

# A Reduced Model for Flame-Flow Interaction

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## Abstract

The paper is concerned with an extension of the classical relation between the flame speed and the curvature-flow stretch, valid only for high Lewis numbers (diffusively stable flames). At low Lewis numbers the corresponding flame-flow system suffers short-wavelength instability, making the associated initial value problem ill-posed. In this study the difficulty is resolved by incorporation of higher-order effects. As a result one ends up with a reduced model based on a coupled system of second-order dynamic equations for the flame interface and its temperature. As an illustration the new model is applied for description of diffusively unstable stagnation-point flow flames.

**AMS subject classification:** 80A25, 35K57, 35B40

**Keywords:** premixed flames, instabilities in flames, flame stretch

## 1 Introduction

In premixed gas combustion it is often reasonable to view a propagating flame as an interface convected by the underlying flow field while advancing towards the unburned gas at a normal velocity  $v_F$ . In the constant density limit when the flow-field  $\mathbf{u}(\mathbf{x}, t)$  is prescribed (passive propagation) the flame dynamics is governed by the equation,

$$v_n - \mathbf{u} \cdot \mathbf{N} = v_F, \quad \nabla \cdot \mathbf{u} = 0 \quad (1.1)$$

Here  $v_n$  is the normal velocity of the interface relative to the laboratory frame of reference,  $\mathbf{N}$  is the unit normal to the interface directed towards the unburned gas, and  $v_F$  is the normal velocity of the interface relative to a local fluid element;  $v_F$  depends on the local curvature of the interface as well as the local strain field.

For weakly curved flames evolving through large-scale flow fields [1], [2],

$$v_F = v_L(1 - l_M \mathcal{K}), \quad (1.2)$$

where  $v_L$  represents the laminar flame velocity in the absence of strain and curvature effects.  $\mathcal{K}$  is the stretch, the combined effect of curvature and strain. The relation specifying the stretch may be written as [2],

$$v_L \mathcal{K} = -v_L \varkappa - \mathbf{N} \cdot \mathbf{E} \cdot \mathbf{N}. \quad (1.3)$$

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Here  $\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$  is the rate of strain tensor evaluated at the interface,  $\varkappa = -\nabla \cdot \mathbf{N}$  is the mean curvature of the interface and  $l_M$  is the Markstein length whose value depends on physico-chemical parameters of the system. In the constant density, high activation energy, near-equidiffusive approximation [3],

$$l_M = l_{th}(1 + \frac{1}{2}\beta(\text{Le} - 1)) \quad (1.4)$$

where  $l_{th}$  is the planar flame thickness;  $\beta$  and  $\text{Le}$  are the Zeldovich and Lewis numbers, respectively. Eqs. (1.1)-(1.3) define a second-order differential equation. For positive  $l_M$  (high Lewis numbers) the short-wavelength disturbances decay exponentially ensuring the well-posedness of the associated initial-value problem. However, if  $l_M$  is negative (small Lewis numbers) the short-wavelength disturbances undergo exponential growth, making the initial-value problem ill-posed, and the whole model incapable of securing a sensible dynamical picture.

The short-wavelength instability is a trace of the well-known flame cellularity occurring at low Lewis numbers. An adequate description of cellular flames requires incorporation of higher-order effects absent in the first-order model based on Eqs. (1.1)-(1.3). Such higher-order models have long been developed for the flames evolving in quiescent mixtures. Here, depending on the asymptotic strategy adopted, one ends up either with a single fourth-order equation for the flame-interface [4], [5] or with a system of second-order equations for the flame interface and its temperature [6],[7]. The objective of the present work is an extension of the existing higher-order models to incorporate effects due to the background flow-fields, which, to our knowledge, has not previously been undertaken. The present paper is a development of our recent study on the subject [8], restricted to the 2D geometry, over the general 3D situation. Moreover, in [8] the problem was tackled heuristically as a geometrically invariant extrapolation of the results obtained for freely evolving flames. In the present paper the problem is solved in a more rational manner by means of an appropriate gradient expansion in the intrinsic coordinates.

The paper is organized as follows. In Section 2 we present basic equations for the flame dynamics in the constant density approximation. In Section 3 we derive a reduced model for the flame interface evolution and discuss basic properties of the system. Section 4 deals with the numerical simulation. In Appendix we present some relations from the differential geometry.

## 2 Formulation

As a starting point we adopt the conventional constant density, high activation energy, reaction-diffusion-advection formulation where the reaction rate is modeled by the surface  $\delta$ -function,  $\delta_\sigma$ . In suitably chosen units, the corresponding system of equations reads,

$$\begin{aligned} T_t + \mathbf{U} \cdot \nabla T &= \Delta T + \exp(\frac{1}{2}\beta(T - 1))\delta_\sigma \\ C_t + \mathbf{U} \cdot \nabla C &= \text{Le}^{-1}\Delta C - \exp(\frac{1}{2}\beta(T - 1))\delta_\sigma \end{aligned} \quad (2.1)$$

Here  $T$  and  $C$  are the appropriately scaled temperature and the deficient reactant concentration. In the constant density approximation the flow field  $\mathbf{U}$  is incompressible, that is,

$$\nabla \cdot \mathbf{U} = 0 \quad (2.2)$$

Far ahead of the interface  $T = 0$  and  $C = 1$ . Far behind the temperature approaches the adiabatic temperature of combustion products  $T = 1$ ;  $C = 0$  on and behind the interface. That is,

$$T(\boldsymbol{\xi}, n, t) \rightarrow 0, \quad C(\boldsymbol{\xi}, n, t) \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

$$\begin{aligned} T(\boldsymbol{\xi}, n, t) &\rightarrow 1 \quad \text{as } n \rightarrow -\infty \\ C(\boldsymbol{\xi}, n, t) &= 0 \quad \text{at } n \leq 0 \end{aligned} \quad (2.3)$$

Here  $n$  stands for the distance from the interface in its normal direction and  $\boldsymbol{\xi} = (\xi_1, \xi_2)$  are local coordinates on the interface.

For further discussion it is convenient to work with a transformed version of the system (2.1) based on the scaled enthalpy,

$$H = T + C, \quad (2.4)$$

rather than on the concentration.

If initially  $H = 1$  and  $\text{Le} = 1$ , then  $H = 1$  for all times. In this paper we consider the near-equidiffusive limit when  $\text{Le}$  is close unity and  $\beta$  is large, but  $\alpha = \frac{1}{2}\beta(1 - \text{Le})$  is finite. In this case  $T$  and  $H$  can be sought as power series of  $2/\beta$ :

$$\begin{aligned} T &= T_0 + \left(\frac{2}{\beta}\right) T_1 + \dots \\ H &= H_0 + \left(\frac{2}{\beta}\right) H_1 + \dots \end{aligned} \quad (2.5)$$

Substituting the ansatz (2.5) into (2.1), for the leading order approximation the original system (2.1) reduces to [3],

$$\begin{aligned} T_t + \mathbf{U} \cdot \nabla T &= \Delta T + \exp(\Theta)\delta_\sigma, \\ S_t + \mathbf{U} \cdot \nabla S &= \Delta(S - \alpha T), \end{aligned} \quad (2.6)$$

Here  $T = T_0$  and  $S = H_1$ .  $\Theta$  is  $S$  evaluated at the interface, and may be viewed as a scaled excess temperature of the interface.

The boundary conditions for the system (2.6) read,

$$\begin{aligned} T(\boldsymbol{\xi}, n, t) &\rightarrow 0, \quad S(\boldsymbol{\xi}, n, t) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \\ T(\boldsymbol{\xi}, n, t) &= 1 \quad \text{and } S(\boldsymbol{\xi}, n, t) \text{ grows no faster than polynomially for all } n \leq 0. \end{aligned} \quad (2.7)$$

Note that the system (2.6) is identical to the following formulation,

$$\begin{aligned} T_t + \mathbf{U} \cdot \nabla T &= \Delta T \\ S_t + \mathbf{U} \cdot \nabla S &= \Delta(S - \alpha T), \end{aligned} \quad (2.8)$$

involving jump and continuity conditions on the interface rather than the surface  $\delta$ -function.

$$\begin{aligned} [\partial_n S(\boldsymbol{\xi}, 0, t)] &= \alpha [\partial_n T(\boldsymbol{\xi}, 0, t)] = -\alpha \exp(\Theta(\boldsymbol{\xi}, t)), \\ [S(\boldsymbol{\xi}, 0, t)] &= 0, \quad T(\boldsymbol{\xi}, 0, t) = 1, \quad \Theta(\boldsymbol{\xi}, t) = S(\boldsymbol{\xi}, 0, t). \end{aligned} \quad (2.9)$$

### 3 Derivation of the reduced model

In this section we derive an asymptotic model for the dynamics of the flame interface governed by the system (2.8), (2.9).

We start from geometrical settings. Let  $(\xi_1, \xi_2, n)$  be an orthogonal system of curvilinear coordinates, and define the position vector (Fig.3.1),

$$\mathbf{r} = \mathbf{R}(\xi_1, \xi_2, t) + n\mathbf{N}(\xi_1, \xi_2, t) \quad (3.1)$$

Here  $\xi_i$  are local coordinates on the interface aligned with the principle directions of curvature, and  $n$  is the distance from the interface.  $\mathbf{N}$  is the unit normal to the interface and  $t$  is time. Thus the interface is described by  $\mathbf{r}$  evaluated at  $n = 0$ , that is by  $\mathbf{R}$ .

Next we introduce an orthonormal basis  $(\mathbf{N}, \mathbf{e}_1, \mathbf{e}_2)$  where  $\mathbf{e}_i$  are defined as,

$$\mathbf{e}_i = \frac{1}{a_i} \frac{\partial \mathbf{R}}{\partial \xi_i}, \quad a_i = \left\| \frac{\partial \mathbf{R}}{\partial \xi_i} \right\| \quad (3.2)$$

We assume that the interface is weakly distorted and thus the mean curvature  $k$  is small (say of

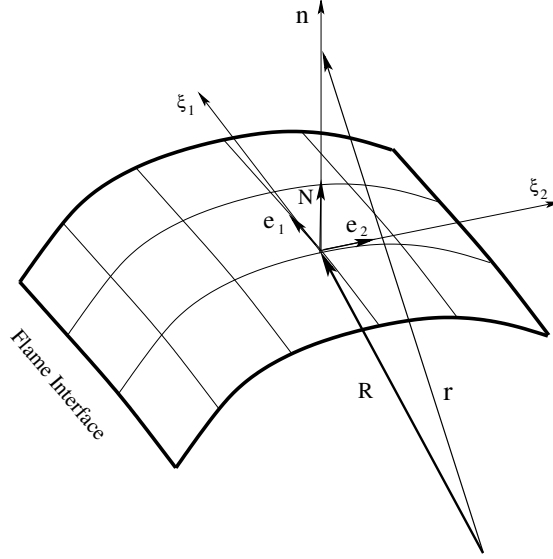


Figure 3.1: Parameterization of the flame interface

the order of  $\varepsilon$  where  $\varepsilon \ll 1$ ), and the flow is slowly varying,

$$\mathbf{U} = \mathbf{U}(\varepsilon t, \varepsilon n, \varepsilon \boldsymbol{\xi}) \quad (3.3)$$

This provides the natural scaling,

$$k = \varepsilon \kappa, \quad \mathbf{s} = \varepsilon \boldsymbol{\xi}, \quad \tau = \varepsilon t \quad (3.4)$$

which allows to expand the differential operators of the model (2.8),(2.9) in power series of  $\varepsilon$ .

We also assume that the impact of terms involving powers of curvature and high order derivatives of the flow-field on the overall dynamics picture is not important and these terms may be ignored. This assumption is based on consideration of certain special flame-flow configurations near the stability threshold [4],[9], [10]. With these premises the differential operators assume the following form (see Appendix for details).

The convective time derivative, taking into account incompressibility condition, reads,

$$\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \approx (U_n - V_n) \frac{\partial}{\partial n} + \varepsilon n (\kappa \bar{U}_n - \bar{\nabla}_\sigma \cdot \bar{\mathbf{U}}_\perp) \frac{\partial}{\partial n} + \varepsilon \mathcal{D}_\tau, \quad (3.5)$$

where

$$\mathcal{D}_\tau = \frac{\partial}{\partial \tau} + (\bar{\mathbf{V}}_\perp + \bar{\mathbf{U}}_\perp) \cdot \bar{\nabla}_\sigma, \quad (3.6)$$

$$\bar{\nabla}_\sigma = \frac{\mathbf{e}_1}{a_1} \frac{\partial}{\partial s_1} + \frac{\mathbf{e}_2}{a_2} \frac{\partial}{\partial s_2}, \quad (3.7)$$

and

$$\kappa = -\bar{\nabla}_\sigma \cdot \mathbf{N}. \quad (3.8)$$

The Laplace operator,

$$\Delta \approx \frac{\partial^2}{\partial n^2} - \varepsilon \kappa \frac{\partial}{\partial n} + \varepsilon^2 \bar{\Delta}_\sigma, \quad (3.9)$$

where

$$\bar{\Delta}_\sigma \Phi = \bar{\nabla}_\sigma \cdot \bar{\nabla}_\sigma \Phi = \frac{1}{a_1 a_2} \left( \frac{\partial}{\partial s_1} \left( \frac{a_2}{a_1} \frac{\partial}{\partial s_1} \Phi \right) + \frac{\partial}{\partial s_2} \left( \frac{a_1}{a_2} \frac{\partial}{\partial s_2} \Phi \right) \right). \quad (3.10)$$

Here  $V_n = \mathbf{R}_\tau \cdot \mathbf{N}$  and  $\bar{\mathbf{V}}_\perp = -(\mathbf{R}_\tau \cdot \mathbf{e}_1)\mathbf{e}_1 - (\mathbf{R}_\tau \cdot \mathbf{e}_2)\mathbf{e}_2$  are normal and tangent components of the velocity of the interface.  $\mathcal{L}_\tau = \partial_\tau + \bar{\mathbf{V}}_\perp \cdot \nabla_S$  is the Lagrangian normal time derivative.  $\bar{U}_n$ ,  $\bar{\mathbf{U}}_\perp$  are normal and tangent component of the velocity of the flow velocity at the interface. That is  $\bar{\mathbf{U}}(\tau, \mathbf{s}) = \mathbf{U}(\tau, n=0, \mathbf{s})$  and  $\bar{\nabla}_\sigma$ ,  $\bar{\Delta}_\sigma$  are the surface gradient and Laplace-Beltrami operators.

It is important to note that the last term  $\mathcal{D}_\tau$  in (3.5) is kept for regularization purposes. A similar role is played by the term  $\bar{\Delta}_\sigma$  in the expansion of the Laplace operator  $\Delta$ , Eq.(3.9).

The problem (2.8),(2.9) then can be solved by iterations. We expand the temperature, enthalpy and velocity in power series of  $\varepsilon$ :

$$\begin{aligned} T &= T^0 + \varepsilon T^1 + \dots, & S &= S^0 + \varepsilon S^1 + \dots, \\ V_n &= V_n^0 + \varepsilon V_n^1 + \dots, & \Theta &= \Theta^0 + \varepsilon \Theta^1 + \dots \end{aligned} \quad (3.11)$$

The problem for the zeroth approximation (basic solution) then can be written as follows,

$$\begin{aligned} \partial_n^2 T^0 + (V_n^0 - \bar{U}_n) \partial_n T^0 &= 0, \\ \partial_n^2 S^0 + (V_n^0 - \bar{U}_n) \partial_n S^0 &= \alpha \partial_n^2 T^0, \end{aligned} \quad (3.12)$$

with boundary, jump and continuity conditions,

$$\begin{aligned} T^0(\mathbf{s}, n, \tau) &\rightarrow 0, & S^0(\mathbf{s}, n, \tau) &\rightarrow 0 & n &\rightarrow \infty, \\ T^0(\mathbf{s}, n, \tau) &= 1, & S^0(\mathbf{s}, n, \tau) & \text{ grows no faster than polynomially for all } & n &\leq 0 \\ [\partial_n S^0(\mathbf{s}, 0, \tau)] &= \alpha [\partial_n T^0(\mathbf{s}, 0, \tau)] = -\alpha, & [S^0(\mathbf{s}, 0, \tau)] &= 0 \end{aligned} \quad (3.13)$$

The associated solution reads,

$$\begin{aligned} T^0 &= e^{-n} \quad \text{for } n > 0 & V_n^0 &= 1 + \bar{U}_n, \\ S^0 &= -\alpha n e^{-n} \quad \text{for } n > 0, & S^0 &= 0, & T^0 &= 1 \quad \text{for } n \leq 0 \end{aligned} \quad (3.14)$$

Higher order terms are obtained iteratively from the system of linear equations,

$$\begin{aligned} \partial_n^2 T^m + \partial_n T^m &= F_m(T^{m-1}, \dots, T^0, S^{m-1}, \dots, S^0, n, \mathbf{s}, \tau) \\ \partial_n^2 S^m + \partial_n S^m &= \alpha \partial_n^2 T^m + G_m(T^{m-1}, \dots, T^0, S^{m-1}, \dots, S^0, n, \mathbf{s}, \tau) \end{aligned} \quad (3.15)$$

Here functions  $F_m$ ,  $G_m$  are obtained by substituting the ansatz (3.11) into the system (2.8) and equating terms with similar powers of  $\varepsilon$ .

The equation for the evolution of the normal velocity  $V_n$  is obtained using the basic solution (3.14). Indeed, since  $T^0$  and  $S^0$  are independent of  $\mathbf{s}$  and  $t$ , one obtains

$$\partial_n^2 T + (V_n - \bar{U}_n) \partial_n T = \varepsilon (\kappa + (\kappa \bar{U}_n - \bar{\nabla}_\sigma \cdot \bar{\mathbf{U}}_\perp) n) \partial_n T^0 + O(\varepsilon^2) \quad (3.16)$$

Integration of (3.16) from 0 to  $\infty$  yields,

$$\begin{aligned} V_n &= e^\Theta + \bar{U}_n - \varepsilon K + O(\varepsilon^2) \\ K &= -(1 + \bar{U}_n)\kappa + \bar{\nabla}_\sigma \cdot \bar{\mathbf{U}}_\perp \end{aligned} \quad (3.17)$$

The derivation of the equation for  $\Theta$  is more involved but still straightforward. It requires solving the system (3.15) for the first and second approximation. Upon matching corresponding solutions on the interface, using jump and continuity conditions (2.9), one obtains,

$$\varepsilon \mathcal{D}_\tau \Theta = \varepsilon^2 \bar{\Delta}_\sigma \Theta + \frac{\alpha}{4} K - \frac{1}{4} \Theta e^\Theta \quad (3.18)$$

It is convenient to return to the original unscaled coordinates, which is formally equivalent to setting  $\varepsilon = 1$ . The resulting system of governing equations then becomes,

$$\begin{aligned} V_n &= U_n + e^\Theta - K \\ \mathcal{D}_t \Theta &= \bar{\Delta}_\xi \Theta + \frac{\alpha}{4} K - \frac{1}{4} \Theta e^\Theta \end{aligned} \quad (3.19)$$

with

$$\begin{aligned} K &= -(1 + \bar{U}_n)k + \bar{\nabla}_\xi \cdot \mathbf{U}_\perp \\ \mathcal{D}_t &= \partial_t + (\bar{\mathbf{U}}_\perp + \bar{\mathbf{V}}_\perp) \cdot \bar{\nabla}_\xi \end{aligned} \quad (3.20)$$

As shown in the Appendix the tangential (induced) velocity field  $\bar{\mathbf{V}}_\perp$  is determined from the separate problem,

$$\begin{aligned} \bar{\nabla}_\xi \cdot \bar{\mathbf{V}}_\perp &= -kV_n \\ \mathbf{e}_1 \cdot \bar{\nabla}_\xi \bar{\mathbf{V}}_\perp \cdot \mathbf{e}_2 + \mathbf{e}_2 \cdot \bar{\nabla}_\xi \bar{\mathbf{V}}_\perp \cdot \mathbf{e}_1 &= 0 \end{aligned} \quad (3.21)$$

with  $\mathbf{V}_\perp = 0$  on the lateral boundary of the interface. Here  $\mathbf{e}_1$  and  $\mathbf{e}_2$  is a basis on the interface associated with the local coordinates  $(s_1, s_2)$ .

In the case of the two dimensional problem the system (3.19) becomes

$$\begin{aligned} V_n &= \bar{U}_n + e^\Theta - \tilde{K} \\ \tilde{\mathcal{D}}_t \Theta &= \partial_s^2 \Theta + \frac{\alpha}{4} \tilde{K} - \frac{1}{4} \Theta e^\Theta \\ \partial_s V_s &= -\tilde{k}V_n \end{aligned} \quad (3.22)$$

with

$$\begin{aligned} \tilde{K} &= -(1 + \bar{U}_n)\tilde{k} + \partial_s \bar{U}_s \\ \tilde{\mathcal{D}}_t &= \partial_t + (\bar{U}_s + V_s)\partial_s \end{aligned} \quad (3.23)$$

Here  $\tilde{k}$  is the curvature and  $s$  is the arc-length.  $\bar{U}_s, V_s = ds/dt$  are tangential components of the external flow (computed at the interface) and the induced flow.

## 4 Cellular flames in the stagnation-point flow

As an illustration, in this section, the model (3.22) is employed to describe the classical system of the diffusively unstable flame ( $\alpha > 1$ ) held in the stagnation-point flow  $\mathbf{U}(x, y) = (q(x - L/2), -qy)$

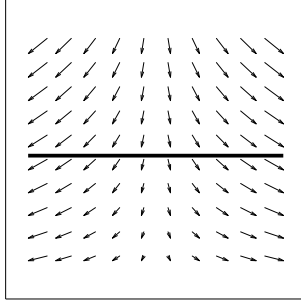


Figure 4.1: Diagram of premixed planar flame (bold line) stabilized in stagnation-point flow field. The upstream flow corresponds to the fresh mixture, the downstream to the burned gas.

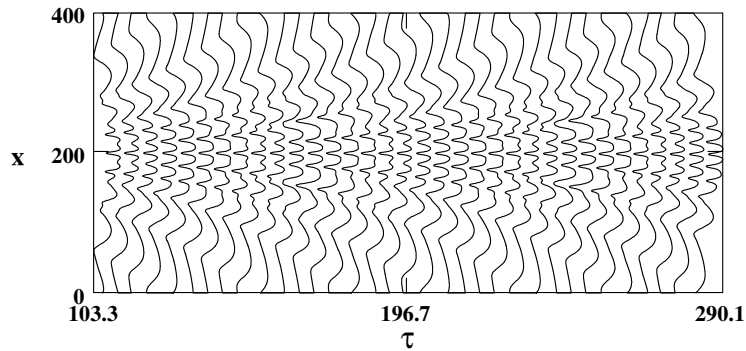


Figure 4.2: Temporal evolution of the cellular flame.

[9][11][12] (Fig. 4.1). Here  $q$  is the appropriately scaled flow intensity. The system is considered over half-strip ( $0 < x < L$ ,  $y < 0$ ) with the zero-gradient boundary conditions for the flame interface  $y = F(x, \tau)$  and its excess temperature  $\Theta(x, \tau)$ ,

$$F_x(0, \tau) = \Theta_x(0, \tau) = 0, \quad F_x(L, \tau) = \Theta_x(L, \tau) = 0. \quad (4.1)$$

The problem is solved numerically for  $\alpha = 2.5$ ,  $L = 400$ ,  $0.01 \leq q \leq 0.02$ . It is found, as expected [12], that for sufficiently high flow intensity ( $q = 0.02$ ) the flame interface is planar and located near  $y = -1/q$ . However, if  $q$  is small enough ( $q = 0.01$ ) the central part of the flame becomes cellular with the cells in a state of permanent self-motion similar to that occurring in freely propagating cellular flames.

Figure 4.2 shows the temporal evolution of the cellular flame at  $q = 0.014$ , Figure 4.3 an instantaneous profiles of the flame interface and Figure 4.4 its temperature.

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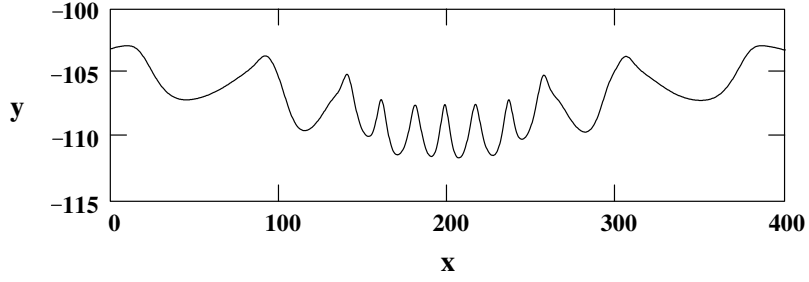


Figure 4.3: An instantaneous profile of the flame interface.

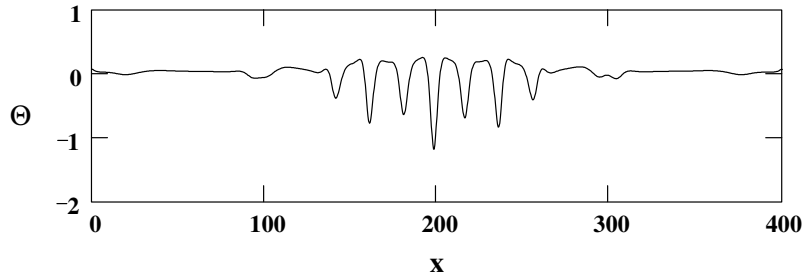


Figure 4.4: Temperature distribution for the profile in Fig.4.3

## Appendix A Expressions for differential operators

In this section we recall some results from the differential geometry Derivation of the basic formulas presented here can be found in [13], [14] and [15].

Let  $(\xi_1, \xi_2, n)$  be an orthogonal system of curvilinear coordinates, and define the position vector,

$$\mathbf{r} = \mathbf{R}(\xi_1, \xi_2, t) + n\mathbf{N}(\xi_1, \xi_2, t) \quad (\text{A.1})$$

In this setting  $\xi_i$  are local coordinates on the interface and  $n$  is the distance from the interface.  $\mathbf{N}$  is the unit normal to the interface and  $t$  is time. Thus, the interface is described by  $\mathbf{r}$  evaluated at  $n = 0$ , that is by  $\mathbf{R}$ . We introduce an orthogonal basis  $(\mathbf{N}, \mathbf{e}_1, \mathbf{e}_2)$  where  $\mathbf{e}_i$  are defined as,

$$\mathbf{e}_i = \frac{1}{a_i} \frac{\partial \mathbf{R}}{\partial \xi_i}, \quad a_i = \left\| \frac{\partial \mathbf{R}}{\partial \xi_i} \right\| \quad (\text{A.2})$$

*Surface gradient and Laplace-Beltrami operator*

Define the gradient

$$\nabla = \mathbf{N} \frac{\partial}{\partial n} + \nabla_\xi, \quad \nabla_\xi = \frac{\mathbf{e}_1}{l_1} \frac{\partial}{\partial \xi_1} + \frac{\mathbf{e}_2}{l_2} \frac{\partial}{\partial \xi_2} \quad (\text{A.3})$$

where  $\nabla_\xi$  and  $l_i$  are the surface gradient and scaling factors,

$$l_i = \left\| \frac{\partial \mathbf{r}}{\partial \xi_i} \right\| = a_i(1 - nk_i), \quad i = 1, 2 \quad \text{and} \quad l_3 = \left\| \frac{\partial \mathbf{r}}{\partial n} \right\| = 1 \quad (\text{A.4})$$

The restriction of the surface gradient on the interface is defined as,

$$\bar{\nabla}_\xi = \frac{\mathbf{e}_1}{a_1} \frac{\partial}{\partial \xi_1} + \frac{\mathbf{e}_2}{a_2} \frac{\partial}{\partial \xi_2} \quad (\text{A.5})$$

The Laplace operator then reads,

$$\Delta = \frac{\partial^2}{\partial n^2} + (\nabla_\xi \cdot \mathbf{N}) \frac{\partial}{\partial n} + \Delta_\xi \quad (\text{A.6})$$

where

$$\Delta_\xi \Phi = \nabla_\xi \cdot \nabla_\xi \Phi = \frac{1}{l_1 l_2} \left( \frac{\partial}{\partial \xi_1} \left( \frac{l_2}{l_1} \frac{\partial}{\partial \xi_1} \Phi \right) + \frac{\partial}{\partial \xi_2} \left( \frac{l_1}{l_2} \frac{\partial}{\partial \xi_2} \Phi \right) \right) \quad (\text{A.7})$$

and

$$\nabla_\xi \cdot \mathbf{N} = - \left( \frac{k_1}{1 - nk_1} + \frac{k_2}{1 - nk_2} \right) \quad (\text{A.8})$$

Restrictions of the above operators on the interface are defined as,

$$\bar{\Delta} = \frac{\partial^2}{\partial n^2} + (\bar{\nabla}_\xi \cdot \mathbf{N}) \frac{\partial}{\partial n} + \bar{\Delta}_\xi, \quad (\text{A.9})$$

$$\bar{\Delta}_\xi \Phi = \bar{\nabla}_\xi \cdot \bar{\nabla}_\xi \Phi = \frac{1}{a_1 a_2} \left( \frac{\partial}{\partial \xi_1} \left( \frac{a_2}{a_1} \frac{\partial}{\partial \xi_1} \Phi \right) + \frac{\partial}{\partial \xi_2} \left( \frac{a_1}{a_2} \frac{\partial}{\partial \xi_2} \Phi \right) \right), \quad (\text{A.10})$$

$$\bar{\nabla}_\xi \cdot \mathbf{N} = -(k_1 + k_2) = -k \quad (\text{A.11})$$

where  $k = k_1 + k_2$  is the twice mean curvature of the interface.

*Incompressibility condition*

Consider an incompressible vector field,

$$\mathbf{U}(n, \xi_1, \xi_2) = U_n \mathbf{N} + \mathbf{U}_\perp, \quad \nabla \cdot \mathbf{U} = 0 \quad (\text{A.12})$$

Then

$$\nabla \cdot \mathbf{U} = \frac{\partial}{\partial n} U_n + (\nabla_\xi \cdot \mathbf{N}) U_n + \nabla_\xi \cdot \mathbf{U}_\perp \quad (\text{A.13})$$

In particular, the incompressibility condition implies,

$$\frac{\partial}{\partial n} U_n = -((\nabla_\xi \cdot \mathbf{N}) U_n + \nabla_\xi \cdot \mathbf{U}_\perp) \quad (\text{A.14})$$

Moreover, for  $n = 0$  we have,

$$\frac{\partial}{\partial n} \bar{U}_n = k \bar{U}_n - \bar{\nabla}_\xi \cdot \bar{\mathbf{U}}_\perp \quad (\text{A.15})$$

where

$$\bar{\mathbf{U}}(\xi_1, \xi_2) = \mathbf{U}(n = 0, \xi_1, \xi_2), \quad (\text{A.16})$$

*Time derivative*

We define normal velocity of an interface ( $V_n$ ) and the restriction of the tangential velocity onto the interface ( $\bar{\mathbf{V}}_\perp$ ) as follows,

$$V_n = \frac{\partial \mathbf{R}}{\partial t} \cdot \mathbf{N}, \quad \bar{V}_{\perp, i} = -\frac{\partial \mathbf{R}}{\partial t} \cdot \mathbf{e}_i, \quad (\text{A.17})$$

In this setting the material time derivative of the scalar quantity  $\phi(n, \xi_1, \xi_2, t)$  takes the following form,

$$\frac{d\phi}{dt} = \mathcal{L}\phi - V_n \frac{\partial \phi}{\partial n} \quad (\text{A.18})$$

where

$$\mathcal{L}_t = \partial_t + \mathbf{V}_\perp \cdot \nabla_\xi \quad (\text{A.19})$$

is the Lagrangian normal time derivative, that is

$$\mathcal{L}_t\phi = \lim_{\Delta t \rightarrow 0} \frac{\phi(n, \xi_1 + \Delta\xi_1, \xi_2 + \Delta\xi_2, t + \Delta t) - \phi(n, \xi_1, \xi_2, t)}{\Delta t} \quad (\text{A.20})$$

*Equation for the induced tangential velocity*

It was shown in [14] that the restriction of the tangential velocity onto the interface  $\bar{\mathbf{V}}_{\perp}$  obeys the following system of equations,

$$\frac{1}{a_1 a_2} \left( \frac{\partial a_2 \bar{V}_{\perp,1}}{\partial \xi_1} + \frac{\partial a_1 \bar{V}_{\perp,2}}{\partial \xi_2} \right) = -k V_n \quad (\text{A.21})$$

$$\frac{a_1}{a_2} \frac{\partial}{\partial \xi_2} \left( \frac{\bar{V}_{\perp,1}}{a_1} \right) + \frac{a_2}{a_1} \frac{\partial}{\partial \xi_1} \left( \frac{\bar{V}_{\perp,2}}{a_2} \right) = 0 \quad (\text{A.22})$$

Equations (A.21) and (A.22) in the coordinate free form read,

$$\bar{\nabla}_{\xi} \cdot \bar{\mathbf{V}}_{\perp} = -k V_n \quad (\text{A.23})$$

$$\mathbf{e}_1 \cdot (\nabla_{\xi} \bar{\mathbf{V}}_{\perp}) \cdot \mathbf{e}_2 + \mathbf{e}_2 \cdot (\nabla_{\xi} \bar{\mathbf{V}}_{\perp}) \cdot \mathbf{e}_1 = 0 \quad (\text{A.24})$$

Therefore, the tangential (induced) flow  $\mathbf{V}_{\perp}$  can be obtained from Eqs.(A.23),(A.24) when  $V_n$  and  $k$  are regarded as prescribed. In addition since the interface is moving in the normal direction  $\bar{\mathbf{V}}_{\perp}$  must be zero at the lateral boundary of the interface.

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