q^* = \mathcal{C}_p$, where $q^{-1} + p^{-1} = 1$ (Rudin 1987, Theorem 6 .16). Also, for $\varphi \in \mathcal{C}_q^* = \mathcal{C}_p$, we denote the action of φ on η by $\langle\varphi,\eta\rangle_q$. Since we may identify φ as an element in $\mathcal{C}_p$, the action $\langle\varphi,\eta\rangle_q$ may be understood as an integral, namely

$$\langle\varphi,\eta\rangle_q = \int_0^\infty \langle\varphi(s),\eta(s)\rangle K(s)\mathrm{d}s. \tag{2.2}$$

The product phase space for the dynamics (1.1) is denoted as

$$\mathcal{X}_q = \mathcal{R}^d \times \mathcal{C}_q, \tag{2.3}$$

where we recall $\mathcal{R}^d$ as in (1.3) is the phase space for the pair (x, v). We will also use $\|\cdot\|_{\mathcal{X}_q}$ to denote the norm in $\mathcal{X}_q$ given by

$$\|(x,v,\eta)\|_{\mathcal{X}_q} = |x| + |v| + \|\eta\|_q, \quad (x,v,\eta) \in \mathcal{X}_q. \tag{2.4}$$

The projections of (x, v, η) onto marginal spaces are denoted by π, namely

$$\pi_x(x,v,\eta) = x, \quad \pi_v(x,v,\eta) = v, \quad \pi_\eta(x,v,\eta) = \eta. \tag{2.5}$$

For any bounded linear map $f \in \mathcal{X}_q^* = L(\mathcal{X}_q; \mathbb{R})$, the action of f on (x, v, η) is denoted by $\langle f, (x, v, \eta)\rangle_{\mathcal{X}_q}$.

Following the framework of Bonaccorsi et al. (2012); Caraballo et al. (2007, 2008); Engel et al. (2000); Miller (1974); Nguyen and Oza (2024), we introduce the memory variable $\eta(t)$ given by

$$\eta(t; s) = x(t - s), \quad t, s \geq 0, \tag{2.6}$$

which captures the past information of $x(t)$. For every $\eta_0 \in \mathcal{C}_q$ and $x \in L^q([0, t]; \mathbb{R}^d)$, observe that formally $\eta(t)$ obeys the following transport equation (Bonaccorsi et al. 2012; Prüss 2013)

$$\partial_t \eta(t; \cdot) = -\partial_s \eta(t; \cdot), \quad \eta(t; 0) = x(t), \quad \eta(0) = \eta_0. \tag{2.7}$$

Since the parameters m and σ in (1.1) do not affect the analysis, we set $m = \sigma = 1$ for the sake of simplicity. From equation (2.7) as well as (1.7), we may recast (1.1) as follows:

$$\begin{aligned}
\mathrm{d}x(t) &= v(t)\mathrm{d}t, \\
\mathrm{d}v(t) &= -v(t)\mathrm{d}t - \nabla U(x(t))\mathrm{d}t - \nabla G(x(t))\mathrm{d}t \\
&\quad - \int_0^\infty H(x(t) - \eta(t; s))K(s)\mathrm{d}s\mathrm{d}t + \mathrm{d}W(t), \qquad (2.8)\\
\mathrm{d}\eta(t) &= -\partial_s \eta(t)\mathrm{d}t, \quad \eta(t; 0) = x(t).
\end{aligned}$$

2.2 Main Assumptions and Well-Posedness

In this subsection, we state the main assumptions that will be employed throughout the analysis. We start with the memory kernel K and impose the following condition:

Assumption 2.1 The memory kernel $K \in C^1([0, \infty); \mathbb{R}^+)$ satisfies

$$K'(t) \leq -\delta K(t), \quad t \geq 0, \tag{2.9}$$

for some positive constant $\delta > 0$.

Remark 2.2 1 . We note that the condition (2.9) is crucial to establish useful moment bounds on the solutions, cf., Lemma 3.1. In turn, the energy estimates will be invoked in Sect. 4 to prove the convergence rate toward equilibrium.

It is also worth mentioning that the differential inequality (2.9) implies that K decays exponentially fast, i.e.,

$$K(t) \leq K(0)e^{-\delta t}, \quad t \geq 0,$$

by virtue of Grönwall's inequality. However, (2.9) is slightly more general than the above estimate and will be directly employed to produce suitable bounds on the variable $\eta(t)$, for instance in Eq. (3.5).

2 . It is also important to point out that Assumption 2.1 does not cover singular memory kernels, e.g., $K(t) = \frac{e^{-\delta t}}{t^\alpha}$, $\alpha \in (0, 1)$. Recently, an alternative method was developed in the work of Liu and Caraballo (2024); Xu et al. (2022, 2024) to handle these types of irregular kernels. However, the approach therein seems to be only applicable to linear convolution, whereas the pilot-wave function H in equation (2.8) is a nonlinear function, cf. Assumption 2.3. It is therefore not immediately clear how to adapt the technique of Liu and Caraballo (2024); Xu et al. (2022, 2024) to our setting. We leave the issue of a singular memory kernel for future work.

With regard to the pilot-wave function H, we assume the following standard condition:

Assumption 2.3 $H \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ satisfies

$$\max\{|\nabla H(x)|, |H(x)|\} \leq a_H(|x|^{p_1} + 1), \quad x \in \mathbb{R}^d, \tag{2.10}$$

for some constants $a_H > 0$ and $p_1 \geq 0$.

We note that the condition we impose on H is quite general. In particular, it includes the function $H(x) = \mathrm{J}_1(|x|)x/|x|$, where J_1 is the Bessel function of the first kind of order one, which has been employed in models and simulations of the pilot-wave dynamics of walkers (Eddi et al. 2011; Moláček and Bush 2013; Oza et al. 2013; Turton et al. 2018).

Next, we state the main assumption on the potential U (Mattingly et al. 2002; Pavliotis 2014):

Assumption 2.4 1 . The potential $U \in C^\infty(\mathbb{R}^d; [1, \infty))$ satisfies, for all $x \in \mathbb{R}^d$,

$$\frac{1}{a_0}|x|^{q_0} - 1 \leq |U(x)| \leq a_0(|x|^{q_0} + 1), \tag{2.11}$$

$$|\nabla U(x)| \leq a_0(|x|^{q_0-1} + 1), \tag{2.12}$$

$$\text{and} \quad |\nabla^2 U(x)| \leq a_0(|x|^{q_0-2} + 1), \tag{2.13}$$

for some positive constants a_0 and q_0, where $\nabla^2 U$ denotes the Hessian of $U(x)$. Furthermore, there exists a positive constant ε_1 such that

$$q_0 \geq 2\max\{1, p_1 + \varepsilon_1\} > \max\{2p_1, p_1 + 1\}, \tag{2.14}$$

where p_1 is as in Assumption 2.3.

2 . There exist positive constants a_1, a_2 such that

$$\langle x, \nabla U(x)\rangle \geq a_1|x|^{q_0} - a_2, \quad x \in \mathbb{R}^d. \tag{2.15}$$

Remark 2.5 While most of the conditions in Assumption 2.4 are standard, the condition (2.14) states that the potential U must dominate the pilot-wave force induced by H. For example, when $H = \mathrm{J}_1$ which is a bounded function ($p_1 = 0$), then U is assumed to have at least quadratic growth. In case H is Lipschitz ($p_1 = 1$), U is required to grow faster than a quadratic function.

Concerning the singular potential G, we will make the following assumption (Duong and Nguyen 2024a, b).

Assumption 2.6 1 . $G \in C^\infty(\mathbb{R}^d \setminus \{0\}; \mathbb{R})$ satisfies $G(x) \to \infty$ as $|x| \to 0$. Furthermore, there exist constants $\beta_1 \geq 1$ and $a_3 > 0$ such that for all $x \in \mathbb{R}^d \setminus \{0\}$

$$|G(x)| \leq a_3\Big(1 + |x| + \frac{1}{|x|^{\beta_1}}\Big), \tag{2.16}$$

$$|\nabla G(x)| \leq a_3\Big(1 + \frac{1}{|x|^{\beta_1}}\Big), \tag{2.17}$$

$$\text{and} \quad |\nabla^2 G(x)| \leq a_3\Big(1 + \frac{1}{|x|^{\beta_1+1}}\Big). \tag{2.18}$$

2 . There exist nonnegative constants $\beta_2 \in [0, \beta_1)$, a_4 and a_5 such that

$$\Big|\nabla G(x) + a_4 \frac{x}{|x|^{\beta_1+1}}\Big| \leq \frac{a_5}{|x|^{\beta_2}} + a_5, \quad x \in \mathbb{R}^d \setminus \{0\}. \tag{2.19}$$

3 . There exists a positive constant a_6 such that

$$1 + \frac{e^{G(x)}}{|x|^{\beta_1}} \geq a_6 |\nabla^2 G(x)|^2, \quad x \in \mathbb{R}^d \setminus \{0\}. \tag{2.20}$$

Remark 2.7 Intuitively, part 1 in Assumption 2.6 controls the behavior of the singular potential G both as $x \to 0$ and $x \to \infty$, whereas part 2 implies that $\nabla G \sim -|x|^{-\beta_1}$ as $x \to 0$. Moreover, part 3 says that the singular potential must be sufficiently steep, a technical condition that allows us to ensure that $x(t)$ does not get too close to the origin.

We note that the condition (2.19) is standard and will be employed to construct Lyapunov functions in Lemma 3.1, part 1 . On the other hand, condition (2.20) is actually crucial and will be employed to produce moment bounds on the derivatives of the solution with respect to initial data, ensuring the asymptotic strong Feller property in Lemma 4.2. See (4.16) in particular. Altogether, they will be invoked to establish the main result on the convergence rate toward equilibrium. Also, while the condition (2.20) is valid for both the Lennard–Jones function (1.4) and the Coulomb function (1.5) in dimension $d \geq 3$, it is not necessarily valid for the Coulomb log function in dimension $d = 2$. In particular, given $G(x) = -\alpha \log |x|$ for some constant $\alpha > 0$, since $\beta_1 = 1$, (2.20) requires that $\alpha \geq 3$.

Turning to the auxiliary space $\mathcal{C}_q$ for the memory variable η, it is crucial to determine a suitable $\mathcal{C}_q$ in which $\eta(t)$ evolves. For this purpose, in view of conditions (2.10) and (2.14), we pick the parameter p_2 as follows:

Choice 2.8 *The constant p_2 satisfies*

$$2\max\{1, p_1 + \varepsilon_1\} > p_2 > \max\{2p_1, p_1 + 1\}, \tag{2.21}$$

where p_1 and ε_1 are as in (2.10) and (2.14), respectively.

Given a test function $\varphi \in C^1$ satisfying $\varphi, \varphi' \in \mathcal{C}^*_{p2}$, we interpret equation (2.7) for $\eta(t) \in \mathcal{C}_{p2}$ as a weak formulation in $\mathcal{C}^*_{p2}$. That is, we have formally

$$
\begin{aligned}
&\int_0^\infty \langle \eta(t;s), \varphi(s)\rangle K(s)\mathrm{d}s \\
&= \int_0^\infty \langle \eta(0;s), \varphi(s)\rangle K(s)\mathrm{d}s - \int_0^t \int_0^\infty \langle \partial_s \eta(r;s), \varphi(s)\rangle K(s)\mathrm{d}s\,\mathrm{d}r \\
&= \int_0^\infty \langle \eta(0;s), \varphi(s)\rangle K(s)\mathrm{d}s + K(0)\int_0^t \langle \eta(r;0), \varphi(0)\rangle \mathrm{d}r \\
&\quad + \int_0^t \int_0^\infty \langle \eta(r;s), \varphi'(s)\rangle K(s)\mathrm{d}s\,\mathrm{d}r \\
&\quad + \int_0^t \int_0^\infty \langle \eta(r;s), \varphi(s)\rangle K'(s)\mathrm{d}s\,\mathrm{d}r.
\end{aligned}
$$

In the last implication above, we employed an integration by parts. Invoking $\eta(r;0) = x(r)$, we deduce that

$$
\begin{aligned}
&\int_0^\infty \langle \eta(t;s), \varphi(t)\rangle K(s)\mathrm{d}s \\
&= \int_0^\infty \langle \eta(0;s), \varphi(t)\rangle K(s)\mathrm{d}s + K(0)\int_0^t \langle x(r), \varphi(0)\rangle \mathrm{d}r \\
&\quad + \int_0^t \int_0^\infty \langle \eta(r;s), \varphi'(s)\rangle K(s)\mathrm{d}s\,\mathrm{d}r \\
&\quad + \int_0^t \int_0^\infty \langle \eta(r;s), \varphi(r)\rangle K'(s)\mathrm{d}s\,\mathrm{d}r.
\end{aligned}
$$

Having performed the above heuristic calculation, we are now in a position to define a solution of (2.8), which is analytically weak and stochastically strong. For this purpose, we will fix a stochastic basis $\mathcal{S} = \big(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0}, W\big)$ satisfying the usual conditions (Karatzas and Shreve 2012).

Definition 2.9 Given p_2 as in Choice 2.8 and an initial condition $X_0 \equiv (x_0, v_0, \eta_0) \in \mathcal{X}_{p_2}$, a process $X(t) = (x(t), v(t), \eta(t))$ is called a solution of (2.8) if for all $T > 0$, $y_1, y_2 \in \mathbb{R}^d$ and test function φ satisfying $\varphi, \varphi' \in \mathcal{C}^*_{p_2}$, the following holds $\mathbb{P}$-a.s. for a.e. $t \in [0, T]$:

$$
\begin{aligned}
\langle x(t), y_1\rangle &= \langle x_0, y_1\rangle + \int_0^t \langle v(r), y_1\rangle \mathrm{d}r, \\
\langle v(t), y_2\rangle &= \langle v_0, y_2\rangle + \int_0^t \big\langle -v(r) - \nabla U(x(r)) - \nabla G(x(r)), y_2\big\rangle \mathrm{d}r + \int_0^t \langle y_2, \mathrm{d}W(r)\rangle, \\
&\quad - \int_0^t \int_0^\infty \langle H(x(r) - \eta(r;s)), y_2\rangle K(s)\mathrm{d}s\,\mathrm{d}r, \\
\langle \eta(t), \varphi\rangle_{p_2} &= \langle \eta(0), \varphi\rangle_{p_2} + K(0)\int_0^t \langle x(r), \varphi(0)\rangle \mathrm{d}r + \int_0^t \int_0^\infty \langle \eta(r;s), \varphi'(s)\rangle K(s)\mathrm{d}s
\end{aligned}
$$

$$+\int_0^t\int_0^\infty \langle \eta(r;s),\varphi(s)\rangle K'(s)\mathrm{d}s,$$

where $\langle\cdot,\cdot\rangle_{p_2}$ is defined in (2.2).

The solutions of (2.8) in the sense of Definition 2.9 can be established following the method in Nguyen and Oza (2024) while making use of the energy estimates in Lemma 3.1. Since the argument is quite standard, we refer the readers to Nguyen and Oza (2024) for the details and will henceforth assume the well-posedness of (2.8) for every initial condition in $\mathcal{X}_{p_2}$. As a consequence of the well-posedness of (2.8), we can thus introduce the Markov transition probabilities of the solution $X(t)$ by

$$P_t(X_0,A) := \mathbb{P}\big(X(t;X_0)\in A\big),$$

which are well defined for $t\ge 0$, initial condition $X_0\in\mathcal{X}_{p_2}$ and Borel sets $A\subset\mathcal{X}_{p_2}$. Letting $\mathcal{B}_b(\mathcal{X}_{p_2})$ denote the set of bounded Borel measurable functions $f:\mathcal{X}_{p_2}\to\mathbb{R}$, the associated Markov semigroup $P_t:\mathcal{B}_b(\mathcal{X}_{p_2})\to\mathcal{B}_b(\mathcal{X}_{p_2})$ is defined as

$$P_tf(X_0)=\mathbb{E}\big[f(X(t;X_0))\big],\ \ f\in\mathcal{B}_b(\mathcal{X}_{p_2}).$$

Letting $\mathcal{P}r(\mathcal{X}_{p_2})$ be the space of probability measures on $\mathcal{X}_{p_2}$, we denote by $P_t\nu$ the push-forward measure of $\nu\in\mathcal{P}r(\mathcal{X}_{p_2})$ under the action of P_t. That is,

$$P_t\nu(A)=\int_{\mathcal{X}_{p_2}}\mathbb{P}(X(t;X_0)\in A)\nu(\mathrm{d}X_0),$$

for all Borel sets $A\subset\mathcal{X}_{p_2}$. Lastly, following the framework of (Da Prato (2004), Sect. 3 .5), we denote by $\mathcal{L}$ the Kolmogorov operator associated with P_t defined for $g\in C^2(\mathcal{X}_{p_2})$ through the relation

$$P_tg(X_0)-g(X_0)=\mathbb{E}\int_0^t\mathcal{L}g(X(r))\mathrm{d}r.$$

Particularly, in view of Itô's formula (Da Prato and Zabczyk 2014, Theorem 3 .2), it holds that

$$\begin{aligned}\mathcal{L}g &:= \langle v,\partial_x g\rangle+\Big\langle -v-\nabla U(x)-\nabla G(x)+\int_0^\infty H(x-\eta(s))K(s)\,\mathrm{d}s,\partial_v g\Big\rangle\\ &\quad+\frac{1}{2}\triangle_v g+\int_0^\infty\langle -\partial_s\eta(s),\partial_\eta g\rangle K(s)\mathrm{d}s. \end{aligned}\tag{2.22}$$

We note that in the last term on the above right-hand side, $\partial_\eta g$ is an element in $\mathcal{C}^*_{p_2}=\mathcal{C}_{\frac{p_2}{p_2-1}}$, the dual space of $\mathcal{C}_{p_2}$.

2.3 Geometric Ergodicity

We now turn to the topic of the unique ergodicity and exponential mixing rate of (2.8). Recall that an element $\nu \in \mathcal{P}r(\mathcal{X}_{p_2})$ is said to be ***invariant*** for the semigroup P_t if

$$P_t \nu = \nu, \quad t \geq 0.$$

Alternatively, invariance is equivalent to the identity

$$\int_{\mathcal{X}_{p_2}} P_t f(X) \nu(\mathrm{d}X) = \int_{\mathcal{X}_{p_2}} f(X) \nu(\mathrm{d}X)$$

being true for every $f \in \mathcal{B}_b(\mathcal{X}_{p_2})$. As mentioned in the Introduction, to address the problem of ergodicity, we will draw upon the framework developed in Hairer and Mattingly (2006, 2008) and later popularized in Butkovsky et al. (2020); Hairer et al. (2011); Hairer and Mattingly (2011); Kulik and Scheutzow (2015); Kulik (2017), tailored to our setting. For the reader's convenience, we briefly review the theory below.

Recall that a function $\varrho : \mathcal{X}_{p_2} \times \mathcal{X}_{p_2} \to [0, \infty)$ is called *distance-like* if it is symmetric, lower semi-continuous and $\varrho(X, \tilde{X}) = 0$ if and only if $X = \tilde{X}$; see (Hairer et al. (2011), Definition 4 .3). Let $\mathcal{W}_\varrho$ be the Wasserstein distance in $\mathcal{P}r(\mathcal{X}_{p_2})$ associated with ϱ given by

$$\mathcal{W}_\varrho(\nu_1, \nu_2) = \inf \mathbb{E}\, \varrho(X, Y), \tag{2.23}$$

where the infimum is taken over all bivariate random variables (X, Y) such that $X \sim \nu_1$ and $Y \sim \nu_2$. If ϱ is a metric in $\mathcal{X}_{p_2}$, by the dual Kantorovich theorem, $\mathcal{W}_\varrho$ is equivalently defined as (Villani (2009), Theorem 5 .10)

$$\mathcal{W}_\varrho(\nu_1, \nu_2) = \sup_{[f]_{\mathrm{Lip},\varrho} \leq 1} \left| \int_{\mathcal{X}_{p_2}} f(X) \nu_1(\mathrm{d}X) - \int_{\mathcal{X}_{p_2}} f(X) \nu_2(\mathrm{d}X) \right|, \tag{2.24}$$

where

$$[f]_{\mathrm{Lip},\varrho} = \sup_{X \neq \tilde{X}} \frac{|f(X) - f(\tilde{X})|}{\varrho(X, \tilde{X})}. \tag{2.25}$$

On the other hand, if ϱ is not a metric but instead is a distance-like function, then the following one-sided inequality holds:

$$\mathcal{W}_\varrho(\nu_1, \nu_2) \geq \sup_{[f]_{\mathrm{Lip},\varrho} \leq 1} \left| \int_{\mathcal{X}_{p_2}} f(X) \nu_1(\mathrm{d}X) - \int_{\mathcal{X}_{p_2}} f(X) \nu_2(\mathrm{d}X) \right|. \tag{2.26}$$

See (Glatt-Holtz and Mondaini (2022), Proposition A.3) for a proof of (2.26).

The type of the Wasserstein distance $\mathcal{W}_\varrho$ we are mainly interested in will be described through suitable Lyapunov structures. For this purpose, we introduce the function

$$\Phi(x, v) = U(x) + G(x) + \frac{1}{2}|v|^2 + \kappa\langle x, v\rangle - \frac{\langle x, v\rangle}{|x|}, \tag{2.27}$$

where $\kappa > 0$ satisfies the following:

Choice 2.10 *The constant $\kappa > 0$ is sufficiently small such that*

$$c_\kappa\Big(U(x) + \frac{1}{2}|v|^2\Big) \le U(x) + \frac{1}{2}|v|^2 + \kappa\langle x, v\rangle - \frac{\langle x, v\rangle}{|x|} \le C_\kappa\Big(U(x) + \frac{1}{2}|v|^2\Big), \tag{2.28}$$

for some positive constants c_κ, C_κ independent of (x, v, η). Furthermore,

$$\kappa < \min\left\{\frac{1}{2(1 + 1/a_1)}, \frac{a_1}{2}\right\}, \tag{2.29}$$

where a_1 is the constant as in condition (2.15).

Remark 2.11 The role of the κ-term in (2.27) is to introduce dissipation due to the potential $U(x)$ for large $|x|$ (Mattingly et al. 2002), while the last term introduces dissipation due to the singular potential G near the origin (Herzog and Mattingly 2019; Lu and Mattingly 2019). The condition (2.29) will be employed to derive an exponential bound on the solution $(x(t), v(t), \eta(t))$, in the argument for ((3.2)) below.

To establish a convergence rate for the semigroup P_t associated with (2.8), let $\Psi : \mathcal{X}_{p_2} \to [0, \infty)$ be defined as

$$\Psi(x, v, \eta) = \Phi(x, v) + \|\eta\|_{p_2}^{p_2} = U(x) + G(x) + \frac{1}{2}|v|^2 + \kappa\langle x, v\rangle - \frac{\langle x, v\rangle}{|x|} + \|\eta\|_{p_2}^{p_2}, \tag{2.30}$$

and let $\varrho : \mathcal{X}_{p_2} \times \mathcal{X}_{p_2} \to [0, \infty)$ be the associated metric given by

$$\varrho(X, \tilde{X}) = \inf \int_0^1 e^{\frac{1}{2}\Psi(\gamma(s))}\|\gamma'(s)\|_{\mathcal{X}_{p_2}} \mathrm{d}s, \tag{2.31}$$

where the infimum is taken over all paths $\gamma \in C^1([0, 1]; \mathcal{X}_{p_2})$ such that $\gamma(0) = X$ and $\gamma(1) = \tilde{X}$. Following the framework of Butkovsky et al. (2020); Glatt-Holtz and Mondaini (2022); Hairer et al. (2011); Kulik (2017); Kulik and Scheutzow (2015); Nguyen (2023b), in our setting, for $N > 0$, we consider the distance

$$\varrho_N(X, \tilde{X}) = N\varrho(X, \tilde{X}) \wedge 1. \tag{2.32}$$

The actual convergence rate of (2.8) toward equilibrium is measured through the distance-like function $\tilde{\varrho}_N$ defined as

$$\tilde{\varrho}_N(X, \tilde{X}) = \sqrt{\varrho_N(X, \tilde{X})\big[1 + e^{\Psi(X)} + e^{\Psi(\tilde{X})}\big]}. \tag{2.33}$$

We now state the main result of the paper, which establishes the unique ergodicity and exponential mixing rate of (2.8) toward equilibrium:

Theorem 2.12 *The semigroup P_t admits a unique invariant probability measure ν. Furthermore, for all N sufficiently large*

$$\mathcal{W}_{\tilde{\varrho}_N}\big(P_t\nu_1, P_t\nu_2\big) \le Ce^{-ct}\mathcal{W}_{\tilde{\varrho}_N}\big(\nu_1, \nu_2\big), \quad t \ge 0,\ \nu_1, \nu_2 \in \mathcal{P}r(\mathcal{X}_{p_2}), \tag{2.34}$$

for some positive constants c and C independent of t, ν_1 and ν_2.

The proof of Theorem 2.12 consists of three important ingredients. The first is the Lyapunov function involving $e^{\Psi(X)}$ which is used to prove that the returning time is exponentially fast. The approach that we employ to construct the Lyapunov function is drawn from the technique in Lu and Mattingly (2019) tailored to our setting. The second is the contracting property of the semigroup P_t with respect to ϱ_N. In other words, if two initial data are close enough, then so are the two solutions for all time $t \ge 0$. The last property is the ϱ-smallness of bounded sets. Heuristically, this means that the distance between two solutions with initial data selected from a bounded set remains uniformly bounded for all time. The Lyapunov structure will be presented in Sect. 3, whereas the latter two properties will be supplied in Sect. 4.3. The proof of Theorem 2.12 will also be provided in Sect. 4.3.

As a consequence of Theorem 2.12 and the relation (2.26), we obtain the following convergence rate in terms of observables.

Corollary 2.13 *For all $X_0 \in \mathcal{X}_{p_2}$ and $f \in C^1(\mathcal{X}_{p_2}; \mathbb{R})$ satisfying $[f]_{\mathrm{Lip},\tilde{\varrho}_N} < \infty$, the following holds:*

$$\Big|P_t f(X_0) - \int_{\mathcal{X}_{p_2}} f(X)\nu(\mathrm{d}X)\Big| \le Ce^{-ct}, \quad t \ge 0, \tag{2.35}$$

for some positive constants c and C independent of t.

The proof of Corollary 2.13 will be given in Sect. 4.3.

Remark 2.14 Following (Glatt-Holtz and Mondaini (2022), Proposition A.9), a sufficient condition for $f \in C^1(\mathcal{X}_{p_2}; \mathbb{R})$ satisfying $[f]_{\mathrm{Lip},\tilde{\varrho}_N} < \infty$ is that

$$\sup_{X \in \mathcal{X}_{p_2}} \frac{\max\{f(X), \|Df(X)\|_{\mathcal{X}^*_{p_2}}\}}{\sqrt{1 + e^{\frac{1}{2}\Psi(X)}}} < \infty.$$

In the above, Df denotes the Fréchet derivative. In particular, it is not difficult to verify that the class of polynomials $f(X)$ satisfies the above condition.

3 Moment Estimates

In this section, we establish moment bounds on the solution of (2.8), which will be employed to prove the main results in Sect. 4. Throughout the rest of the paper, c and C denote generic positive constants. The main parameters that they depend on will appear between parentheses, e.g., $c(T, q)$ is a function of T and q.

Lemma 3.1 *Given $(x_0, v_0, \eta_0) \in \mathcal{X}_{p_2} = \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{C}_{p_2}$ where p_2 is as in Choice 2.8, let $(x(t), v(t), \eta(t))$ be the solution of (2.8). Then, the following holds:*

1 . There exist positive constants c_1 and C_1 such that

$$\begin{aligned}&\mathbb{E}\exp\left\{\Phi(x(t), v(t)) + \|\eta(t)\|_{p_2}^{p_2}\right\}\\&\quad\le C_1 e^{-c_1 t}\exp\left\{\Phi(x_0, v_0) + \|\eta_0\|_{p_2}^{p_2}\right\} + C_1, \quad (x_0, v_0, \eta_0) \in \mathcal{X}_{p_2},\ t \ge 0, \qquad (3.1)\end{aligned}$$

where Φ is the function defined in (2.27).

2 . For all α sufficiently large, it holds that

$$\int_0^\infty e^{-\alpha t}\,\mathbb{E}\left[\frac{e^{G(x(t))}}{|x(t)|^{\beta_1}}\right]\mathrm{d}t < C_\alpha \exp\left\{\Phi(x_0, v_0) + \|\eta_0\|_{p_2}^{p_2}\right\}, \quad (x_0, v_0, \eta_0) \in \mathcal{X}_{p_2} \qquad (3.2)$$

for some positive constant C_α independent of (x_0, v_0, η_0). In the above, β_1 is the constant from Assumption 2.6.

Proof 1 . Recalling the operator $\mathcal{L}$ from (2.22), we first apply $\mathcal{L}$ to $U(x) + G(x) + \frac{1}{2}|v|^2 + \kappa\langle x, v\rangle$ and obtain the identity

$$\begin{aligned}\mathcal{L}\Big(U(x) + G(x) + \frac{1}{2}|v|^2 + \kappa\langle x, v\rangle\Big) &= -(1-\kappa)|v|^2\\&\quad + \int_0^\infty \langle H(x - \eta(s)), v + \kappa x\rangle K(s)\mathrm{d}s + \frac{d}{2}\\&\quad - \kappa\langle x, \nabla U(x) + \nabla G(x)\rangle. \qquad (3.3)\end{aligned}$$

Since H satisfies condition (2.10), we employ the Cauchy–Schwarz inequality to estimate the integral on the above right-hand side as follows:

$$\begin{aligned}&\left|\int_0^\infty \langle H(x - \eta(s)), \kappa x + v\rangle K(s)\mathrm{d}s\right|\\&\quad\le c\big(|x|^{1+p_1} + |v|\,|x|^{p_1} + |x| + |v|\big)\|K\|_{L^1(\mathbb{R}^+)} + c\big(|x| + |v|\big)\int_0^\infty |\eta(s)|^{p_1} K(s)\mathrm{d}s.\end{aligned}$$

Letting ε be small and chosen later, by the ε-Young inequality and the condition (2.14) on q_0, we observe that

$$c\big(|x|^{1+p_1} + |v|\,|x|^{p_1} + |x| + |v|\big)\|K\|_{L^1(\mathbb{R}^+)} \le \frac{1}{4}\varepsilon|v|^2 + \frac{1}{8}\varepsilon|x|^{q_0} + C,$$

for some positive constant $C = C(\varepsilon)$ that might grow arbitrarily large as $\varepsilon \to 0$. Likewise,

$$\begin{aligned} c(|x|+|v|)\int_0^\infty |\eta(s)|^{p_1} K(s)\mathrm{d}s &\le \frac{1}{8}\varepsilon|x|^2 + \frac{1}{4}\varepsilon|v|^2 + \frac{1}{4}\varepsilon\int_0^\infty |\eta(s)|^{p_2}K(s)\mathrm{d}s + C \\ &\le \frac{1}{8}\varepsilon|x|^{q_0} + \frac{1}{4}\varepsilon|v|^2 + \frac{1}{4}\varepsilon\|\eta\|_{p_2}^{p_2} + C, \end{aligned}$$

where, again, $C = C(\varepsilon)$ may be arbitrarily large. Concerning the cross term $\langle x, \nabla U(x) + \nabla G(x)\rangle$ on the right-hand side of (3.3), we invoke conditions (2.15) and (2.17) to infer the bound

$$\begin{aligned} -\kappa\langle x, \nabla U(x) + \nabla G(x)\rangle &\le -\kappa a_1|x|^{q_0} + C|x| + C\frac{1}{|x|^{\beta_1 - 1}} + C \\ &\le -\Big(\kappa a_1 - \frac{1}{4}\varepsilon\Big)|x|^{q_0} + \frac{1}{4}\varepsilon\frac{1}{|x|^{\beta_1}} + C, \end{aligned}$$

where the last inequality follows from the fact that $q_0 \ge 2$, cf. condition (2.14), and $\beta_1 \ge 1$, cf. Assumption 2.6. Together with the identity (3.3), we deduce the estimate

$$\begin{aligned} &\mathcal{L}\Big(U(x) + G(x) + \frac{1}{2}|v|^2 + \kappa\langle x, v\rangle\Big) \\ &\quad \le -\Big(1 - \kappa - \frac{1}{2}\varepsilon\Big)|v|^2 - \Big(\kappa a_1 - \frac{1}{2}\varepsilon\Big)|x|^{q_0} + \frac{1}{4}\varepsilon\|\eta\|_{p_2}^{p_2} + \frac{1}{4}\varepsilon\frac{1}{|x|^{\beta_1}} + C. \end{aligned} \tag{3.4}$$

Next, considering $\|\eta\|_{p_2}^{p_2}$, note that

$$\partial_\eta\big(\|\eta\|_{p_2}^{p_2}\big) = p_2|\eta|^{p_2-2}\eta,$$

from which it follows that

$$\begin{aligned} \mathcal{L}\|\eta\|_{p_2}^{p_2} &= -\int_0^\infty \langle \partial_s\eta(s), \eta(s)\rangle \cdot p_2|\eta(s)|^{p_2-2}K(s)\mathrm{d}s \\ &= -\int_0^\infty \tfrac{\mathrm{d}}{\mathrm{d}s}\big(|\eta(s)|^{p_2}\big)K(s)\mathrm{d}s. \end{aligned}$$

Using integration by parts on the above right-hand side yields

$$\begin{aligned} \mathcal{L}\|\eta\|_{p_2}^{p_2} &= |\eta(0)|^{p_2}K(0) + \int_0^\infty |\eta(s)|^{p_2}K'(s)\mathrm{d}s \\ &\le |\eta(0)|^{p_2}K(0) - \delta\int_0^\infty |\eta(s)|^{p_2}K(s)\mathrm{d}s \\ &= |\eta(0)|^{p_2}K(0) - \delta\|\eta\|_{p_2}^{p_2} \\ &\le \frac{1}{4}\varepsilon|\eta(0)|^{q_0} - \delta\|\eta\|_{p_2}^{p_2} + C, \end{aligned} \tag{3.5}$$

where the first inequality follows from Assumption 2.1, namely $K'(s) \leq -\delta K(s)$.

Turning to the cross term $-\frac{\langle x, v\rangle}{|x|}$, applying $\mathcal{L}$ to $-\frac{\langle x, v\rangle}{|x|}$ gives

$$-\mathcal{L}\left(\frac{\langle x, v\rangle}{|x|}\right) = -\frac{|v|^2}{|x|} + \frac{|\langle x, v\rangle|^2}{|x|^3} + \frac{\langle x, v\rangle}{|x|} + \frac{\langle x, \nabla U(x) + \nabla G(x)\rangle}{|x|}$$
$$- \frac{1}{|x|}\int_0^\infty \langle H(x-\eta(s)), x\rangle K(s)\mathrm{d}s. \tag{3.6}$$

Using the Cauchy–Schwarz inequality, we observe that

$$-\frac{|v|^2}{|x|} + \frac{|\langle x, v\rangle|^2}{|x|^3} \leq 0,$$

and that

$$\begin{aligned}
&\frac{\langle x, v\rangle}{|x|} + \frac{\langle x, \nabla U(x)\rangle}{|x|} - \frac{1}{|x|}\int_0^\infty \langle H(x-\eta(s)), x\rangle K(s)\mathrm{d}s \\
&\leq |v| + |\nabla U(x)| + \int_0^\infty |H(x-\eta(s))| K(s)\mathrm{d}s \\
&\leq \frac{1}{2}\varepsilon|v|^2 + C|x|^{q_0-1} + C|x|^{p_1} + C\int_0^\infty |\eta(s)|^{p_1} K(s)\mathrm{d}s + C \\
&\leq \frac{1}{2}\varepsilon|v|^2 + \frac{1}{4}\varepsilon|x|^{q_0} + \frac{1}{4}\varepsilon\|\eta\|_{p_2}^{p_2} + C,
\end{aligned}$$

where, in the second to last implication above, we employed (2.12) and (2.10). Concerning the last term involving ∇G on the right-hand side of (3.6), we invoke (2.19) as follows:

$$\begin{aligned}
\frac{\langle x, \nabla G(x)\rangle}{|x|} &= -a_4\frac{1}{|x|^{\beta_1}} + \frac{1}{|x|}\left\langle x, \nabla G(x) + a_4\frac{x}{|x|^{\beta_1+1}}\right\rangle \\
&\leq -a_4\frac{1}{|x|^{\beta_1}} + a_5\frac{1}{|x|^{\beta_2}} + a_5 \\
&\leq -\left(a_4 - \frac{1}{4}\varepsilon\right)\frac{1}{|x|^{\beta_1}} + C,
\end{aligned}$$

where in the last implication, we employed the fact that $\beta_1 > \beta_2$. Together with the identity (3.6), we obtain

$$-\mathcal{L}\left(\frac{\langle x, v\rangle}{|x|}\right) \leq \frac{1}{2}\varepsilon|v|^2 + \frac{1}{4}\varepsilon|x|^{q_0} + \frac{1}{4}\varepsilon\|\eta(s)\|_{p_2}^{p_2} - \left(a_4 - \frac{1}{4}\varepsilon\right)\frac{1}{|x|^{\beta_1}} + C. \tag{3.7}$$

Now, we collect (3.4), (3.5) and (3.7) to arrive at the bound

$$\begin{aligned}\mathcal{L}\big[\Phi(x,v)+\|\eta\|_{p_2}^{p_2}\big]\leq&-(1-\kappa-\varepsilon)|v|^2\\&-\Big(\kappa a_1-\frac{3}{4}\varepsilon\Big)|x|^{q_0}-(\delta-\varepsilon)\|\eta\|_{p_2}^{p_2}-(a_4-\varepsilon)\frac{1}{|x|^{\beta_1}}\\&+\frac{1}{4}\varepsilon|\eta(0)|^{q_0}+C.\end{aligned}$$

As a consequence, we invoke Itô's formula while making use of the boundary condition $\eta(t;0)=x(t)$ to see that

$$\begin{aligned}\mathrm{d}\big[\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big]\leq&-(1-\kappa-\varepsilon)|v(t)|^2\mathrm{d}t\\&-(\kappa a_1-\varepsilon)|x(t)|^{q_0}\mathrm{d}t-(\delta-\varepsilon)\|\eta(t)\|_{p_2}^{p_2}\mathrm{d}t\\&-(a_4-\varepsilon)\frac{1}{|x(t)|^{\beta_1}}\mathrm{d}t+C\mathrm{d}t+\mathrm{d}M(t),\end{aligned}\tag{3.8}$$

where the semi-Martingale process $M(t)$ is defined as

$$M(t)=\int_0^t\Big\langle v(r)+\kappa x(r)+\frac{x(r)}{|x(r)|},\mathrm{d}W(r)\Big\rangle,\tag{3.9}$$

whose quadratic variation process is given by

$$\langle M\rangle(t)=\int_0^t\Big|v(r)+\kappa x(r)+\frac{x(r)}{|x(r)|}\Big|^2\mathrm{d}r.\tag{3.10}$$

Turning back to (3.1), we apply Itô's formula to $\exp\big\{\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}$ and obtain the identity

$$\begin{aligned}&\frac{\mathrm{d}\exp\big\{\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}}{\exp\big\{\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}}\\&\quad=\mathrm{d}\big[\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big]+\frac{1}{2}\langle\mathrm{d}\big[\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big],\mathrm{d}\big[\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big]\rangle\\&\quad=\mathrm{d}\big[\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big]+\frac{1}{2}\Big|v(t)+\kappa x(t)+\frac{x(t)}{|x(t)|}\Big|^2\mathrm{d}t,\end{aligned}$$

where the last equality follows from the expression (3.10). By the ε-Young inequality, we have

$$\begin{aligned}
\frac{1}{2}\Big|v+\kappa x+\frac{x}{|x|}\Big|^2 &\leq \frac{1+\frac{2\kappa}{a_1}}{2}|v|^2+\frac{1+\frac{a_1}{2\kappa}}{2}\Big|\kappa x+\frac{x}{|x|}\Big|^2\\
&\leq \frac{1+\frac{2\kappa}{a_1}}{2}|v|^2+\Big(1+\frac{a_1}{2\kappa}\Big)\kappa^2|x|^2+1+\frac{a_1}{2\kappa}\\
&\leq \frac{1+\frac{2\kappa}{a_1}}{2}|v|^2+\Big(1+\frac{a_1}{2\kappa}\Big)\kappa^2|x|^{q_0}+C,
\end{aligned}$$

for some positive constant C dependent on κ and a_1. We combine the above estimate with (3.8) and the choice of κ in (2.29) to deduce that

$$\begin{aligned}
&\frac{\mathrm{d}\exp\big\{\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}}{\exp\big\{\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}}\\
&\quad=\mathrm{d}\big[\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big]+\frac{1}{2}\Big|v(t)+\kappa x(t)+\frac{x(t)}{|x(t)|}\Big|^2\mathrm{d}t\\
&\quad\leq-\Big(\frac{1}{2}-\kappa-\varepsilon-\frac{\kappa}{a_1}\Big)|v(t)|^2\mathrm{d}t-\Big(\frac{\kappa a_1}{2}-\varepsilon-\kappa^2\Big)|x(t)|^{q_0}\mathrm{d}t\\
&\qquad-(a_4-\varepsilon)\frac{1}{|x(t)|^{\beta_1}}\mathrm{d}t-(\delta-\varepsilon)\|\eta(t)\|_{p_2}^{p_2}\mathrm{d}t+C\mathrm{d}t+\mathrm{d}M(t)\\
&\quad\leq-c\big[\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big]\mathrm{d}t+C\mathrm{d}t+\mathrm{d}M(t).
\end{aligned}\tag{3.11}$$

In particular, we also have

$$\mathcal{L}\exp\big\{\Phi(x,v)+\|\eta\|_{p_2}^{p_2}\big\}\leq\exp\big\{\Phi(x,v)+\|\eta\|_{p_2}^{p_2}\big\}\Big[-c\exp\big\{\Phi(x,v)+\|\eta\|_{p_2}^{p_2}\big\}+C\Big].$$

Since

$$\mathbb{E}\exp\big\{\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}-\mathbb{E}\exp\big\{\Phi(0)+\|\eta_0\|_{p_2}^{p_2}\big\}=\int_0^t\mathbb{E}\Big[\mathcal{L}\exp\big\{\Phi(r)+\|\eta(r)\|_{p_2}^{p_2}\big\}\Big]\mathrm{d}r,$$

we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\exp\big\{\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}\\
&\quad\leq\mathbb{E}\Big[\mathcal{L}\exp\big\{\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}\Big]\\
&\quad\leq\mathbb{E}\Big[\exp\big\{\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}\cdot\big(-c\big[\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big]+C\big)\Big].
\end{aligned}$$

We note that the following holds:

$$e^r(-cr+C)\leq-\tilde{c}e^r+\tilde{C},\quad r\geq0,$$

for some positive constants $\tilde{c}$ and $\tilde{C}$ that may depend on c and C. In turn, this implies that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\exp\big\{\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}\le -c\mathbb{E}\exp\big\{\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}+C.$$

By Grönwall's inequality, the above immediately produces the estimate (3.1), which completes the proof of part 1 .

2 . With regard to (3.2), letting $\alpha\ge 0$ be given and chosen later, we apply Itô's formula to $\exp\big\{-\alpha t+\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}$ and obtain the identity

$$\begin{aligned}&\frac{\mathrm{d}\exp\big\{-\alpha t+\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}}{\exp\big\{-\alpha t+\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}}\\ &\quad=-\alpha\mathrm{d}t+\mathrm{d}\big[\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big]+\frac{1}{2}\langle \mathrm{d}\big[\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big],\mathrm{d}\big[\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big]\rangle\\ &\quad=-\alpha\mathrm{d}t+\mathrm{d}\big[\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big]+\frac{1}{2}\Big|v(t)+\kappa x(t)+\frac{x(t)}{|x(t)|}\Big|^2\mathrm{d}t,\end{aligned}$$

where the last equality follows from the expression (3.10). From (2.28) and (3.11), we have

$$\begin{aligned}&\frac{\mathrm{d}\exp\big\{-\alpha t+\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}}{\exp\big\{-\alpha t+\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}}\\ &\quad=-\alpha\mathrm{d}t+\mathrm{d}\big[\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big]+\frac{1}{2}\Big|v(t)+\kappa x(t)+\frac{x(t)}{|x(t)|}\Big|^2\mathrm{d}t\\ &\quad\le -\alpha\mathrm{d}t+C\mathrm{d}t-(a_4-\varepsilon)\frac{1}{|x(t)|^{\beta_1}}\mathrm{d}t+\mathrm{d}M(t).\end{aligned}$$

As a consequence, we may take α large enough to infer

$$\frac{\mathrm{d}\exp\big\{-\alpha t+\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}}{\exp\big\{-\alpha t+\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}}\le -(a_4-\varepsilon)\frac{1}{|x(t)|^{\beta_1}}\mathrm{d}t+\mathrm{d}M(t).$$

In particular, we obtain the bound in expectation

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\exp\big\{-\alpha t+\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}\\ &\quad\le -(a_4-\varepsilon)\mathbb{E}\Big[\frac{\exp\big\{-\alpha t+\Phi(t)+\|\eta(t)\|_{p_2}^{p_2}\big\}}{|x(t)|^{\beta_1}}\Big].\end{aligned}\tag{3.12}$$

By integrating (3.12) with respect to time, it follows immediately that

$$\int_0^t(a_4-\varepsilon)\mathbb{E}\Big[\frac{\exp\big\{-\alpha r+\Phi(r)+\|\eta(r)\|_{p_2}^{p_2}\big\}}{|x(r)|^{\beta_1}}\Big]\mathrm{d}r\le \exp\big\{\Phi(x_0,v_0)+\|\eta_0\|_{p_2}^{p_2}\big\}.$$

Since Φ as in (2.27) satisfies $\Phi(x,v)\ge G(x)$, we arrive at the estimate

$$\int_0^t (a_4-\varepsilon)\mathbb{E}\left[\frac{\exp\big\{-\alpha r+G(x(r))\big\}}{|x(r)|^{\beta_1}}\right]\mathrm{d}r\le \exp\big\{\Phi(x_0,v_0)+\|\eta_0\|_{p_2}^{p_2}\big\}.$$

Sending t to infinity, this establishes (3.2) by virtue of the monotone convergence theorem. The proof is thus finished. □

4 Geometric Ergodicity

In this section, we establish the main results of the paper by proving the existence and uniqueness of the invariant probability measure ν, as well as establishing the exponential mixing rate toward ν. As mentioned in the Introduction, we draw upon the framework of Butkovsky et al. (2020); Hairer et al. (2011) tailored to our setting. In Sect. 4.1, we establish the asymptotic strong Feller property, which will be employed to prove the contracting property of the semigroup P_t. In Sect. 4.2, we derive the irreducibility property, which is a control problem proving that the solutions can always be driven back to the center of the phase space. In turn, this will be invoked to deduce the smallness property of any bounded sets in $\mathcal{X}_{p_2}$. In Sect. 4.3, we will establish Theorem 2.12 by using the auxiliary results collected in Sect. 3, Sect. 4.1 and Sect. 4.2.

4.1 Asymptotic Strong Feller Property

For any unit element $\xi\in\mathcal{X}_{p_2}$, we denote by $J_{0,t}\xi$ the derivative of the solution $(x(t),v(t),\eta(t))$ with respect to the initial condition ξ. Recalling the projection π defined in (2.5), observe that $J_{0,t}\xi=(\pi_x J_{0,t}\xi,\pi_v J_{0,t}\xi,\pi_\eta J_{0,t}\xi)$ satisfies the following equation with random coefficients:

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\pi_x J_{0,t}\xi &= \pi_v J_{0,t}\xi,\\
\tfrac{\mathrm{d}}{\mathrm{d}t}\pi_v J_{0,t}\xi &= -\pi_v J_{0,t}\xi-\nabla^2 U(x(t))\pi_x J_{0,t}\xi-\nabla^2 G(x(t))\pi_x J_{0,t}\xi\\
&\qquad -\int_0^\infty \nabla H(x(t)-\eta(t;s))(\pi_x J_{0,t}\xi-\pi_\eta J_{0,t}\xi(s))K(s)\mathrm{d}s, \qquad (4.1)\\
\tfrac{\mathrm{d}}{\mathrm{d}t}\pi_\eta J_{0,t}\xi &= -\partial_s\pi_\eta J_{0,t}\xi,\quad \pi_\eta J_{0,t}\xi(0)=\pi_x J_{0,t}\xi,\ t>0,
\end{aligned}$$

with the initial condition

$$J_{0,0}\xi=\xi.$$

Next, for any adapted process $\zeta\in L^2(0,t;\mathbb{R}^d)$, we introduce the Malliavin derivative $A_{0,t}\zeta$ of $(x(t),v(t),\eta(t))$ with respect to W along the path ζ. Observe that $A_{0,t}\zeta$

obeys the equation

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\pi_x A_{0,t}\zeta = \pi_v A_{0,t}\zeta,\\
&\tfrac{\mathrm{d}}{\mathrm{d}t}\pi_v A_{0,t}\zeta = -\pi_v A_{0,t}\zeta - \nabla^2 U(x(t))\pi_x A_{0,t}\zeta - \nabla^2 G(x(t))\pi_x A_{0,t}\zeta + \zeta(t)\\
&\qquad\qquad - \int_0^\infty \nabla H(x(t)-\eta(t;s))(\pi_x A_{0,t}\zeta - \pi_\eta A_{0,t}\zeta(s))K(s)\mathrm{d}s,
\end{aligned} \tag{4.2}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\pi_\eta A_{0,t}\zeta = -\partial_s \pi_\eta A_{0,t}\zeta, \quad \pi_\eta A_{0,t}\zeta(0) = \pi_x A_{0,t}\zeta,\ t>0,$$

with the initial condition

$$A_{0,0}\zeta = 0.$$

Remark 4.1 We recall that when ζ is an adapted process, the Malliavin derivative $A_{0,t}\zeta$ satisfies the identity

$$\mathbb{E}\langle Df(x(t),v(t),\eta(t)), A_{0,t}\zeta\rangle_{\mathcal{X}_{p_2}} = \mathbb{E}\left[f(x(t),v(t),\eta(t))\int_0^t \langle \zeta(r), \mathrm{d}W(r)\rangle\right], \tag{4.3}$$

for every $f \in C_b^1(\mathcal{X}_{p_2})$. We note that the above equation is the so-called Malliavin integration by parts where the stochastic integral on the right-hand side is understood in the usual Itô sense. In case ζ is non-adapted, then one interprets this as a Skorokhod integral. See (Hairer and Mattingly (2011), Eq. (3 .2)) and (Nualart (2013), Chapter 1 .3) for further discussions of this point. Later in the proof of Lemma 4.2, we will carefully pick ζ to prove that P_t satisfies the asymptotic Feller property.

Now, denoting

$$\rho_t = J_{0,t}\xi - A_{0,t}\zeta, \tag{4.4}$$

we observe that ρ_t solves

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\pi_x \rho_t = \pi_v \rho_t, \qquad\qquad \rho_0 = \xi,\\
&\tfrac{\mathrm{d}}{\mathrm{d}t}\pi_v \rho_t = -\pi_v \rho_t - \nabla^2 U(x(t))\pi_x \rho_t - \nabla^2 G(x(t))\pi_x \rho_t - \zeta(t)\\
&\qquad\qquad - \int_0^\infty \nabla H(x(t)-\eta(t;s))(\pi_x \rho_t - \pi_\eta \rho_t(s))K(s)\mathrm{d}s,\\
&\tfrac{\mathrm{d}}{\mathrm{d}t}\pi_\eta \rho_t = -\partial_s \pi_\eta \rho_t, \quad \pi_\eta \rho_t(0) = \pi_x \rho_t,\ t>0.
\end{aligned} \tag{4.5}$$

Having introduced the needed processes, we now state and prove the required asymptotic strong Feller property in Lemma 4.2.

Lemma 4.2 *There exist positive constants c and C such that for all $(x, v, \eta) \in \mathcal{X}_{p_2}$ and $f \in C_b^1(\mathcal{X}_{p_2})$, $C_b^1(\mathcal{X}_{p_2})$ being the space of bounded differentiable functions on $\mathcal{X}_{p_2}$, the following holds:*

$$\|DP_t f(x, v, \eta)\|_{\mathcal{X}^*_{p_2}} \leq Ce^{-ct}\sqrt{P_t\|Df\|^2_{\mathcal{X}^*_{p_2}}(x, v, \eta)} + Ce^{\frac{1}{2}\Psi(x,v,\eta)}\sqrt{P_t|f|^2(x, v, \eta)}, \quad t \geq 0, \tag{4.6}$$

where $\Psi(x, v, \eta)$ is as in (2.30).

Proof Let $(x(t), v(t), \eta(t))$ be the solution of (2.8) with initial condition $(x, v, \eta) \in \mathcal{X}_{p_2}$. For $f \in C_b^1(\mathcal{X}_{p_2})$, a unit element $\xi \in \mathcal{X}_{p_2}$ and a path $\zeta \in L^2(0, t; \mathbb{R}^d)$, from the relation (4.4), we have the following chain of implications:

$$\begin{aligned}\langle DP_t f(x, v, \eta), \xi\rangle_{\mathcal{X}_{p_2}} &= \mathbb{E}\langle Df(x(t), v(t), \eta(t)), J_{0,t}\xi\rangle_{\mathcal{X}_{p_2}} \\ &= \mathbb{E}\langle Df(x(t), v(t), \eta(t)), \rho_t\rangle_{\mathcal{X}_{p_2}} \\ &\quad + \mathbb{E}\langle Df(x(t), v(t), \eta(t)), A_{0,t}\zeta\rangle_{\mathcal{X}_{p_2}} \\ &= \mathbb{E}\langle Df(x(t), v(t), \eta(t)), \rho_t\rangle_{\mathcal{X}_{p_2}} \\ &\quad + \mathbb{E}\left[f(x(t), v(t), \eta(t))\int_0^t\langle\zeta(r), \mathrm{d}W(r)\rangle\right],\end{aligned}$$

where the last implication follows from identity (4.3). In order to establish (4.6), it suffices to determine a suitable path ζ such that

$$\mathbb{E}\langle Df(x(t), v(t), \eta(t)), \rho_t\rangle_{\mathcal{X}_{p_2}} \leq Ce^{-ct}\sqrt{P_t\|Df\|^2_{\mathcal{X}^*_{p_2}}(x, v, \eta)}, \tag{4.7}$$

and that

$$\mathbb{E}\left[f(x(t), v(t), \eta(t))\int_0^t\langle\zeta(r), \mathrm{d}W(r)\rangle\right] \leq Ce^{\frac{1}{2}[\Phi(x,v)+\|\eta\|^{p_2}_{p_2}]}\sqrt{P_t|f|^2(x, v, \eta)}, \tag{4.8}$$

for some positive constants C, c independent of (x, v, η), t, f and ξ. To this end, we let $\alpha > 0$ be an arbitrary positive constant. In view of (4.5), we choose the control path ζ as follows:

$$\begin{aligned}\zeta(t) = &-\pi_v\rho_t + 5\alpha\pi_v\rho_t + 6\alpha^2\pi_x\rho_t - \nabla^2U(x(t))\pi_x\rho_t - \nabla^2G(x(t))\pi_x\rho_t \\ &- \int_0^\infty \nabla H(x(t) - \eta(t; s))(\pi_x\rho_t - \pi_\eta\rho_t(s))K(s)\mathrm{d}s.\end{aligned} \tag{4.9}$$

With this choice of ζ, observe that (4.5) is reduced to

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\pi_x\rho_t = \pi_v\rho_t, \qquad\qquad \rho_0 = \xi,\\
&\tfrac{\mathrm{d}}{\mathrm{d}t}\pi_v\rho_t = -5\alpha\pi_v\rho_t - 6\alpha^2\pi_x\rho_t,\\
&\tfrac{\mathrm{d}}{\mathrm{d}t}\pi_\eta\rho_t = -\partial_s\pi_\eta\rho_t, \quad \pi_\eta\rho_t(0) = \pi_x\rho_t,\ t > 0.
\end{aligned} \tag{4.10}$$

On the one hand, by a routine calculation, the first two equations of (4.10) yield the solution

$$\begin{pmatrix}\pi_x\rho_t\\ \pi_v\rho_t\end{pmatrix} = \left(3\pi_x\xi + \frac{1}{\alpha}\pi_v\xi\right)e^{-2\alpha t}\begin{pmatrix}1\\ -2\alpha\end{pmatrix} - \left(2\pi_x\xi + \frac{1}{\alpha}\pi_v\xi\right)e^{-3\alpha t}\begin{pmatrix}1\\ -3\alpha\end{pmatrix}.$$

In particular, this implies the decay rate (recalling ξ is a unit element in $\mathcal{X}_{p_2}$)

$$|\pi_x\rho_t| + |\pi_v\rho_t| \le Ce^{-2\alpha t}\big(|\pi_x\xi| + |\pi_v\xi|\big) \le C\,e^{-2\alpha t}, \quad t \ge 0, \tag{4.11}$$

for some positive constant C independent of t and ξ. On the other hand, concerning the last equation of (4.10), we employ an argument similar to (3.5) and obtain

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|\pi_\eta\rho_t\|_{p_2}^{p_2} \le |\pi_x\rho_t|^{p_2}K(0) - \delta\|\pi_\eta\rho_t\|_{p_2}^{p_2},$$

whence

$$\|\pi_\eta\rho_t\|_{p_2}^{p_2} \le e^{-\delta t}\|\pi_\eta\xi\|_{p_2}^{p_2} + K(0)\int_0^t e^{-\delta(t-r)}|\pi_x\rho_r|^{p_2}\mathrm{d}r.$$

Together with (4.11), we infer the existence of positive constants C and c such that

$$\|\pi_\eta\rho_t\|_{p_2}^{p_2} \le Ce^{-ct}\big(|\pi_x\xi| + |\pi_v\xi| + \|\pi_\eta\xi\|_{p_2}\big)^{p_2} = Ce^{-ct}. \tag{4.12}$$

Turning back to (4.7), by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\mathbb{E}\langle Df(x(t), v(t), \eta(t)), \rho_t\rangle_{\mathcal{X}_{p_2}} &\le \sqrt{\mathbb{E}\|Df(x(t), v(t), \eta(t))\|^2_{\mathcal{X}^*_{p_2}}}\cdot\sqrt{\mathbb{E}\|\rho_t\|^2_{\mathcal{X}_{p_2}}}\\
&\le \sqrt{P_t\|Df\|^2_{\mathcal{X}^*_{p_2}}(x, v, \eta)}\cdot Ce^{-ct},
\end{aligned}$$

where, in the last implication, we invoked (4.11) and (4.12). This establishes (4.7).

With regard to (4.8), we employ the Cauchy–Schwarz inequality with Itô's isometry to see that

$$\begin{aligned}
&\mathbb{E}\left[f(x(t), v(t), \eta(t))\int_0^t\langle\zeta(r), \mathrm{d}W(r)\rangle\right]\\
&\quad\le \sqrt{\mathbb{E}|f(x(t), v(t), \eta(t))|^2}\cdot\sqrt{\mathbb{E}\int_0^\infty|\zeta(r)|^2\mathrm{d}r}.
\end{aligned} \tag{4.13}$$

To estimate the integral involving ζ on the above right-hand side, we recall the choice of ζ in (4.9) and observe that

$$|\zeta(t)|^2 \le C\Big[|\pi_v\rho_t|^2 + |\pi_x\rho_t|^2 + \big(|\nabla^2 U(x(t))|^2 + |\nabla^2 G(x(t))|^2\big)|\pi_x\rho_t|^2\Big] + C\Big|\int_0^\infty \nabla H(x(t)-\eta(t;s))(\pi_x\rho_t - \pi_\eta\rho_t(s))K(s)\mathrm{d}s\Big|^2. \tag{4.14}$$

Concerning the term involving $\nabla^2 U$ on the right-hand side of (4.14), from condition (2.11), condition (2.13) and the choice of the function Φ in (2.27),

$$|\nabla^2 U(x)|^2 \le C(|U(x)|^2+1) \le Ce^{\Phi(x,v)}.$$

Together with (3.1) and (4.11), we see that

$$\mathbb{E}\int_0^\infty |\nabla^2 U(x(r))|^2\cdot|\pi_x\rho_r|^2\mathrm{d}r \le C\int_0^\infty e^{-2\alpha r}\,\mathbb{E}e^{\Phi(r)}\mathrm{d}r \le Ce^{\Phi(x,v)+\|\eta\|_{p_2}^{p_2}}. \tag{4.15}$$

To estimate the term involving $\nabla^2 G$ on the right-hand side of (4.14), we combine Eqs. (2.20), (3.2) and (4.11) to deduce that

$$\begin{aligned}&\mathbb{E}\int_0^\infty |\nabla^2 G(x(r))|^2\cdot|\pi_x\rho_r|^2\mathrm{d}r\\ &\le C\int_0^\infty e^{-2\alpha r}\,\mathbb{E}\left[\frac{e^{G(x(r))}}{|x(r)|^{\beta_1}}\right]\mathrm{d}r + C\int_0^\infty e^{-2\alpha r}\mathrm{d}r \le Ce^{\Phi(x,v)+\|\eta\|_{p_2}^{p_2}}.\end{aligned} \tag{4.16}$$

Next, considering the integral on the right-hand side of (4.14), we employ condition (2.10) to see that

$$\begin{aligned}&\Big|\int_0^\infty \nabla H(x(t)-\eta(t;s))(\pi_x\rho_t - \pi_\eta\rho_t(s))K(s)\mathrm{d}s\Big|^2\\ &\le C\Big|\int_0^\infty \big(|x(t)|^{p_1}+|\eta(t;s)|^{p_1}+1\big)\cdot\big(|\pi_x\rho_t|+|\pi_\eta\rho_t(s)|\big)K(s)\mathrm{d}s\Big|^2\\ &\le C\Big(|x(t)|^{2p_1}|\pi_x\rho_t|^2 + |\pi_x\rho_t|^2\Big|\int_0^\infty |\eta(t;s)|^{p_1}K(s)\mathrm{d}s\Big|^2 + |\pi_x\rho_t|^2\\ &\quad + \big(|x(t)|^{2p_1}+1\big)\Big|\int_0^\infty |\pi_\eta\rho_t(s)|K(s)\mathrm{d}s\Big|^2\\ &\quad + \Big|\int_0^\infty |\eta(t;s)|^{p_1}|\pi_\eta\rho_t(s)|K(s)\mathrm{d}s\Big|^2\Big).\end{aligned} \tag{4.17}$$

Recalling the choice of p_2 as in (2.21), we have

$$\Big|\int_0^\infty |\eta(t;s)|^{p_1}K(s)\mathrm{d}s\Big|^2 \le \int_0^\infty |\eta(t;s)|^{2p_1}K(s)\mathrm{d}s\int_0^\infty K(s)\mathrm{d}s \le C\|\eta(t)\|_{p_2}^{p_2}+C,$$

and that

$$\Big|\int_0^\infty |\pi_\eta\rho_t(s)|K(s)\mathrm{d}s\Big|^2 \le \Big[\int_0^\infty |\pi_\eta\rho_t(s)|^{p_2}K(s)\mathrm{d}s\Big]^{\frac{2}{p_2}}\cdot\Big[\int_0^\infty K(s)\mathrm{d}s\Big]^{\frac{2}{p_2^*}}$$
$$= C\|\pi_\eta\rho_t\|_{p_2}^2,$$

where p_2^* denotes the conjugate of p_2, i.e., $\frac{1}{p_2}+\frac{1}{p_2^*}=1$. Also,

$$\Big|\int_0^\infty |\eta(t;s)|^{p_1}|\pi_\eta\rho_t(s)|K(s)\mathrm{d}s\Big|^2$$
$$\le \Big|\int_0^\infty |\eta(t;s)|^{p_1p_2^*}K(s)\mathrm{d}s\Big|^{\frac{2}{p_2^*}}\Big|\int_0^\infty |\pi_\eta\rho_t(s)|^{p_2}K(s)\mathrm{d}s\Big|^{\frac{2}{p_2}}.$$

Since $p_2>p_1+1$, note that $p_1p_2^*<p_2$. As a consequence,

$$\Big|\int_0^\infty |\eta(t;s)|^{p_1p_2^*}K(s)\mathrm{d}s\Big|^{\frac{2}{p_2^*}}\Big|\int_0^\infty |\pi_\eta\rho_t(s)|^{p_2}K(s)\mathrm{d}s\Big|^{\frac{2}{p_2}}$$
$$\le C\|\eta(t)\|_{p_2}^{\frac{2p_2}{p_2^*}}\|\pi_\eta\rho_t\|_{p_2}^2 + C\|\pi_\eta\rho_t\|_{p_2}^2.$$

Combining with (4.17), we deduce the bound

$$\mathbb{E}\Big|\int_0^\infty \nabla H(x(t)-\eta(t;s))(\pi_x\rho_t-\pi_\eta\rho_t(s))K(s)\mathrm{d}s\Big|^2$$
$$\le C|\pi_x\rho_t|^2\mathbb{E}\big[|x(t)|^{2p_1}+\|\eta(t)\|_{p_2}^{p_2}+1\big]+C\|\pi_\eta\rho_t\|_{p_2}^2\mathbb{E}\big[|x(t)|^{2p_1}+\|\eta(t)\|_{p_2}^{\frac{2p_2}{p_2^*}}+1\big].$$

Since $U(x)$ dominates $|x|^{q_0}$, cf. (2.11), we have

$$|x|^{2p_1}+\|\eta\|_{p_2}^{p_2}+\|\eta\|_{p_2}^{\frac{2p_2}{p_2^*}}+1\le Ce^{\Phi(x,v)+\|\eta\|_{p_2}^{p_2}}.$$

It follows that

$$\mathbb{E}\Big|\int_0^\infty \nabla H(x(t)-\eta(t;s))(\pi_x\rho_t-\pi_\eta\rho_t(s))K(s)\mathrm{d}s\Big|^2$$
$$\le C\big(|\pi_x\rho_t|^2+\|\pi_\eta\rho_t\|_{p_2}^2\big)\mathbb{E}e^{\Phi(x(t),v(t))+\|\eta(t)\|_{p_2}^{p_2}}.$$

In light of (3.1), we deduce further that

$$\mathbb{E}\Big|\int_0^\infty \nabla H(x(t)-\eta(t;s))(\pi_x\rho_t-\pi_\eta\rho_t(s))K(s)\mathrm{d}s\Big|^2$$
$$\le C\big(|\pi_x\rho_t|^2+\|\pi_\eta\rho_t\|_{p_2}^2\big)e^{\Phi(x,v)+\|\eta\|_{p_2}^{p_2}},$$

whence

$$
\begin{aligned}
&\int_0^\infty \mathbb{E}\Big|\int_0^\infty \nabla H(x(t)-\eta(t;s))(\pi_x\rho_t-\pi_\eta\rho_t(s))K(s)\mathrm{d}s\Big|^2\mathrm{d}t \\
&\quad\le Ce^{\Phi(x,v)+\|\eta\|_{p2}^{p2}}\int_0^\infty \big(|\pi_x\rho_t|^2+\|\pi_\eta\rho_t\|_{p2}^2\big)\mathrm{d}t\le Ce^{\Phi(x,v)+\|\eta\|_{p2}^{p2}},
\end{aligned} \tag{4.18}
$$

where the last implication follows from (4.11) and (4.12).

Now, we collect (4.14), (4.15), (4.16) and (4.18) while making use of (4.11) again to arrive at the bound

$$
\mathbb{E}\int_0^\infty |\zeta(t)|^2\mathrm{d}t\le C(1+e^{\Phi(x,v)+\|\eta\|_{p2}^{p2}})\le Ce^{\Phi(x,v)+\|\eta\|_{p2}^{p2}}.
$$

This together with (4.13) implies that

$$
\mathbb{E}\Big[f(x(t),v(t),\eta(t))\int_0^t\langle\zeta(r),\mathrm{d}W(r)\rangle\Big]\le Ce^{\frac12[\Phi(x,v)+\|\eta\|_{p2}^{p2}]}\sqrt{P_t|f|^2(x,v,\eta)},
$$

which produces (4.8). The proof is thus finished.

□

4.2 Irreducibility

For $R>0$, let A_R be the bounded set in $\mathcal{X}_{p_2}$ given by

$$
A_R=\{(x,v,\eta)\in\mathcal{X}_{p_2}:|x|+|x|^{-1}+|v|+\|\eta\|_{p_2}\le R\}. \tag{4.19}
$$

Also, letting $e_1=(1,0,\dots,0)$ be a unit vector in $\mathbb{R}^d$, for $r\in(0,1)$ we denote by B_r the disk centered at $(e_1,0,0)\in\mathcal{X}_{p_2}$ with radius r, i.e.,

$$
B_r=\{(x,v,\eta)\in\mathcal{X}_{p_2}:|x-e_1|+|v|+\|\eta\|_{p_2}\le r\}. \tag{4.20}
$$

In Proposition 4.3, we assert that the probability of the solutions eventually entering B_r is uniform with respect to initial data in A_R. In turn, this will be employed to establish the ϱ_R-small property of A_R in Sect. 4.3.

Proposition 4.3 *Given every initial condition $(x,v,\eta)\in\mathcal{X}_{p_2}$, let $(x(t),v(t),\eta(t))$ be the solution of (2.8). Then, for all positive constants $R>2$, $r\in(0,1)$, there exists $T=T(R,r)>0$ such that*

$$
\inf_{(x,v,\eta)\in A_R}\mathbb{P}\Big(\big(x(T),v(T),\eta(T)\big)\in B_r\Big)>0, \tag{4.21}
$$

where A_R and B_r are defined in (4.19) and (4.20), respectively.

Owing to the combination of nonlinearities and memory in (2.8), we will not directly establish (4.21). Instead, we will compare the dynamics (2.8) with the following simpler system:

$$\mathrm{d}x_1(t) = v_1(t)\mathrm{d}t, \tag{4.22a}$$

$$\mathrm{d}v_1(t) = -v_1(t)\mathrm{d}t - \nabla U(x_1(t))\mathrm{d}t - \nabla G(x_1(t))\mathrm{d}t + \mathrm{d}W(t), \tag{4.22b}$$

$$\mathrm{d}\eta_1(t) = -\partial_s\eta_1(t)\mathrm{d}t, \quad \eta_1(t;0) = x_1(t),\ t > 0. \tag{4.22c}$$

We note that the first two equations (4.22a)-(4.22b) are the classical Langevin dynamics, which is decoupled from the η_1−equation. The irreducibility proof of Proposition 4.3 essentially consists of two main steps. Firstly, we prove an analogous result for (4.22), while making use of the fact that the Langevin dynamics (4.22a)-(4.22b) itself is well behaved. This is summarized in Lemma 4.4. Then, we will employ Girsanov's theorem to show that the law of (2.8) is equivalent to (4.22), allowing us to deduce (4.21).

Lemma 4.4 *For every initial condition $(x, v, \eta) \in \mathcal{X}_{p_2}$, let $(x_1(t), v_1(t), \eta_1(t)$ be the solution of (4.22). Then, for all positive constants $R > 2$, $r \in (0, 1)$, there exists $T = T(R, r) > 0$ such that for all $t \geq T$,*

$$\inf_{(x,v,\eta)\in A_R} \mathbb{P}\Big(\big(x_1(t), v_1(t), \eta_1(t)\big) \in B_r\Big) > 0, \tag{4.23}$$

where A_R and B_r are defined in (4.19) and (4.20), respectively.

The proof of Lemma 4.4 will be presented at the end of this subsection. We now invoke Lemma 4.4 to conclude Proposition 4.3, whose argument is standard and can be found in the literature (Földes et al. 2019; Nguyen 2023b; Seong 2023).

Proof of Proposition 4.3 We first rewrite the auxiliary system (4.22) as follows:

$$\begin{aligned}
\mathrm{d}x_1(t) &= v_1(t)\mathrm{d}t,\\
\mathrm{d}v_1(t) &= -v_1(t)\mathrm{d}t - \nabla U(x_1(t))\mathrm{d}t - \nabla G(x_1(t))\mathrm{d}t\\
&\quad - \int_0^\infty H(x_1(t) - \eta_1(t;s))K(s)\mathrm{d}s\mathrm{d}t + \mathrm{d}W(t),\\
&\quad + \int_0^\infty H(x_1(t) - \eta_1(t;s))K(s)\mathrm{d}s\mathrm{d}t,\\
\mathrm{d}\eta_1(t) &= -\partial_s\eta_1(t)\mathrm{d}t, \quad \eta_1(t;0) = x_1(t),\ t > 0.
\end{aligned}$$

Observe that the above system only differs from (2.8) by the appearance of the term $+\int_0^\infty H(x_1(t) - \eta_1(t;s))K(s)\mathrm{d}s\mathrm{d}t$. In order to produce the uniform bound (4.21) by making use of (4.23), we first show that $\big(x(\cdot), v(\cdot), \eta(\cdot)\big)$ and $\big(x_1(\cdot), v_1(\cdot), \eta_1(\cdot)\big)$ are equivalent in law on $C([0, t]; \mathcal{X}_{p_2})$. In turn, this can be established by Girsanov's theorem provided the following Novikov's condition is verified:

$$\mathbb{E}\exp\left\{\frac{1}{2}\int_0^t \left|\int_0^\infty H\big(x_1(r) - \eta_1(r;s)\big)K(s)\mathrm{d}s\right|^2 \mathrm{d}r\right\} < \infty. \tag{4.24}$$

To establish (4.24), we recall condition (2.10) and condition (2.21) to estimate

$$\begin{aligned}\Big|\int_0^\infty H\big(x_1(r)-\eta_1(r;s)\big)K(s)\mathrm{d}s\Big|^2 &\le c\Big(|x_1(r)|^{p_1}+\int_0^\infty \big|\eta_1(r;s)\big|^{p_1}K(s)\mathrm{d}s+1\Big)^2\\ &\le c|x_1(r)|^{2p_1}+c\|\eta_1(r)\|^{2p_1}_{2p_1}+c\\ &\le c|x_1(r)|^{p_2}+c\|\eta_1(r)\|^{p_2}_{p_2}+c.\end{aligned}$$

To further bound the above right-hand side, we employ the argument as in (3.5) while making use of condition (2.9) to see that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|\eta_1(t)\|^{p_2}_{p_2}\le K(0)|\eta_1(t;0)|^{p_2}-\delta\|\eta_1(t)\|^{p_2}_{p_2},$$

whence (recalling $\eta_1(t;0)=x_1(t)$)

$$\|\eta_1(t)\|^{p_2}_{p_2}\le e^{-\delta t}\|\eta\|^{p_2}_{p_2}+K(0)\int_0^t|x_1(s)|^{p_2}\mathrm{d}s. \tag{4.25}$$

As a consequence, for all $r\in[0,t]$, it holds that

$$\begin{aligned}\Big|\int_0^\infty H\big(x_1(r)-\eta_1(r;s)\big)K(s)\mathrm{d}s\Big|^2 &\le c|x_1(r)|^{p_2}+c\|\eta_1(r)\|^{p_2}_{p_2}+c\\ &\le c|x_1(r)|^{p_2}+\|\eta\|^{p_2}_{p_2}+c\int_0^t|x_1(s)|^{p_2}\mathrm{d}s+c\\ &\le c\|\eta\|^{p_2}_{p_2}+c\sup_{r\in[0,t]}|x_1(r)|^{p_2}+c,\end{aligned} \tag{4.26}$$

for some positive constant c that possibly depends on time t. Next, since $U(x)$ dominates $|x|^{p_2}$, cf. conditions (2.14) and (2.21), we infer that, for all β sufficiently small,

$$c\sup_{r\in[0,t]}|x_1(r)|^{p_2}\le\beta\sup_{r\in[0,t]}U(x_1(r))+C\le\beta\sup_{r\in[0,t]}\Phi(x_1(r),v_1(r))+C,$$

where the last implication follows from the definition of Φ in (2.27). Together with (4.26), we obtain the a.s. bound

$$\int_0^t\Big|\int_0^\infty H\big(x_1(r)-\eta_1(r;s)\big)K(s)\mathrm{d}s\Big|^2\mathrm{d}r\le c\|\eta\|^{p_2}_{p_2}+\beta\sup_{r\in[0,t]}\Phi(x_1(r),v_1(r))+C.$$

In view of Lemma A.1, we deduce that

$$\begin{aligned}&\mathbb{E}\exp\Big\{\frac12\int_0^t\Big|\int_0^\infty H\big(x_1(r)-\eta_1(r;s)\big)K(s)\mathrm{d}s\Big|^2\mathrm{d}r\Big\}\\ &\le\exp\big\{c\|\eta\|^{p_2}_{p_2}+C\big\}\cdot\mathbb{E}\exp\Big\{\beta\sup_{r\in[0,t]}\Phi(x_1(r),v_1(r))\Big\}\le C(x,v,\eta,t)<\infty,\end{aligned} \tag{4.27}$$

which verifies the Novikov condition (4.24), thereby establishing the equivalence in law.

Turning back to (4.21), we introduce the change-of-measure function

$$Q(t)=\exp\Big\{\int_0^t\Big\langle\int_0^\infty H\big(x_1(r)-\eta_1(r;s)\big)K(s)\mathrm{d}s,\mathrm{d}W(r)\Big\rangle \\ -\frac{1}{2}\int_0^t\Big|\int_0^\infty H\big(x_1(r)-\eta_1(r;s)\big)K(s)\mathrm{d}s\Big|^2\mathrm{d}r\Big\}.$$

In light of Girsanov's theorem, for all time $t>0$, the following holds:

$$\mathbb{P}\big((x(t),v(t),\eta(t))\in B_r\big)=\mathbb{E}\big[Q(t)\cdot\mathbf{1}\big\{\big(x_1(t),v_1(t),\eta_1(t)\big)\in B_r\big\}\big],$$

where B_r is defined in (4.20). On the one hand, letting $\lambda\in(0,1)$ be given and chosen later, we have

$$\mathbb{P}\big((x(t),v(t),\eta(t))\in B_r\big)=\mathbb{E}\big[Q(t)\cdot\mathbf{1}\big\{\big(x_1(t),v_1(t),\eta_1(t)\big)\in B_r\big\}\big] \\ \geq\lambda\mathbb{P}\big(Q(t)>\lambda,\big(x_1(t),v_1(t),\eta_1(t)\big)\in B_r\big).$$

On the other hand,

$$\mathbb{P}\big(\big(x_1(t),v_1(t),\eta_1(t)\big)\in B_r\big)\leq\mathbb{P}\big(Q(t)>\lambda,\big(x_1(t),v_1(t),\eta_1(t)\big)\in B_r\big) \\ +\mathbb{P}\big(Q(t)\leq\lambda\big).$$

We deduce from the two estimates above that

$$\frac{1}{\lambda}\mathbb{P}\big((x(t),v(t),\eta(t))\in B_r\big)\geq\mathbb{P}\big(\big(x_1(t),v_1(t),\eta_1(t)\big)\in B_r\big)-\mathbb{P}\big(Q(t)\leq\lambda\big),$$

whence (recalling the set A_R in (4.19))

$$\frac{1}{\lambda}\inf_{(x,v,\eta)\in A_R}\mathbb{P}\big((x(t),v(t),\eta(t))\in B_r\big) \\ \geq\inf_{(x,v,\eta)\in A_R}\mathbb{P}\big(\big(x_1(t),v_1(t),\eta_1(t)\big)\in B_r\big)-\sup_{(x,v,\eta)\in A_R}\mathbb{P}\big(Q(t)\leq\lambda\big). \tag{4.28}$$

To estimate the supremum on the above right-hand side, we note that

$$\mathbb{P}\big(Q(t)\leq\lambda\big)=\mathbb{P}\Big(\frac{1}{2}\int_0^t\Big|\int_0^\infty H\big(x_1(r)-\eta_1(r;s)\big)K(s)\mathrm{d}s\Big|^2\mathrm{d}r \\ -\int_0^t\Big\langle\int_0^\infty H\big(x_1(r)-\eta_1(r;s)\big)K(s)\mathrm{d}s,\mathrm{d}W(r)\Big\rangle\geq-\log(\lambda)\Big).$$

By Markov's inequality, we deduce that

$$\mathbb{P}\big(Q(t)\le\lambda\big)\le\frac{1}{-\log(\lambda)}\Big(\mathbb{E}\Big[\frac{1}{2}\int_0^t\Big|\int_0^\infty H\big(x_1(r)-\eta_1(r;s)\big)K(s)\mathrm{d}s\Big|^2\mathrm{d}r\Big]$$
$$+\mathbb{E}\Big|\int_0^t\Big\langle\int_0^\infty H\big(x_1(r)-\eta_1(r;s)\big)K(s)\mathrm{d}s,\mathrm{d}W(r)\Big\rangle\Big|\Big).$$

Also, Hölder's inequality and Itô's isometry yield

$$\mathbb{E}\Big|\int_0^t\Big\langle\int_0^\infty H\big(x_1(r)-\eta_1(r;s)\big)K(s)\mathrm{d}s,\mathrm{d}W(r)\Big\rangle\Big|$$
$$\le\sqrt{t}\cdot\sqrt{\int_0^t\mathbb{E}\Big|\int_0^\infty H\big(x_1(r)-\eta_1(r;s)\big)K(s)\mathrm{d}s\Big|^2\mathrm{d}r}.$$

So,

$$\mathbb{P}\big(Q(t)\le\lambda\big)\le\frac{1}{-\log(\lambda)}\Big(\mathbb{E}\Big[\int_0^t\Big|\int_0^\infty H\big(x_1(r)-\eta_1(r;s)\big)K(s)\mathrm{d}s\Big|^2\mathrm{d}r\Big]+t\Big)$$
$$\le\frac{1}{-\log(\lambda)}\cdot\exp\Big\{c\|\eta\|_{p_2}^{p_2}+\beta\Phi(x,v)+C\Big\},$$

where the last implication follows from (4.27) and Lemma A.1. Since $(x,v,\eta)\in A_R$, we infer that

$$\sup_{(x,v,\eta)\in A_R}\mathbb{P}\big(Q(t)\le\lambda\big)\le\frac{C}{-\log(\lambda)},$$

for some positive constant $C=C(R,t)$ independent of λ. From (4.28) together with Lemma 4.4, we obtain

$$\frac{1}{\lambda}\inf_{(x,v,\eta)\in A_R}\mathbb{P}\big(\big(x(t),v(t),\eta(t)\big)\in B_r\big)$$
$$\ge\inf_{(x,v,\eta)\in A_R}\mathbb{P}\big(\big(x_1(t),v_1(t),\eta_1(t)\big)\in B_r\big)-\sup_{(x,v,\eta)\in A_R}\mathbb{P}\big(Q(t)\le\lambda\big)\ge c-\frac{C}{|\log(\lambda)|}.$$

We emphasize again that the positive constants c, C do not depend on the choice of λ. Lastly, by taking λ small enough, we immediately obtain the uniform lower bound

$$\inf_{(x,v,\eta)\in A_R}\mathbb{P}\big(\big(x(t),v(t),\eta(t)\big)\in B_r\big)\ge\lambda\Big(c-\frac{C}{|\log(\lambda)|}\Big)>0,$$

thereby finishing the irreducibility argument.

□

We now turn to Lemma 4.4, whose argument will employ the Support Theorem (Stroock and Varadhan 1972, 1970) as well as Lemma A.2.

Proof of Lemma 4.4 Firstly, we consider the pair (x_1, v_1) solving (4.22a)-(4.22b). To derive an estimate in probability for (x_1, v_1), we introduce the deterministic control problem

$$\begin{aligned} \mathrm{d}\tilde{x}_1(t) &= \tilde{v}_1(t)\mathrm{d}t, \\ \mathrm{d}\tilde{v}_1(t) &= -\tilde{v}_1(t)\mathrm{d}t - \nabla U(\tilde{x}_1(t))\mathrm{d}t - \nabla G(\tilde{x}_1(t))\mathrm{d}t + \mathrm{d}\Gamma(t), \\ \tilde{x}_1(0) &= x, \quad \tilde{v}_1(0) = v, \end{aligned} \tag{4.29}$$

where $\Gamma \in C^1([0,\infty);\mathbb{R}^d)$ is a control function. By the Stroock–Varadhan support theorem, for each $t,\ \varepsilon > 0$ and (x_0, v_0), there exists $\varepsilon_1 = \varepsilon_1(t,\varepsilon)$ such that

$$\inf_{(x,v)\in E\big((x_0,v_0),\varepsilon_1\big)} \mathbb{P}\Big(\sup_{\ell\in[0,t]} |x_1(\ell) - \tilde{x}_1(\ell)| + |v_1(\ell) - \tilde{v}_1(\ell)| \le \varepsilon\Big) > 0,$$

where

$$E\big((x_0,v_0),\varepsilon_1\big) = \Big\{(x,v) \in \mathbb{R}^d\setminus\{0\}\times\mathbb{R}^d : |x-x_0| + |v-v_0| < \varepsilon_1\Big\}.$$

In view of Lemma A.2, for each $R > 2$, $\varepsilon > 0$ and (x_0, v_0) such that $|x_0| + |x_0|^{-1} + |v_0| \le R$, there exists a path $(\tilde{x}_1, \tilde{v}_1, \Gamma)$ satisfying

$$(\tilde{x}_1(t), \tilde{v}_1(t)) = (e_1, 0) \quad \text{and} \quad \int_0^t |\tilde{x}_1(s)|^{p_2}\mathrm{d}s < \varepsilon.$$

It follows that

$$\inf_{(x,v)\in E\big((x_0,v_0),\varepsilon_1\big)} \mathbb{P}\Big(|x_1(t) - e_1| + |v_1(t)| \le \varepsilon, \quad \text{and} \quad \int_0^t |x_1(s)|^{p_2}\mathrm{d}s \le C\,t\,\varepsilon\Big) > 0,$$

for some positive constant C independent of $t, \varepsilon, \varepsilon_1$ and R. Since the set $\{|x| + |x|^{-1} + |v| \le R\}$ is compact and thus can be covered by finitely many open disks $E((x_0, v_0), \varepsilon_1)$, we obtain the uniform bound

$$\inf_{(x,v):|x|+|x|^{-1}+|v|\le R} \mathbb{P}\Big(|x_1(t) - e_1| + |v_1(t)| \le \varepsilon, \quad \text{and} \quad \int_0^t |x_1(s)|^{p_2}\mathrm{d}s \le C\,t\,\varepsilon\Big) > 0. \tag{4.30}$$

Turning to the η_1-equation (4.22c), we combine (4.25) and (4.30) to see that for all $R > 2$, $t > 1$ and $\varepsilon > 0$, it holds that

$$\inf_{(x,v,\eta)\in A_R} \mathbb{P}\Big(|x_1(t) - e_1| + |v_1(t)| \le \varepsilon, \quad \text{and} \quad \|\eta_1(t)\|_{p_2}^{p_2} \le e^{-\delta t}R^{p_2} + C\,t\,\varepsilon\Big) > 0.$$

By taking t sufficiently large and then shrinking ε small enough, this produces the bound (4.23), as claimed.

□

4.3 Proof of the main results

In this section, we establish the exponential mixing rate of P_t toward the unique invariant probability measure ν in terms of Wasserstein distances and observables, as presented in Theorem 2.12 and Corollary 2.13, respectively. The argument follows the approach of Butkovsky et al. (2020); Hairer et al. (2011), which consists of constructing Lyapunov functions, ϱ_N-*contracting* property and ϱ_N-*small* sets where ϱ_N is the metric defined in (2.32). For the reader's convenience, we recall these notions below.

Definition 4.5 1 . A function $V : \mathcal{X}_{p_2} \to [0, \infty)$ is called a *Lyapunov* function for P_t if

$$P_t V(X) \le Ce^{-ct} V(X) + C, \quad t \ge 0,\ X \in \mathcal{X}_{p_2},$$

for some positive constants c, C independent of X and t.

2 . A distance-like function ϱ bounded by 1 is called *contracting* for P_t if there exists $\lambda_1 \in (0, 1)$ such that for any $X, \tilde{X}$ with $\varrho(X, \tilde{X}) < 1$, it holds that

$$\mathcal{W}_\varrho\big(P_t(X, \cdot), P_t(\tilde{X}, \cdot)\big) \le \lambda_1 \varrho(X, \tilde{X}).$$

3 . A set $A \subset \mathcal{X}_{p_2}$ is called ϱ-*small* for P_t if for some $\lambda_2 = \lambda_2(A) > 0$,

$$\sup_{X, \tilde{X} \in A} \mathcal{W}_\varrho\big(P_t(X, \cdot), P_t(\tilde{X}, \cdot)\big) \le 1 - \lambda_2.$$

On the one hand, the existence of a Lyapunov function is guaranteed by the energy estimates in Lemma 3.1. On the other hand, in Proposition 4.6, we will demonstrate that one can always tune ϱ_N defined in (2.32) such that ϱ_N is contracting for P_t and that the bounded set A_R defined in (4.19) is small with respect to ϱ_N.

Proposition 4.6 *Let ϱ_N and A_R be defined in* (2.32) *and* (4.19)*, respectively. Then,*

1 . There exist N_1 and t_1 sufficiently large such that for all $N > N_1$ and $t > t_1$, ϱ_N is contracting for P_t in the sense of Definition 4.5, part 2 ., with $\lambda_1 = \frac{1}{2}$, i.e.,

$$\mathcal{W}_{\varrho_N}\big(P_t(X, \cdot), P_t(\tilde{X}, \cdot)\big) \le \frac{1}{2}\varrho_N(X, \tilde{X}), \tag{4.31}$$

whenever $\varrho_N(X, \tilde{X}) < 1$.

2 . For all $N > 0$, $R > 2$, there exists $t_2 = t_2(N, R)$ such that for all $t \ge t_2$, the set A_R is ϱ_N-small in the sense of Definition 4.5, part 3 ., with a constant $\lambda_2 = \lambda_2(R) \in (0, 1)$, i.e.,

$$\sup_{X, \tilde{X} \in A_R} \mathcal{W}_{\varrho_N}\big(P_t(X, \cdot), P_t(\tilde{X}, \cdot)\big) \le 1 - \lambda_2. \tag{4.32}$$

The proof of Proposition 4.6 is quite standard (Hairer et al. 2011) and makes use of the auxiliary results presented in Sect. 4.1 and Sect. 4.2. For the sake of clarity, we will defer the explicit argument to the end of this section. We are now in a position to conclude the proof of Theorem 2.12 by verifying the conditions of (Hairer et al. (2011), Theorem 4 .8)

Proof of Theorem 2.12 On the one hand, from Lemma 3.1, cf., (3.1), the function $V = e^{\Psi}$ plays the role of the Lyapunov function as in Definition 4.5, part 1 . On the other hand, in view of Proposition 4.6, the distance ϱ_N defined in (2.32) is contracting for P_t, and any bounded set A_R ($R > 2$) is ϱ_N-small. In other words, all of the conditions of (Hairer et al. (2011), Theorem 4 .8) are met. In light of (Hairer et al. (2011), Theorem 4 .8), we obtain the existence and uniqueness of the invariant probability measure ν for P_t as well as the exponential convergence rate (2.34).

□

As a consequence of Theorem 2.12, we deduce the mixing rate with respect to observables in Corollary 2.13. Since the proof of Corollary 2.13 is short, we include it here for the sake of completeness.

Proof of Corollary 2.13 Let $f \in C(\mathcal{X}_{p_2}; \mathbb{R})$ be such that $[f]_{\text{Lip},\tilde{\varrho}_N} < \infty$. Given $X_0 \in \mathcal{X}_{p_2}$, in view of estimate (2.26), it holds that

$$\mathcal{W}_{\tilde{\varrho}_N}\big(P_t\delta_{X_0}, \nu\big) \geq \frac{1}{[f]_{\text{Lip},\tilde{\varrho}_N}}\Big|P_t f(X_0) - \int_{\mathcal{X}_{p_2}} f(X)\nu(\mathrm{d}X)\Big|.$$

From Theorem 2.12, we deduce that

$$\Big|P_t f(X_0) - \int_{\mathcal{X}_{p_2}} f(X)\nu(\mathrm{d}X)\Big| \leq [f]_{\text{Lip},\tilde{\varrho}_N}\mathcal{W}_{\tilde{\varrho}_N}\big(\delta_{X_0}, \nu\big)Ce^{-ct}.$$

This produces (2.35), thereby finishing the proof. □

We now turn to Proposition 4.6, which was ultimately invoked to conclude the main result in Theorem 2.12. The proof of Proposition 4.6 will employ the auxiliary estimates performed in Sect. 4.1 and Sect. 4.2. See also Butkovsky et al. (2020); Hairer et al. (2011).

Proof of Proposition 4.6 1 . Letting $N > 0$ be given and chosen later, consider $X, \tilde{X} \in \mathcal{X}_{p_2}$ such that $\varrho_N(X, \tilde{X}) < 1$. From the expression (2.32), this implies that

$$\varrho_N(X, \tilde{X}) = N\varrho(X, \tilde{X}) \wedge 1 = N\varrho(X, \tilde{X}),$$

where ϱ is defined in (2.31). By the dual Kantorovich formula (2.24), the inequality (4.31) is equivalent to

$$|P_t f(X) - P_t f(\tilde{X})| \leq \frac{1}{2}N\varrho(X, \tilde{X}), \tag{4.33}$$

which holds for all $f \in C_b^1(\mathcal{X}_{p2})$ satisfying $[f]_{\text{Lip},\varrho_N} \le 1$. Here, we recall $[f]_{\text{Lip},\varrho_N}$ is the Lipschitz norm with respect to ϱ_N in (2.25). Note also that (2.25) may be recast as

$$[f]_{\text{Lip},\varrho_N} = \sup_{Y\neq\tilde{Y}} \frac{f(Y)-f(\tilde{Y})}{\varrho_N(Y,\tilde{Y})} = \sup_{Y\neq\tilde{Y}} \frac{[f(Y)-f(e_1,0,0)]-[f(\tilde{Y})-f(e_1,0,0)]}{\varrho_N(Y,\tilde{Y})}.$$

In the above, $e_1 = (1,0,\dots,0)$ is the unit element in $\mathbb{R}^d$. It therefore suffices to prove (4.33) for those functions f such that $f(e_1,0,0)=0$. In particular, this implies that $\|f\|_\infty \le 1$ since

$$|f(Y)| = |f(Y)-f(e_1,0,0)| \le \varrho_N\big(Y,(e_1,0,0)\big) \le 1, \quad Y \in \mathcal{X}_{p2}.$$

Furthermore, for $\tilde{Y}$ sufficiently close to Y such that the path

$$\gamma_*(s) = s\cdot Y + (1-s)\cdot\tilde{Y} \quad \text{for} \quad 0\le s\le 1$$

belongs to $\mathcal{X}_{p2}$, we observe that

$$\begin{aligned}|f(Y)-f(\tilde{Y})| &\le \varrho_N(Y,\tilde{Y}) \le N\varrho(Y,\tilde{Y})\\ &\le N\int_0^1 e^{\frac12\Psi(\gamma_*(s))}\mathrm{d}s\,\|Y-\tilde{Y}\|_{\mathcal{X}_{p2}}.\end{aligned}$$

In the above, we recall $\Psi(x,v,\eta) = \Phi(x,v)+\|\eta\|_{p2}^{p2}$ defined in (2.30). It follows that

$$\frac{|f(Y)-f(\tilde{Y})|}{\|Y-\tilde{Y}\|_{\mathcal{X}_{p2}}} \le N\int_0^1 e^{\frac12\Psi(\gamma_*(s))}\mathrm{d}s \to Ne^{\frac12\Psi(Y)}$$

as $\tilde{Y}\to Y$ in $\mathcal{X}_{p2}$ by virtue of the dominated convergence theorem. As a consequence,

$$\|Df(Y)\|_{\mathcal{X}_{p2}^*} \le Ne^{\frac12\Psi(Y)}, \quad Y\in\mathcal{X}_{p2}.$$

It follows that

$$\begin{aligned}P_t\|Df\|^2_{\mathcal{X}^*_{p2}}(x,v,\eta) &= \mathbb{E}\|Df(x(t),v(t),\eta(t))\|^2_{\mathcal{X}^*_{p2}}\\ &\le N^2\mathbb{E}e^{\Psi(x(t),v(t),\eta(t))}\\ &\le C\,N^2 e^{\Psi(x,v,\eta)},\end{aligned}$$

where in the last implication above, we invoke (3.1) with some positive constant C independent of N and (x,v,η). Together with the asymptotic strong Feller property

(4.6), we deduce that

$$
\begin{aligned}
&\|DP_t f(x,v,\eta)\|_{\mathcal{X}^*_{p2}}\\
&\quad\le Ce^{-ct}\sqrt{P_t\|Df\|^2_{\mathcal{X}^*_{p2}}(x,v,\eta)}+Ce^{\frac{1}{2}\Psi(x,v,\eta)}\sqrt{P_t|f|^2(x,v,\eta)}\\
&\quad\le Ce^{-ct}\cdot Ne^{\frac{1}{2}\Psi(x,v,\eta)}+Ce^{\frac{1}{2}\Psi(x,v,\eta)}\sqrt{P_t|f|^2(x,v,\eta)}\\
&\quad\le Ne^{\frac{1}{2}\Psi(x,v,\eta)}\Big[Ce^{-ct}+\frac{C}{N}\Big].
\end{aligned}
$$

We emphasize again that on the above right-hand side, C and c are independent of N, t and (x,v,η). By choosing t and N sufficiently large such that

$$
Ce^{-ct}+\frac{C}{N}<\frac{1}{2},
$$

we obtain

$$
\|DP_t f(x,v,\eta)\|_{\mathcal{X}^*_{p2}}\le\frac{1}{2}Ne^{\frac{1}{2}\Psi(x,v,\eta)}. \tag{4.34}
$$

Now, returning to (4.33), letting $\gamma\in C([0,1];\mathcal{X}_{p2})$ be an arbitrary path connecting X and $\tilde{X}$, we recast the left-hand side of (4.33) as

$$
P_t f(X)-P_t f(\tilde{X})=\int_0^1\langle DP_t f(\gamma(s)),\gamma'(s)\rangle_{\mathcal{X}_{p_2}}\mathrm{d}s.
$$

Using Hölder's inequality and (4.34), we infer that

$$
\begin{aligned}
|P_t f(X)-P_t f(\tilde{X})|&\le\int_0^1\|DP_t f(\gamma(s))\|_{\mathcal{X}^*_{p2}}\|\gamma'(s)\|_{\mathcal{X}_{p_2}}\mathrm{d}s\\
&\le\frac{1}{2}N\int_0^1 e^{\frac{1}{2}\Psi(\gamma(s))}\|\gamma'(s)\|_{\mathcal{X}_{p_2}}\mathrm{d}s.
\end{aligned}
$$

Equation (4.33) then follows from the definition of the distance ϱ given in (2.31). In turn, this establishes the contracting property (4.31), as claimed, for all pairs $(X,\tilde{X})$ for which $\varrho_N(X,\tilde{X})<1$.

2 . With regard to the ϱ_N-small property (4.32), let $r\in(0,\frac{1}{2})$ be given and chosen later. Since the set B_r defined in (4.20) is bounded away from the origin in $\mathcal{X}_{p_2}$, we note that for every $Z_0,\tilde{Z}_0\in B_r$, the path

$$
\gamma_*=s\cdot Z_0+(1-s)\cdot\tilde{Z}_0\quad\text{for}\quad 0\le s\le 1
$$

is guaranteed to belong to $C^1([0,1]; B_r)$. Indeed, by the triangle inequality,

$$\begin{aligned}
&|s\cdot\pi_x Z_0 + (1-s)\cdot\pi_x\tilde{Z}_0 - e_1| + |s\cdot\pi_v Z_0 + (1-s)\cdot\pi_v\tilde{Z}_0| \\
&\quad + \|s\cdot\pi_\eta Z_0 + (1-s)\cdot\pi_\eta\tilde{Z}_0\|_{p_2} \\
&\le s\big(|\pi_x Z_0 - e_1| + |\pi_v Z_0| + \|\pi_\eta Z_0\|_{p_2}\big) \\
&\quad + (1-s)\big(|\pi_x\tilde{Z}_0 - e_1| + |\pi_v\tilde{Z}_0| + \|\pi_\eta\tilde{Z}_0\|_{p_2}\big) \le r.
\end{aligned}$$

In particular, this implies that

$$\varrho(Z_0,\tilde{Z}_0) \le \int_0^1 e^{\frac12\Psi(\gamma_*(s))}\|\gamma_*'(s)\|_{\mathcal{X}_{p_2}}\mathrm{d}s \le \sup_{Z\in B_r} e^{\frac12\Psi(Z)}\cdot\|Z_0-\tilde{Z}_0\|_{\mathcal{X}_{p_2}} \le Cr,$$

for some positive constant C that does not depend on $r\in(0,\frac12)$. In other words,

$$\sup_{Z_0,\tilde{Z}_0\in B_r}\varrho(Z_0,\tilde{Z}_0)\le Cr,\quad r\in\left(0,\frac12\right). \tag{4.35}$$

Returning to (4.32), given $X,\tilde{X}\in A_R$, we denote by $(Y,\tilde{Y})$ a coupling of $\big(P_t(X,\cdot), P_t(\tilde{X},\cdot)\big)$ such that Y and $\tilde{Y}$ are independent. By the definition (2.23) and the expression (2.32), we have the following chain of implications:

$$\begin{aligned}
&\mathcal{W}_{\varrho_N}\big(P_t(X,\cdot), P_t(\tilde{X},\cdot)\big) \\
&\quad\le \mathbb{E}\varrho_N(Y,\tilde{Y}) = \mathbb{E}\big[N\varrho(Y,\tilde{Y})\wedge 1\big] \\
&\quad\le \mathbb{E}\Big[\big(N\varrho(Y,\tilde{Y})\wedge 1\big)\mathbf{1}\big\{\{Y\in B_r\}\cap\{\tilde{Y}\in B_r\}\big\}\Big] \\
&\qquad + \mathbb{E}\Big[\big(N\varrho(Y,\tilde{Y})\wedge 1\big)\mathbf{1}\big\{\{Y\notin B_r\}\cup\{\tilde{Y}\notin B_r\}\big\}\Big] \\
&\quad\le N\sup_{Z_0,\tilde{Z}_0\in B_r}\varrho(Z_0,\tilde{Z}_0)\cdot\mathbb{P}\big(\{Y\in B_r\}\cap\{\tilde{Y}\in B_r\}\big) + \mathbb{P}\big(\{Y\notin B_r\}\cup\{\tilde{Y}\notin B_r\}\big).
\end{aligned}$$

We invoke (4.35) to further estimate

$$\begin{aligned}
\mathcal{W}_{\varrho_N}\big(P_t(X,\cdot), P_t(\tilde{X},\cdot)\big) &\le NCr\cdot\mathbb{P}\big(\{Y\in B_r\}\cap\{\tilde{Y}\in B_r\}\big) \\
&\quad + 1 - \mathbb{P}\big(\{Y\in B_r\}\cap\{\tilde{Y}\in B_r\}\big).
\end{aligned}$$

Since N and C are independent of $r\in(0,\frac12)$, we pick r sufficiently small such that $NCr<\frac12$ and obtain

$$\begin{aligned}
\mathcal{W}_{\varrho_N}\big(P_t(X,\cdot), P_t(\tilde{X},\cdot)\big) &\le 1-\frac12\mathbb{P}\big(\{Y\in B_r\}\cap\{\tilde{Y}\in B_r\}\big) \\
&= 1-\frac12\mathbb{P}\big(Y\in B_r\big)\mathbb{P}\big(\tilde{Y}\in B_r\big),
\end{aligned}$$

where the last identity follows from the choice of Y and $\tilde{Y}$ being independent. In light of the irreducibility condition (4.21), we infer the existence of a time $t_2 = t_2(R, r)$ such that for all $t \geq t_2$

$$\inf_{X \in A_R} P_t(X, B_r) > c = c(t) > 0,$$

whence

$$\mathcal{W}_{\varrho_N}\big(P_t(X,\cdot), P_t(\tilde{X},\cdot)\big) \leq 1 - \frac{1}{2}\mathbb{P}\big(Y \in B_r\big)\mathbb{P}\big(\tilde{Y} \in B_r\big) \leq 1 - \frac{1}{2}c^2.$$

This produces (4.32), thereby finishing the proof. □

5 Numerical Simulations of the Invariant Measure

We conclude the paper by presenting numerical examples for the invariant measure obtained from the stroboscopic model in dimension $d = 2$, specifically, for which the walker evolves under the influence of an attractive harmonic potential $U(x) = |x|^2/2$ and repulsive Coulomb potential $G(x) = -\alpha \log(|x|)$. The purpose of this section is to give the reader an intuition about the qualitative structure of the walker's invariant measure, the existence and uniqueness of which we established in the preceding sections by proving Theorem 2.12. To that end, we employ the functional forms of the pilot-wave force $H(x) = \mathrm{J}_1(|x|)x/|x|$ (c.f. Remark 2.5) and memory kernel $K(t) = \mathrm{e}^{-t}$ (c.f. Remark 2.2), fixing the dimensionless walker mass $m = 1$ and noise strength $\sigma = 1$. The forms of $H(x)$ and $K(t)$ are motivated by prior work on the topic (Eddi et al. 2011; Moláček and Bush 2013), in which approximations were derived for the spatiotemporal evolution of the wave field generated by a walker bouncing on a vertically vibrating fluid bath.

Equation (1.1) is solved numerically using an Euler-Maruyama time-stepping scheme with time step $\Delta t = 2^{-6}$ up to a final time $t_{\max} = 10^4$, with the integral computed using the trapezoidal rule. To initialize the simulations, we set the *initial past* corresponding to the stationary state $x(t) = (2\pi, 0)$ for $t < 0$. We consider three different values of α: $\alpha = 1, 3$ and 5 . Recall from Remark 2.7 that $\alpha = 3$ is the "edge case": While we have strictly speaking proven Theorem 2.12 under the assumption $\alpha \geq 3$, we expect that the theorem is true for all values of α. Indeed, the case $\alpha = 0$, in which there is no singular potential, was covered by our prior work (Nguyen and Oza 2024).

The results of the numerical simulations are shown in Fig. 1. The first two columns show the numerical solutions, while the last column shows the walker's radial position probability density function $p(r)$, where $r = |x|$. Panels (a) through (d) correspond to $\alpha = 1$, (e) through (h) to $\alpha = 3$ and (i) through (l) to $\alpha = 5$. It is evident from the leftmost panels that the walker executes an erratic trajectory in a roughly annular region in the plane, with the radius of the inner circle increasing with the strength α of the repulsive potential. In all three cases, the probability density function $p(r)$ eventually reaches a steady state, which we ascertained by doubling the simulation time $t_{\max}$ and

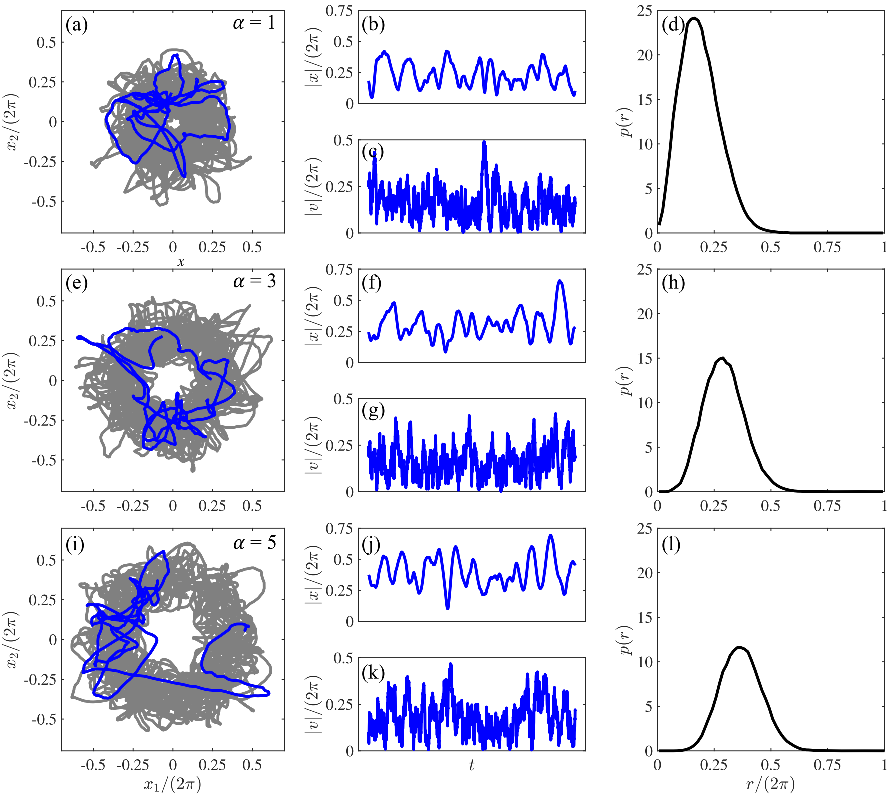

Fig. 1 Numerical simulations of Eq. (1.1) for $m = 1$, $\sigma = 1$, $U(x) = |x|^2/2$ and $G(x) = -\alpha \log(|x|)$, for $\alpha = 1$ (panels **a** through **d**), $\alpha = 3$ (panels **e** through **h**) and $\alpha = 5$ (panels **i** through **l**). In the leftmost panels, the gray curves show the two-dimensional walker position $x(t) = (x_1(t), x_2(t))$ over the time interval $t_{\max} - 1000 < t < t_{\max}$. The blue curve highlights a subset of this trajectory over the interval $t_{\max} - 50 < t < t_{\max}$. The corresponding time series for the radius $r = |x(t)|$ and speed $|v(t)|$ are shown in the middle panels. The rightmost panels show the walker's radial position probability density function $p(r)$. The variables are divided by 2π, which corresponds to the approximate periodicity of the pilot-wave force function $H(x) = \mathrm{J}_1(x)x/|x|$

verifying that $p(r)$ remained virtually unchanged. We make the empirical observation that an invariant measure seems to exist even for $\alpha < 3$, and that its form does not change qualitatively in the neighborhood of $\alpha = 3$. Our simulations thus suggest that condition (2.20) is a technical condition that likely could be relaxed in future work.

For the values of m, σ, α, and the forms of the potential $U(x)$ and pilot-wave force $H(x)$ shown here, the invariant measure $p(r)$ evidently has a relatively simple form with a single peak roughly located at $r \approx \sqrt{\alpha}$, the distance from the origin at which the attractive and repulsive forces on the walker are balanced, $r \approx \alpha/r$. In future work, we will seek to characterize the dependence of the form of $p(r)$ on the strengths of the regular and singular potentials, as well as the walker mass and noise strength.

A Auxiliary Results

In this section, we collect suitable estimates on the auxiliary system (4.22) through Lemma A.1 and Lemma A.2. In particular, the former establishes an exponential moment bound that is directly employed in the proof of Proposition 4.3, whereas the latter provides a control argument which appears in the proof of Lemma 4.4.

Lemma A.1 *Let (x_1, v_1) be the solution of the Langevin system (4.22a)-(4.22b) with initial data $(x, v) \in \mathbb{R}^d \setminus \{0\} \times \mathbb{R}^d$. Then, for all $\beta > 0$ sufficiently small*

$$\mathbb{E} \exp\Big\{\beta \sup_{s\in[0,t]} \Phi(x_1(s), v_1(s))\Big\} \le 2\exp\big\{\beta\Phi(x, v) + C\beta t\big\}, \quad t \ge 0. \tag{A.1}$$

In the above, Φ is defined in (2.27) and C is a positive constant independent of β and t.

Proof Denote by $\mathcal{L}_1$ the Kolmogorov operator associated with (4.22a)-(4.22b), namely

$$\mathcal{L}_1 g := \langle v, \partial_x g\rangle + \big\langle -v - \nabla U(x) - \nabla G(x(t)), \partial_v g\big\rangle + \frac{1}{2}\triangle_v g.$$

Similarly to the proof of Lemma 3.1, part 1 ., cf. (3.8), we apply $\mathcal{L}_1$ to $\Phi(x, v)$ and obtain the estimate

$$\mathcal{L}_1\Phi(x, v) \le -c\Big(|v|^2 + |x|^{q_0} + |x|^{-\beta_1}\Big) + C.$$

In particular, for all $\beta > 0$, Itô's formula yields

$$\begin{aligned}\mathrm{d}\big(\beta\Phi(x_1(t), v_1(t))\big) &= \beta\mathcal{L}_1\Phi(x_1(t), v_1(t))\mathrm{d}t + \mathrm{d}M_1(t)\\ &\le -c\beta\Big(|v_1(t)|^2 + |x_1(t)|^{q_0} + |x_1(t)|^{-\beta_1}\Big)\mathrm{d}t + C\beta\mathrm{d}t + \mathrm{d}M_1(t),\end{aligned}$$

where the semi-Martingale process $M_1(t)$ is defined as

$$M_1(t) = \beta\int_0^t \Big\langle v_1(r) + \kappa x_1(r) + \frac{x_1(r)}{|x_1(r)|}, \mathrm{d}W(r)\Big\rangle, \tag{A.2}$$

whose quadratic variation process is given by

$$\langle M_1\rangle(t) = \beta^2\int_0^t \Big|v_1(r) + \kappa x_1(r) + \frac{x_1(r)}{|x_1(r)|}\Big|^2 \mathrm{d}r. \tag{A.3}$$

Since $q_0 \ge 2$, cf. condition (2.14), we note that

$$\mathrm{d}\langle M_1\rangle(t) \le c\beta^2\big(|v_1(t)|^2 + |x_1(t)|^{q_0} + 1\big)\mathrm{d}t.$$

As a consequence, we obtain

$$\mathrm{d}\big(\beta\Phi(x_1(t), v_1(t))\big) \le C\beta\mathrm{d}t + \mathrm{d}M_1(t) - \frac{c}{\beta}\mathrm{d}\langle M_1\rangle(t), \tag{A.4}$$

for some positive constants C, c independent of β. Next, we recall the exponential Martingale inequality:

$$\mathbb{P}\left(\sup_{t\ge 0}\left[M_1(t) - \frac{1}{2}\cdot\lambda\cdot\langle M_1\rangle(t)\right] \ge R\right) \le e^{-\lambda R}, \quad \lambda > 0,\ R > 0.$$

Based on the right-hand side of (A.4), we choose $\lambda = c/\beta$ and observe that

$$\mathbb{P}\left(\exp\left\{\sup_{t\ge 0}\left[M_1(t) - \frac{c}{\beta}\langle M_1\rangle(t)\right]\right\} \ge R\right) \le R^{-2c/\beta}, \quad R > 0.$$

By taking β sufficiently small, the above estimate implies

$$\begin{aligned}&\mathbb{E}\exp\left\{\sup_{t\ge 0}\left[M_1(t) - \frac{c}{\beta}\langle M_1\rangle(t)\right]\right\}\\ &\quad= \int_0^\infty \mathbb{P}\left(\exp\left\{\sup_{t\ge 0}\left[M_1(t) - \frac{c}{\beta}\langle M_1\rangle(t)\right]\right\} \ge R\right)\mathrm{d}R \le 2.\end{aligned}$$

It follows from (A.4) that

$$\mathbb{E}\exp\left\{\sup_{s\in[0,t]} \beta\Phi(x_1(s), v_1(s))\right\} \le 2\exp\left\{\beta\Phi(x, v) + C\beta\, t\right\}.$$

This produces (A.1), as claimed.

□

Lemma A.2 *Given $R > 2, t > 1, r \in (0, 1)$ and $(x_0, v_0) \in \mathbb{R}^d \setminus \{0\} \times \mathbb{R}^d$ such that*

$$|x_0| + |x_0|^{-1} + |v_0| \le R,$$

there exists a triple $(\tilde{x}_1, \tilde{v}_1, \Gamma)$ satisfying the deterministic control problem (4.29) with

$$(\tilde{x}_1(0), \tilde{v}_1(0)) = (x_0, v_0), \quad (\tilde{x}_1(t), \tilde{v}_1(t)) = (e_1, 0), \tag{A.5}$$

and

$$\int_0^t |\tilde{x}_1(s)|^{p_2}\mathrm{d}s < r, \tag{A.6}$$

where $e_1 = (1, 0, \ldots, 0)$ is a unit element in $\mathbb{R}^d$.

Proof There are two cases to be considered depending on the relationship between x_0 and e_1:

Case 1 : x_0 is not a multiple of e_1, i.e.,

$$x_0 \notin \{z \in \mathbb{R}^d : z = ce_1 \text{ for some } c \in \mathbb{R}\}.$$

Letting ε small be given and chosen later, we first consider the time interval $[0, \varepsilon]$ and construct a pair $(\tilde{x}_1, \tilde{v}_1)$ on this interval satisfying

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\tilde{x}_1(s) = \tilde{v}_1(s), \quad 0 \le s \le \varepsilon,$$

with boundary conditions

$$\begin{aligned} &\tilde{x}_1(0) = x_0, \quad \tilde{x}_1(\varepsilon) = e_1, \\ &\tilde{v}_1(0) = v_0, \quad \tilde{v}_1(\varepsilon) = 0. \end{aligned}$$

To do so, letting $y \in \mathbb{R}^d \setminus \{0\}$ and $a \in (0, \varepsilon)$ be given and chosen later, we consider a path $\tilde{v}_1 \in C([0, \varepsilon]; \mathbb{R}^d)$ defined as

$$\tilde{v}_1(s) = y\frac{s(s-\varepsilon)}{a(a-\varepsilon)} + v_0\frac{(s-a)(s-\varepsilon)}{a\varepsilon}, \quad 0 \le s \le \varepsilon.$$

Note that with the above choice, we readily have

$$\tilde{v}_1(0) = v_0, \quad \tilde{v}_1(\varepsilon) = 0.$$

Setting

$$\tilde{x}_1(s) = x_0 + \int_0^s \tilde{v}_1(s)\mathrm{d}s,$$

a routine calculation yields

$$\begin{aligned} \tilde{x}_1(s) &= x_0 + \int_0^s y\frac{s(s-\varepsilon)}{a(a-\varepsilon)} + v_0\frac{(s-a)(s-\varepsilon)}{a\varepsilon}\mathrm{d}s \\ &= x_0 + \frac{y}{a(a-\varepsilon)}\Big(\frac{s^3}{3} - \frac{\varepsilon s^2}{2}\Big) + \frac{v_0}{a\varepsilon}\Big(\frac{s^3}{3} - (a+\varepsilon)\frac{s^2}{s} + a\varepsilon s\Big). \end{aligned}$$

Picking

$$a = \frac{\varepsilon}{2}, \quad \text{and} \quad y = \Big(e_1 - x_0 - \frac{\varepsilon}{6}v_0\Big)\frac{3}{2\varepsilon},$$

we note that $\tilde{x}_1(\varepsilon) = e_1$. Furthermore, we obtain the following expressions for $s \in [0, \varepsilon]$:

$$\begin{aligned}\tilde{x}_1(s) &= x_0 - \Big(e_1 - x_0 - \frac{\varepsilon}{6}v_0\Big)\Big(2\frac{s^3}{\varepsilon^3} - 3\frac{s^2}{\varepsilon^2}\Big) + \frac{2v_0}{\varepsilon^2}\Big(\frac{s^3}{3} - \frac{3}{4}\varepsilon s^2\Big) + v_0 s,\\ \tilde{v}_1(s) &= -\Big(e_1 - x_0 - \frac{\varepsilon}{6}v_0\Big)\frac{6}{\varepsilon}\Big(\frac{s^2}{\varepsilon^2} - \frac{s}{\varepsilon}\Big) + 2v_0\Big(\frac{s^2}{\varepsilon^2} - \frac{3}{2}\frac{s}{\varepsilon}\Big) + v_0.\end{aligned} \tag{A.7}$$

It is important to point out that the choice of $\tilde{x}_1$ in (A.7) satisfies that $\tilde{x}_1(s) \neq 0$ for $s \in [0, \varepsilon]$. To see this, we recast $\tilde{x}_1$ as follows:

$$\tilde{x}_1(s) = x_0\Big[1 - \Big(3\frac{s^2}{\varepsilon^2} - 2\frac{s^3}{\varepsilon^3}\Big)\Big] + e_1\Big[3\frac{s^2}{\varepsilon^2} - 2\frac{s^3}{\varepsilon^3}\Big] + \varepsilon v_0\Big(\frac{s^3}{\varepsilon^3} - 2\frac{s^2}{\varepsilon^2} + \frac{s}{\varepsilon}\Big). \tag{A.8}$$

On the one hand, since $s \in [0, \varepsilon]$ and $|v_0| < R$, it holds that

$$\Big|\varepsilon v_0\Big(\frac{s^3}{\varepsilon^3} - 2\frac{s^2}{\varepsilon^2} + \frac{s}{\varepsilon}\Big)\Big| \leq 4\varepsilon R.$$

On the other hand, since in this case x_0 is not a multiple of e_1, we have

$$\begin{aligned}&\min_{s\in[0,\varepsilon]}\Big|x_0\Big[1 - \Big(3\frac{s^2}{\varepsilon^2} - 2\frac{s^3}{\varepsilon^3}\Big)\Big] + e_1\Big[3\frac{s^2}{\varepsilon^2} - 2\frac{s^3}{\varepsilon^3}\Big]\Big|\\ &\quad= \min_{r\in[0,1]}|x_0(1-r) + e_1 r| = c > 0,\end{aligned}$$

for some positive constant $c = c(x_0, e_1)$ independent of ε. Taking ε sufficiently small produces the bound

$$|\tilde{x}_1(s)| \geq c - 4\varepsilon R > 0,$$

implying that $\tilde{x}_1(s) \neq 0$ for $0 \leq s \leq \varepsilon$.

Next, let $\psi : \mathbb{R} \to \mathbb{R}$ be an arbitrarily smooth cutoff function satisfying

$$\psi_1(t) = \begin{cases}\varepsilon, & t \leq 0,\\ \text{monotone}, & 0 \leq t \leq 1,\\ 1, & t \geq 1.\end{cases} \tag{A.9}$$

On the interval $[\varepsilon, t]$, we set

$$\begin{aligned}\tilde{x}_1(s) &= \begin{cases}\Big(1 + \varepsilon - \psi_1\big(\frac{s}{\varepsilon} - 1\big)\Big)e_1, & \varepsilon \leq s \leq 2\varepsilon,\\ \varepsilon e_1, & 2\varepsilon \leq s \leq t - \varepsilon,\\ \psi_1\big(\frac{s-t+\varepsilon}{\varepsilon}\big)e_1, & t - \varepsilon \leq s \leq t.\end{cases}\\ \tilde{v}_1(s) &= \tilde{x}_1'(s).\end{aligned} \tag{A.10}$$

From (A.7)–(A.10), we see that $\tilde{x}_1 \in C^1([0, t]; \mathbb{R})$ and $\tilde{v}_1 \in C([0, t]; \mathbb{R})$ satisfy (A.5).

Next, with regard to the condition (A.6), we employ expression (A.8) to see that

$$\int_0^\varepsilon |\tilde{x}_1(s)|^{p_2}\mathrm{d}s \le C\varepsilon R^{p_2},$$

for some positive constant C independent of ε and R. Also, from (A.10), we have

$$\int_\varepsilon^t |\tilde{x}_1(s)|^{p_2}\mathrm{d}s = \Big\{\int_\varepsilon^{2\varepsilon} + \int_{2\varepsilon}^{t-\varepsilon} + \int_{t-\varepsilon}^t\Big\}|\tilde{x}_1(s)|^{p_2}\mathrm{d}s \le \varepsilon + t\varepsilon^{p_2}\varepsilon \le 3\varepsilon t.$$

It follows that

$$\int_0^t |\tilde{x}_1(s)|^{p_2}\mathrm{d}s \le C\varepsilon t R^{p_2}.$$

By shrinking ε further to zero if necessary, the above estimate produces (A.6), as claimed.

Turning back to the control problem (4.29), it remains to pick a suitable control process Γ so as to ensure that the triple $(\tilde{x}_1, \tilde{v}_1, \Gamma)$ satisfies (4.29). To this end, let Γ be defined as

$$\Gamma(s) = \tilde{v}_1(s) - v_0 + \int_0^s \tilde{v}_1(\ell) + \nabla U(\tilde{x}_1(\ell)) + \nabla G(\tilde{x}_1(\ell))\mathrm{d}\ell. \tag{A.11}$$

It is clear that $(\tilde{x}_1, \tilde{v}_1, \Gamma)$ solves (4.29) on the time interval $[0, t]$, thereby finishing the proof in Case 1 .

Case 2 : x_0 is a multiple of e_1. There are two sub-cases depending on the dimension d.

Case 2a: dimension $d = 1$. Since in this case the initial condition satisfies $x_0 > 0$, we may employ the same argument as in Case 1 to obtain the triple $(\tilde{x}_1, \tilde{v}_1, \Gamma)$ satisfying (4.29) as well as the conditions (A.5) and (A.6).

Case 2b: dimension $d \ge 2$. In this case, denoting $e_2 = (0, 1, 0, \dots, 0)$, we observe that x_0 is not a multiple of e_2. Following the same procedure for (A.7), we first drive (x_0, v_0) at time 0 to $(e_2, 0)$ at time ε. That is, we obtain the following path:

$$\begin{aligned}\tilde{x}_1(s) &= x_0 - \Big(e_2 - x_0 - \frac{\varepsilon}{6}v_0\Big)\Big(2\frac{s^3}{\varepsilon^3} - 3\frac{s^2}{\varepsilon^2}\Big) + \frac{2v_0}{\varepsilon^2}\Big(\frac{s^3}{3} - \frac{3}{4}\varepsilon s^2\Big) + v_0 s,\\ \tilde{v}_1(s) &= -\Big(e_2 - x_0 - \frac{\varepsilon}{6}v_0\Big)\frac{6}{\varepsilon}\Big(\frac{s^2}{\varepsilon^2} - \frac{s}{\varepsilon}\Big) + 2v_0\Big(\frac{s^2}{\varepsilon^2} - \frac{3}{2}\frac{s}{\varepsilon}\Big) + v_0, \quad 0 \le s \le \varepsilon,\end{aligned} \tag{A.12}$$

satisfying the boundary condition

$$\begin{aligned}\tilde{x}_1(0) &= x_0, \quad \tilde{x}_1(\varepsilon) = e_2,\\ \tilde{v}_1(0) &= v_0, \quad \tilde{v}_1(\varepsilon) = 0.\end{aligned}$$

On the interval $[\varepsilon, 2\varepsilon]$, denoting

$$\psi_2(s) = \frac{1}{2}\Big[\cos\Big(\pi\frac{s}{\varepsilon}\Big) + 1\Big], \quad \varepsilon \le s \le 2\varepsilon,$$

we choose

$$\begin{aligned} \tilde{x}_1(s) &= \big(\psi_2(s), 1 - \psi_2(s), 0\ldots, 0\big), \\ \tilde{v}_1(s) &= \tilde{x}_1'(s), \quad \varepsilon \le s \le 2\varepsilon. \end{aligned} \tag{A.13}$$

Observe that with this choice,

$$\tilde{x}_1(2\varepsilon) = e_1, \quad \tilde{v}_1(2\varepsilon) = 0.$$

In other words, $(\tilde{x}_1, \tilde{v}_1)$ drives $(e_2, 0)$ at time ε to $(e_1, 0)$ at time 2ε. Next, concerning the interval $[2\varepsilon, t]$, similar to (A.10), we obtain the following expression:

$$\begin{aligned} \tilde{x}_1(s) &= \begin{cases} \Big(1 + \varepsilon - \psi_1\big(\frac{s}{\varepsilon} - 2\big)\Big)e_1, & 2\varepsilon \le s \le 3\varepsilon, \\ \varepsilon e_1, & 3\varepsilon \le s \le t - \varepsilon, \\ \psi_1\big(\frac{s-t+\varepsilon}{\varepsilon}\big)e_1, & t - \varepsilon \le s \le t, \end{cases} \\ \tilde{v}_1(s) &= \tilde{x}_1'(s), \end{aligned} \tag{A.14}$$

where ψ_1 is as in (A.9).

Now, from (A.12), (A.13) and (A.14), it is not difficult to verify that $(\tilde{x}_1, \tilde{v}_1)$ satisfies the estimate (A.6) as well as the boundary condition (A.5). Lastly, by choosing Γ as in (A.11), we obtain the triple $(\tilde{x}_1, \tilde{v}_1, \Gamma)$ solving (4.29), thereby finishing the construction in Case 2 . The proof is thus complete. □

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Declarations

Conflict of interest The authors declare no conflict of interest.

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