Traveling waves in porous media combustion: uniqueness of waves for small thermal diffusivity

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May 30, 2006

Abstract

We study traveling waves solutions arising in Sivashinsky's model of subsonic detonation which describes combustion processes in inert porous media. Subsonic (shockless) detonation waves tend to assume the form of reaction front propagating with a well defined speed. It is known that traveling waves exist for any value of thermal diffusivity. Moreover, it has been shown that, when the thermal diffusivity is neglected, the traveling wave is unique. The question of whether the wave is unique in the presence of thermal diffusivity has remained open. For the subsonic regime, the underlying physics might suggest that the effect of small thermal diffusivity is insignificant. On the other hand, numerical evidence has been obtained recently which shows that presence of non-zero diffusivity, regardles of how small it is, can have a significant effect on the qualitative behaviour of the wave. We analytically resolve the issue of uniqueness of the wave in the presence of non-zero diffusivity through applying geometric singular perturbation theory.

1 Introduction

Gaseous detonation is one of the classical problems of combustion theory. In the past decade, there has been significant progress in understanding the phenomenon. The problem is, however, far from being completely resolved. Recently, Sivashinsky proposed a model of subsonic detonation that describes propagation of the combustion fronts in highly resistible media [8]. The assumption of high resistance of the media provides a natural simplification of the system of the governing equations while still preserving many of the qualitative features of the original system. In particular, the model is capable of describing the transition from deflagration to detonation that remains one of the major challenges in the combustion theory. The model reads [1]:

$$\begin{aligned} T_t - (1-\gamma^{-1})P_t &= \varepsilon T_{xx} + Y\Omega(T), \\ P_t - T_t &= P_{xx}, \\ Y_t &= \varepsilon \mathrm{Le}^{-1} Y_{xx} - \gamma Y \Omega(T). \end{aligned} \tag{1}$$

Here P, T and Y are the appropriately scaled pressure, temperature and concentration of the deficient reactant; $\gamma > 1$ is the specific heat ratio, ε is a ratio of molecular and pressure diffusivities,

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Le is a Lewis number and $Y\Omega(T)$ is the reaction rate. The first and third equations of the system (1) represent the partially linearized equations for the conservation of energy and deficient reactant, while the second one is a linearized continuity equation taking into account the equations of state and momentum (Darcy's law).

One of the most distinctive features of premixed combustion is its ability to form a reaction wave that assumes the shape of a sharp front propagating subsonically or supersonically at a well defined speed. Therefore, traveling wave solutions of the system (1) are indispensable for understanding the underling physical process. In this paper we are interested in the traveling wave solutions of (1), that is solutions of the form $T(x,t) = T(\xi)$, $P(x,t) = P(\xi)$, $Y(x,t) = Y(\xi)$ where $\xi = x - ct$ and c is the a priori unknown front speed. Substituting this ansatz into (1) we obtain a reduced system of ODE's

$$\begin{aligned} -cT' + c(1-\gamma^{-1})P' &= \varepsilon T'' + Y\Omega(T), \\ P'' &= c(T' - P'), \\ cY' + \varepsilon Y'' &= \gamma Y\Omega(T) \end{aligned} \tag{2}$$

with the front-like boundary conditions:

$$\begin{aligned} &P(-\infty) = 1, \quad T(-\infty) = 1, \quad Y(-\infty) = 0, \\ &T(+\infty) = 0, \quad P(+\infty) = 0, \quad Y(+\infty) = 1. \end{aligned} \tag{3}$$

We have set Le $= 1$ in (2) for simplicity. Note that, unlike the situation in some other thermo-diffusive systems, a Lewis number not equal to one would not change the results of this paper. We assume that the function $\Omega(T)$ is of the Arrhenius type with an ignition cut-off, that is, $\Omega(T)$ vanishes on an interval $[0, \Theta]$ and is positive for $T > \Theta$:

$$\Omega(T) = 0 \quad \text{for} \quad 0 \le T < \Theta < 1. \tag{4}$$

Moreover, $\Omega(T)$ is an increasing Lipschitz continuous function, except for a possible discontinuity at the ignition temperature $T = \Theta$. In what follows, we fix the translational invariance of the system by assuming that fronts in consideration reach the ignition temperature Θ at $\xi = 0$.

Most relevant for applications is the case when ε is small (for realistic materials ε varies in the range $\varepsilon \sim 10^{-2} - 10^{-5}$). This observation makes ε a natural small parameter of the problem. For ε small one may formally distinguish two separate regimes of propagation: deflagration associated with the small (order of $\sqrt{\varepsilon}$) thermal diffusivity and detonation associated with order one diffusion of pressure [8]. Setting $\varepsilon = 0$, that is ignoring the thermal diffusivity, is very attractive for studying subsonic detonation regime and is believed to reflect the correct phenomenon [8]. For the leading order asymptotics, the system of governing equations (2) reduces to the following one [8]

$$\begin{aligned} -cT' + c(1-\gamma^{-1})P' &= Y\Omega(T), \\ P'' &= c(T' - P'), \\ cY' &= \gamma Y\Omega(T). \end{aligned} \tag{5}$$

On the other hand, there exists numerical evidence that, in some limiting situations, the details of propagation of the detonation waves are very sensitive to the presence of nonzero thermal diffusivity [9],[6].

Solutions of the problem (5) are well understood. In particular, it is known that there is a unique value of $c = c_0$ for which the solution exists [4]. Moreover, as was shown in [5], solutions of the system (2) converge to that of (5) as $\varepsilon \to 0$. Uniqueness of solutions of the system (2) has not however yet been established. As we have mentioned above, the presence of non-zero thermal diffusivity can lead to a significant change in the qualitative behavior of the solutions. Therefore, it is of interest to see if uniqueness of the wave solution of (5) is robust under perturbation (2) with non-zero ε. We show here that the solution of the system (2) is, indeed, unique for small $\varepsilon > 0$. Our main result can be stated as follows:

Theorem 1 *There exist $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$, there is a unique value of c, depending on ε, for which system (2) has an orbit satisfying (3). Moreover the orbit is unique, and hence so is the traveling wave up to translation.*

The system of ordinary differential equations (2), governing the traveling wave solution, is a singular perturbation of (5) as the higher order derivatives are being added in (5). Thus it is natural to attempt to study the problem using the techniques of geometric singular perturbation theory. Singular perturbations can be studied using Fenichel's invariant manifold theory [3]. This approach allows us to show that the traveling wave of (5) perturbs to a unique wave of (2) for $\varepsilon > 0$ small enough.

The strategy of the proof of our main result can be described as follows. We first construct a smooth manifold M_0 on which the traveling wave of the unperturbed system (5) lives. This manifold is normally hyperbolic and therefore, by [2] for small enough $\varepsilon > 0$, perturbs to a unique invariant manifold M_ε of the system (2). It is worth mentioning that, because of the discontinuity of the nonlinear terms, generally speaking, M_0 can perturb to a discontinuous manifold. That significantly complicates further analysis. To overcome this difficulty, it helpful to notice that equivalent formulations of (2), although having the same set of solutions, do not yield the same M_0, when ε is set to be zero, and, consequently, M_ε. We find a system equivalent to (2) with a discontinuous vector field which nevertheless results in a smooth invariant manifold M_ε.

As a next step we show that in a neighborhood of M_ε no traveling wave can exist off M_ε. Here the continuity of M_ε is important. Restricting the flow to M_ε brings us to a lower dimensional problem. On the other hand, the smooth dependence of M_ε on ε ($0 \leq \varepsilon \ll 1$) allows us to extrapolate the information about the existence of a unique front on M_0 to M_ε.

More precisely, we extend the phase space of the system describing the flow on M_0 by adding a direction corresponding to the velocity c. Using properties obtained in [4] we show that the front is represented as a transversal intersection of two invariant manifolds suspended in $M_0 \times \{c \text{ near } c_0\}$. The intersection occurs at a unique value $c = c_0$. Upon switching on a sufficiently small $\varepsilon > 0$ the transversal intersection perturbs with a nearby c_ε replacing c_0, thus proving the existence of a, unique up to translation, front solution of (2).

The construction of the front using methods of geometric singular perturbation theory also implies that the ε-dependent family of fronts supported by (2) converge to the front of (5) as $\varepsilon \to 0$, thus providing a proof alternative to the one presented in [5] .

2 Construction of the orbit in $\varepsilon = 0$ case

It has been shown in [5] that there are no traveling wave solutions to the system (2) with $c = 0$. Therefore, in what follows, $c > 0$. We introduce a new variable

$$Q = \frac{1}{c} \int_{\xi}^{+\infty} Y \Omega(T) \, \mathrm{d}x.$$

We integrate each of the equations of (2) using the boundary conditions (3) at $+\infty$ to obtain an equivalent system:

$$\begin{aligned} Q' &= -c^{-1}Y\Omega(T), \\ P' &= c(T-P), \\ \varepsilon T' &= c(1-\gamma^{-1})P - cT + cQ, \\ \varepsilon Y' &= c(1-Y) - \gamma c Q. \end{aligned} \tag{6}$$

We call (6) the slow system as opposed to the fast system obtained from (6) in the fast scaling $\eta = \frac{1}{\varepsilon}\xi$

$$\begin{aligned} \dot{Q} &= -\varepsilon c^{-1}Y\Omega(T), \\ \dot{P} &= \varepsilon c(T-P), \\ \dot{T} &= c(1-\gamma^{-1})P - cT + cQ, \\ \dot{Y} &= c(1-Y) - \gamma c Q. \end{aligned} \tag{7}$$

The two systems (6) and (7) are equivalent when $\varepsilon > 0$. The problem amounts to finding values of c for which a certain heteroclinic orbit, in either of these systems, exists.

Satisfying the boundary conditions (3) determines the critical points at the ends of the desired heteroclinic orbit. We seek an orbit (Q,P,T,Y) such that

$$(Q,P,T,Y) \to (\gamma^{-1},1,1,0) \ \text{ at } \ -\infty, \tag{8}$$

and

$$(Q,P,T,Y) \to (0,0,0,1) \ \text{ at } \ +\infty. \tag{9}$$

When $\varepsilon = 0$, a reduced system can be derived formally as follows: In (6), the last two equations reduce to algebraic relations when $\varepsilon = 0$:

$$\begin{aligned} (1-\gamma^{-1})P - T + Q &= 0, \\ 1 - Y - \gamma Q &= 0, \end{aligned} \tag{10}$$

which, when inserted into the equations for P and Y, give the limiting slow equations:

$$\begin{aligned} Q' &= c^{-1}(\gamma Q - 1)\Omega(Q + (1-\gamma^{-1})P), \\ P' &= -c\gamma^{-1}P + cQ. \end{aligned} \tag{11}$$

From the results found in [5] we know that (11) possesses a heteroclinic orbit which connects the equilibrium at $(Q,P) = (\gamma^{-1},1)$ as $\xi \to -\infty$ to $(Q,P) = (0,0)$ as $\xi \to +\infty$. Moreover, the speed $c = c_0 > 0$ is unique at which such a heteroclinic exists. The linearization of (11) at $(\gamma^{-1},1)$ has eigenvalues $-\frac{c}{\gamma} < 0$ and $\frac{\gamma}{c}\Omega(1) > 0$, thus, is a saddle and so has a 1-dimensional unstable manifold (denoted W^u). At $(0,0)$ the eigenvalues are 0 and $-\frac{c}{\gamma}$. So, there is a 1-dimensional stable manifold W^s and a 1-dimensional center manifold W^c. The center manifold W^c is due to the presence of the curve of equilibria $P = \gamma Q$ which contains $(0,0)$. Since the equation is actually linear in a neighborhood of $(0,0)$, due to the ignition cut-off, the flow is easily analyzed. The heteroclinic solution is formed as an intersection of $W^u(\gamma^{-1},1)$ with $W^s(0,0)$. The intersection will be formed at some fixed value of c, say $c = c_0$.

Appending an equation for the parameter c to (11), we obtain:

$$\begin{aligned} Q' &= c^{-1}(\gamma Q - 1)\Omega(Q + (1-\gamma^{-1})P), \\ P' &= -c\gamma^{-1}P + cQ, && (12) \\ c' &= 0, && (13) \end{aligned}$$

which has as critical points: $(\gamma^{-1}, 1, c)$ and $(0, 0, c)$ for any c. We form a center-unstable manifold for $(\gamma^{-1}, 1, c)$, denoted W^{cu}. The situation at $(0, 0, c)$ is more complicated. We are not interested in the full center-stable manifold. Rather we form a stable manifold for an appropriate submanifold of the center manifold of $(0, 0, c)$ in (12). More presicely, this submanifold does not include any equilibria from the curve $P = \gamma Q$ other than $(Q, P) = (0, 0)$. With an abuse of notation, we call this manifold W^{cs}. The following lemma is an extension of [5, Theorem 2]:

Lemma 2 *$W^{cu}(\gamma^{-1}, 1, c_0)$ transversely intersects $W^{cs}(0, 0, c_0)$.*

Proof. We consider the intersections of W^{cu} and W^{cs} with the plane $T = \Theta$, where Θ is the ignition temperature (see Fig. 1). From [5] we know that the temperature T is a monotone function. The ignition value is reached at some finite ξ, which without loss of generality we can assume to be $\xi = 0$. The description of the plane in terms of Q and P is

$$Q = \Theta - (1 - \gamma^{-1})P. \tag{14}$$

These intersections are curves in (14) and we denote them $P = h^u(c)$ and $P = h^s(c)$ respectively. Then the proof amounts to checking the transversality condition [7, Sect. 1.4]:

$$\left(\frac{\partial h^u}{\partial c} - \frac{\partial h^s}{\partial c} \right)\Bigg|_{c=c_0} \neq 0. \tag{15}$$

For temperatures below Θ, the system (11) is linear. The solution is explicitly described as

$$Q = 0, \qquad P = \frac{\Theta}{1-\gamma^{-1}} e^{-\frac{\gamma}{c}\xi},$$

where ξ is scaled so the ignition temperature is reached at $\xi = 0$. Then at (14)

$$\frac{\partial h^s}{\partial c}\Bigg|_{c=c_0} = 0. \tag{16}$$

Next we concentrate on the temperatures above ignition. We start by identifying vectors tangent to W^{cu} and W^{cs}. One of them is

$$H_1 = (\frac{\partial Q}{\partial c}, \frac{\partial P}{\partial c}, 1) = (\delta Q^-, \delta P^-, 1).$$

Another one is the vector field of (12)

$$H_2 = (f_1(Q, P, c), f_2(Q, P, c), 0), \tag{17}$$

where

$$\begin{aligned} f_1(Q, P, c) &= c^{-1}(\gamma Q - 1)\Omega(Q + (1-\gamma^{-1})P), \\ f_2(Q, P, c) &= -c\gamma^{-1}P + cQ. \end{aligned}$$

To find the sign of $\frac{\partial h^u}{\partial c}$ at $c = c_0$ we look at the following vector product on the interval $\xi \le 0$

$$\begin{vmatrix} i & j & k \\ f_1 & f_2 & 0 \\ \delta Q^- & \delta P^- & 1 \end{vmatrix} = f_2 i - f_1 j + (f_1 \delta P^- - f_2 \delta Q^-)k. \tag{18}$$

Note that on the plane (14)

$$H_1 = (-(1-\gamma^{-1})\frac{\partial h^u}{\partial c}, \frac{\partial h^u}{\partial c}, 1). \tag{19}$$

Therefore, at $\xi = 0$, (18) reads

$$\begin{vmatrix} i & j & k \\ f_1 & f_2 & 0 \\ -(1-\gamma^{-1})\frac{\partial h^u}{\partial c} & \frac{\partial h^u}{\partial c} & 1 \end{vmatrix} = f_2 i - f_1 j + (f_1 + (1-\gamma^{-1} f_2)\frac{\partial h^u}{\partial c} k,$$

thus imposing a condition

$$f_1 \delta P^- - f_2 \delta Q^- = (f_1 + (1-\gamma^{-1}) f_2)\frac{\partial h^u}{\partial c}. \tag{20}$$

Differentiating the quantity

$$w = f_1 \delta P^- - f_2 \delta Q^-, \tag{21}$$

with respect to ξ we obtain

$$\begin{aligned} w' &= (\frac{\partial f_1}{\partial Q} f_1 + \frac{\partial f_1}{\partial P} f_2)\delta P^- + f_1(\frac{\partial f_2}{\partial Q}\delta Q^- + \frac{\partial f_2}{\partial P}\delta P^- + \frac{\partial f_2}{\partial c}) \\ &\quad -(\frac{\partial f_2}{\partial Q} f_1 + \frac{\partial f_2}{\partial P} f_2)\delta Q^- - f_2(\frac{\partial f_1}{\partial Q}\delta Q^- + \frac{\partial f_1}{\partial P}\delta P^- + \frac{\partial f_1}{\partial c}) \\ &= (\frac{\partial f_1}{\partial Q} + \frac{\partial f_2}{\partial P})w + f_1 \frac{\partial f_2}{\partial c} - f_2 \frac{\partial f_1}{\partial c} \end{aligned} \tag{22}$$

Then w satisfies a differential equation

$$w' = \left(-\frac{c}{\gamma} + \frac{\gamma}{c}\Omega(T) - \frac{1}{c}(1-\gamma Q)\Omega'(T)\right) w + \frac{2}{c}(Q - \frac{1}{\gamma}P)(\gamma Q - 1)\Omega(T), \tag{23}$$

where $T = T(P,Q) = Q + (1-\gamma^{-1})P$. We know from [4] that if a solution asymptotically connecting $(\gamma^{-1}, 1)$ and $(0,0)$ exists, then for the temperatures above the ignition temperature

$$T < P, \quad \text{and} \quad Y > 0.$$

These inequalities imply

$$Q - \gamma^{-1}P = T - P < 0, \quad \gamma Q - 1 = -Y < 0$$

and therefore

$$F = \frac{2}{c}(Q - \frac{1}{\gamma}P)(\gamma Q - 1)\Omega(T) > 0, \tag{24}$$

as well as

$$\begin{aligned} f_1 = c^{-1}(\gamma Q - 1)\Omega(Q + (1-\gamma^{-1})P) &< 0, \\ f_2 = -c\gamma^{-1}P + cQ &< 0. \end{aligned} \tag{25}$$

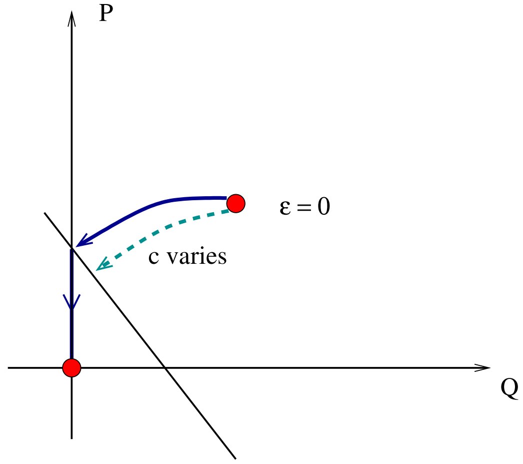

Figure 1: The invariant manifolds intersect transversally as c varies near c_0.

Functions Q and P at $-\infty$ decay to $(\gamma^{-1}, 1)$ at the rate $e^{\frac{\gamma}{c}\Omega(1)\xi}$. Therefore $w\, e^{\left(\frac{c}{\gamma}-\frac{\gamma}{c}\Omega(1)\right)\xi}$ approaches 0 at $-\infty$ and we can write the solution to (23) as

$$w(z) = e^{a_-(\xi)} \int_{-\infty}^{\xi} e^{-a_-(z)} F(z)\, \mathrm{d}z, \tag{26}$$

where

$$a_-(\xi) = [-\frac{c}{\gamma} + \frac{\gamma}{c}\Omega(1)]\xi - \int_{-\infty}^{\xi} \left(\frac{1}{c} Y\Omega(T(z)) - \frac{\gamma}{c}[\Omega(T(z)) - \Omega(1)]\right) \mathrm{d}z.$$

From (26) we obtain that

$$w(0) > 0.$$

On the other hand, (25) implies that

$$f_1 + (1-\gamma^{-1}) f_2 < 0$$

for any $\xi \le 0$. Using the last two inequalities in (20) we conclude that

$$\left.\frac{\partial h^u}{\partial c}\right|_{c=c_0} < 0, \tag{27}$$

which combined with (16) completes the proof of the transversality condition (15).

■

3 Construction of the orbit for a small ε

Geometric singular perturbation theory, see [7], gives a clear prescription as to how to construct a heteroclinic orbit (traveling wave) for the system (2) in the case when ε is small. The idea is to construct a manifold on which a perturbation of equation (11) governs the flow. This manifold will be invariant under the full equations (7) or, equivalently, (6). Since the intersection that creates the heteroclinic in (12) is transverse, it will perturb to (6).

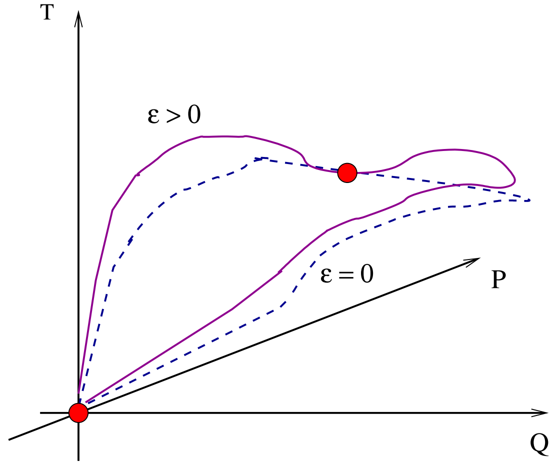

Figure 2: The critical manifold M_0, at least over compact, perturbs to an invariant manifold M_ε.

The manifold in question is constructed as a perturbation of the critical manifold for the limiting ($\varepsilon = 0$) system. Indeed, in the fast scaling (7) with $\varepsilon = 0$ reads

$$\begin{aligned} \dot{Q} &= 0, \\ \dot{P} &= 0, \\ \dot{T} &= c(1-\gamma^{-1})P - cT + cQ, \\ \dot{Y} &= c(1-Y) - c\gamma Q. \end{aligned} \tag{28}$$

There is then a large set of critical points given by the condition:

$$\begin{aligned} (1-\gamma^{-1})P - T + Q &= 0, \\ 1 - Y - \gamma Q &= 0. \end{aligned} \tag{29}$$

We denote the set where (29) is satisfied as M_0. The manifold M_0 is a manifold of critical points for the system (28). This system linearized about each critical point on M_0 has two double eigenvalues: 0 and $-c$. The double zero eigenvalue corresponds to the dimension of M_0. The other eigenvalues being negative ($c > 0$) implies that M_0 is attracting. Therefore each point on M_0 has a two-dimensional stable manifold and a two-dimensional center manifold, which is M_0 itself. Therefore M_0 is an invariant and normally hyperbolic manifold for (28).

Under these conditions invariant manifold theory by Fenichel is applicable. More precisely, by Fenichel's First Theorem, see [3] or [7], the critical manifold M_0, at least over compact sets, perturbs to an invariant manifold for (7) with $\varepsilon > 0$ but small (see Fig. 2). We call this manifold M_ε. The distance between M_0 and M_ε is of order ε. If ε is small enough, M_ε is normally hyperbolic and attracting on the fast scale $\eta = \xi/\varepsilon$.

On M_ε the flow on the slow scale ξ is determined by equations that are an $O(\varepsilon)$ perturbation of (11). Indeed, by Fenichel's First Theorem, M_ε is given by

$$\begin{aligned} T &= \Gamma_1(Q,P,\varepsilon), \\ Y &= \Gamma_2(Q,P,\varepsilon). \end{aligned} \tag{30}$$

with $\Gamma_1(Q,P,\varepsilon) = (1-\gamma^{-1})P + Q + O(\varepsilon)$, $\Gamma_2(Q,P,\varepsilon) = 1 - \gamma Q + O(\varepsilon)$. The equations for the flow on M_ε are then given by the following system

$$\begin{aligned} Q' &= -c^{-1}\Gamma_2(Q,P,\varepsilon)\Omega(\Gamma_1(Q,P,\varepsilon)), \\ P' &= c(\Gamma_1(Q,P,\varepsilon) - P). \end{aligned} \tag{31}$$

According to [3] any invariant set for (7) which is sufficiently close to M_0 is located on M_ε. Therefore the equilibria $(Q,P,T,Y) = (\gamma^{-1},1,1,0)$ and $(Q,P,T,Y) = (0,0,0,1)$ belong to M_ε.

The heteroclinic orbit of interest is now constructed as a value of c at which the unstable manifold W^u of $(Q,P) = (\gamma^{-1},1)$ intersects transversely the stable manifold W^s of $(Q,P) = (0,0)$ as the speed parameter c varies. This will be done on M_ε as a perturbation of the same construction on M_0. The following lemma guaranties the validity of the reduction of the problem to M_ε.

Lemma 3 *For sufficiently small $\varepsilon > 0$, any heteroclinic connecting (8) to (9) must lie in M_ε.*

Proof. In deriving the critical manifold M_0 for the case $\varepsilon = 0$, we obtained that M_0 is an attracting set. The slow manifold M_ε exists for sufficiently small ε and is also attracting. Therefore the unstable manifold of (8) must lie in M_ε for ε sufficiently small. ■

Next let us show that the structure of the flow at $\xi \to \pm\infty$ for small ε is similar to one for $\varepsilon = 0$.

The flow at $\xi \to -\infty$, i.e. at the critical point $(\gamma^{-1},1)$ in M_0, is a saddle with a one-dimensional unstable manifold W^u and a one-dimensional stable manifold W^s. This structure perturbs to the case $\varepsilon > 0$. We thus want to follow $W^u(\gamma^{-1},1)$ as c is varied in M_ε.

The situation is more complicated at the boundary $\xi \to \infty$. This boundary correspond to the critical point $(Q,P) = (0,0)$ in both cases $\varepsilon = 0$ and $\varepsilon > 0$, on M_0 and M_ε respectively.

First, when $\varepsilon = 0$, M_0 for temperatures below the ignition temperatures can be described as

$$\begin{aligned} T &= (1-\gamma^{-1})P - \gamma^{-1}(Y-1), \\ Y &= 1. \end{aligned}$$

The equation on M_0 is very simple as it is given by a linear vector field:

$$\begin{aligned} Q' &= 0, \\ P' &= -c\gamma^{-1}P + cQ. \end{aligned} \tag{32}$$

The eigenvalues of the linearization of (32) about $(Q,P) = (0,0)$ are 0 and $\frac{-c}{\gamma}$. As pointed out above, it is the stable manifold W^s of $(Q,P) = (0,0)$, that we want to hit with $W^u(\gamma^{-1},1)$. The flow of (32) can be easily analyzed and it is not hard to see that the only way to decay to $(0,0)$ is by being on $W^s(0,0)$.

The behavior near $(0,0)$ when $\varepsilon > 0$ is essentially identical. Indeed it is easy to check that for low temperatures

$$\begin{aligned} Y &= 1, \\ T &= \frac{(\varepsilon-1) + \sqrt{(\varepsilon-1)^2 + 4\varepsilon(1-\gamma^{-1})}}{2\varepsilon} P. \end{aligned}$$

define an invariant manifold for (6), therefore it coincides with M_ε. The flow on M_ε near $(0,0)$ is given by

$$\begin{aligned} Q' &= 0, \\ P' &= c\left(\frac{(\varepsilon-1) + \sqrt{(\varepsilon-1)^2 + 4\varepsilon(1-\gamma^{-1})}}{2\varepsilon} - 1\right) P + cQ. \end{aligned} \tag{33}$$

Since

$$\frac{(\varepsilon-1) + \sqrt{(\varepsilon-1)^2 + 4\varepsilon(1-\gamma^{-1})}}{2\varepsilon} - 1 = -\gamma^{-1} + O(\varepsilon),$$

the flow near $(0,0)$ is structurally the same as for the $\varepsilon = 0$ case.

It suffices then to prove that for every $0 < \varepsilon \ll 1$ there exists c_ε such that

$$W^{cu}(\gamma^{-1}, 1, c_\varepsilon) \pitchfork W^{cs}(0, 0, c_\varepsilon) \neq \varnothing. \tag{34}$$

Lemma 2 shows that (34) is true when $\varepsilon = 0$. The transversality condition (15) proved for the case $\varepsilon = 0$ also allows us to claim that for the perturbed problem the intersection persists [7]. Let $W^{cu}(\gamma^{-1}, 1, c) \cup \{T = \Theta\}$ and $W^{cs}(0, 0, c) \cup \{T = \Theta\}$ on the manifold M_ε be given by $P = h^u(c, \varepsilon)$ and $P = h^s(c, \varepsilon)$ respectively. An intersection point $(c_\varepsilon, P_\varepsilon)$ of h^u and h^s is found by solving

$$\begin{aligned} P &= h^u(c, \varepsilon), \\ P &= h^s(c, \varepsilon) \end{aligned}$$

for P and c as functions of ε (see Fig. 1). This system has a unique solution if the determinant

$$\det \begin{vmatrix} -1 & \frac{\partial h^u}{\partial c} \\ -1 & \frac{\partial h^s}{\partial c} \end{vmatrix} = \frac{\partial h^u}{\partial c} - \frac{\partial h^s}{\partial c}$$

is nonzero at $c = c_0$ and $\varepsilon = 0$. This condition coincides with (15). By the Implicit Function Theorem, (34) then holds when $\varepsilon > 0$ also but with a nearby c_ε replacing c_0. The Theorem 1 then easily follows.

Acknowledgment. AG and CJ were supported in part by the NSF grant DMS-0410267, PG was partially supported by the NSF grant DMS-0554775.

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