ON INITIATION OF SUBSONIC DETONATION IN POROUS MEDIA COMBUSTION

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Abstract

The long time behavior of the system $\gamma\Theta_t - (\gamma - 1)\Pi_t = \Omega(\Psi, \Theta)$, $\Psi_t = -\Omega(\Psi, \Theta)$, $\Pi_t - \Theta_t = \Pi_{xx}$ is explored. The model describes gaseous detonation subjected to strong hydraulic resistance, e.g. gaseous explosion in inert porous media. It is shown that successful initiation of a self-sustaining detonation wave is not always attainable and depends on the initial conditions. The current work is an extension of the previous study of the problem based on a simplified formulation assuming a similarity relation between the temperature and concentration fields.

1 Introduction

Since the early works of Laffitte [8] and Schelkin [10], it has been known that hydraulic resistance (frictional drag) may have a profound effect on gaseous detonation markedly reducing its propagation velocity compared to the associated Chapman - Jouguet (CJ) value. Moreover, it has been recently realized that at sufficiently high level of resistance the conventional sub - CJ detonation may convert into the subsonic (shockless) combustion wave driven by the drag - induced diffusion of pressure [1] - [5]. This regime which may be termed "subsonic detonation" is the main concern of the current study.

Perhaps the physically simplest and yet experimentally quite feasible system for studying subsonic detonation is combustion in the inert porous medium [9]. In this case, on one hand, the distortions introduced by the porous matrix may be ignored while, on the other hand, the resistance of the matrix to the gas flow is often so strong that one may neglect the inertial effects and take Darcy's law as the momentum equation. In this high drag limit the shocks are ruled out and the pressure nonuniformities are equalized not by the acoustic waves but rather through the diffusion of pressure associated with low Reynolds number creeping flows. For many gas-porous medium systems the gas pressure diffusivity exceeds its thermal diffusivity by several orders of magnitude thereby emerging as the principal transport agency controlling the reaction spread.

Similar to conventional detonation the subsonic detonation may be initiated by a localized energy deposition, e.g., by a localized temperature elevation (hot spot). Depending on its profile (initial data) the detonation wave forms either immediately or only after a long induction time as a result of an abrupt transition from slowly spreading deflagration driven by thermal diffusivity. In the limit where thermal diffusivity may be regarded as negligibly small the predetonation time becomes infinitely long and the transition to detonation is ruled out altogether. One thus ends up with two alternatives: successful initiation of detonation or its failure. The numerical simulation of the present model shows that for successful initiation the temperature gradient of the hot spot should be sufficiently low [1],[3],[4]. The current paper deals with the mathematical aspects of this intriguing phenomenon.

To single out the impact of momentum loss the effective features of the reactive gas-porous medium system will be assumed to be controlled exclusively by its gaseous phase subjected to the resistance of the porous medium matrix. The thermal and molecular diffusivities will be regarded as negligibly small (compared to the pressure diffusivity) which indeed holds for many real world systems. As an additional simplification the so-called small-heat-release (SHR) approximation will be employed where variations of temperature, pressure, density and gas velocity are regarded as small and, hence, the nonlinear effects are ignored everywhere but in the reaction rate term, generally highly sensitive even to minor temperature changes. In the non-dimensional formulation the resulting model reads [1],

$$\gamma\Theta_t - (\gamma - 1)\Pi_t = \Omega(\Psi, \Theta), \tag{1}$$

$$\Psi_t = -\Omega(\Psi, \Theta), \tag{2}$$

$$\Pi_t - \Theta_t = \Pi_{xx}, \tag{3}$$

Here Π, Θ, Ψ, x, t are the appropriately scaled pressure, temperature, reactant concentration and spatio-temporal coordinates; $\gamma > 1$ is the specific heat ratio, Ω is the scaled reaction rate satisfying the following condition,

$$\Omega(\Psi, \Theta) = 0 \text{ for } 0 < \Theta < \beta < 1, \tag{4}$$

where β is the ignition temperature

$$\Omega(0, \Theta) = 0, \ \partial\Omega/\partial\Psi \geq 0, \ \partial\Omega/\partial\Theta \geq 0. \tag{5}$$

Prior to ignition

$$\Pi = 0, \ \Theta = \Theta_0(x) \geq 0, \ \Psi = 1. \tag{6}$$

Equations (1), (2) represent the partially linearized conservation equations for energy and the deficient reactant respectively. Equation (3) is a linearized continuity equation, taking into account the equations of state and momentum (Darcy's law). Derivation of the model and its numerical simulation are comprehensively discussed in [1],[4].

For the sake of mathematical simplicity in the recent paper [6] equation (2) was replaced by the similarity relation, $\Psi = 1 - \Theta$, which preserves much of the problem's character. In the current work this constrain is removed and the problem is tackled within its original formulation involving equation (2). It transpires that with only a slight modification the basic approach developed for the simplified formulation [6] may be effectively employed for the complete model as well.

The paper is organized as follows. Sec. 2 deals with the existence and uniqueness of traveling wave solutions (detonation wave). Sufficient conditions ensuring detonation failure (unsuccessful initiation) are discussed in Sec. 3. Sec. 4 is concerned with the initial conditions presumably sufficient for a successful initiation.

2 Traveling wave solution

In this section the traveling wave solution for the problem (1)-(3) is studied. The solution $(\Theta, \Pi, \Psi) \geq 0$ is sought in the following form,

$$\Theta(x,t) = \Theta(\eta), \quad \Pi(x,t) = \Pi(\eta), \quad \Psi(x,t) = \Psi(\eta), \tag{7}$$

where $\eta = x - ct$, with conditions

$$\lim_{\eta \to -\infty} (\Theta, \Pi, \Psi) = (A, B, 0), \qquad \lim_{\eta \to \infty} (\Theta, \Pi, \Psi) = (0, 0, 1). \tag{8}$$

Here $c > 0$ is a velocity of propagation viewed as unknown, A, B are some positive constants. Substituting (7) into (1)-(3) one has

$$\begin{aligned} - c\gamma\Theta' + c(\gamma - 1)\Pi' &= \Omega(\Psi, \Theta), \\ c\Psi' &= \Omega(\Psi, \Theta), \\ -c\Pi' + c\Theta' &= \Pi''. \end{aligned} \tag{9}$$

Henceforth $(\cdot)'$ denotes the derivative with respect to η.

Proposition 1. *If the traveling wave exists, then*

$$A = B = 1, \quad \beta < (1 - \gamma^{-1}),$$

the functions Θ, Π *are nonincreasing and function* Ψ *is nondecreasing in* η. *Moreover there exists a constant* $a \in (-\infty, \infty)$ *such that*

$$\begin{aligned} \Theta(\eta) &= \beta \exp\left(-c\gamma^{-1}(\eta - a)\right), \\ \Pi(\eta) &= \frac{\beta}{1 - \gamma^{-1}} \exp\left(-c\gamma^{-1}(\eta - a)\right), \\ \Psi(\eta) &= 1 \qquad \textit{for all} \qquad \eta \geq a, \end{aligned}$$

and a constant $b \in [-\infty, \infty)$, $b < a$ *such that*

$$\Theta(\eta) = \Pi(\eta) = 1, \quad \Psi(\eta) = 0 \quad \textit{for all} \quad \eta \leq b$$

and

$$\Theta' < 0, \quad \Pi' < 0, \quad \Psi' > 0, \quad \Psi > 1 - \Pi, \quad \Pi > \Theta \quad \text{for} \quad \eta \in (b, a).$$

Proof. Combining the first two equations in (9) and integrating them from $\eta = -\infty$ we have

$$\Theta = A - (1-\gamma^{-1})B + (1-\gamma^{-1})\Pi - \gamma^{-1}\Psi. \tag{10}$$

Passing to the limit as $\eta \to \infty$ we obtain

$$A - (1-\gamma^{-1})B = \gamma^{-1}. \tag{11}$$

Integration of the last equation in (9) from $\eta = -\infty$ gives

$$\Pi' = c(\Theta - \Pi - A + B). \tag{12}$$

Passing to the limit as $\eta \to \infty$ in (12) we have $A = B$. Substituting this result into (11) we finally deduce that $A = B = 1$. Thus from the second equation in (9), (10) and (12) we have

$$\begin{aligned} \Theta = (1-\gamma^{-1})\Pi + \gamma^{-1}(1-\Psi), \\ c\Psi' = \Omega(\Psi, \Theta), \\ \Pi' = c(\Theta - \Pi). \end{aligned} \tag{13}$$

Note that since $c > 0$ and $\Omega \geq 0$, $\Psi' \geq 0$ by the second equation in (13) and thus Ψ is a nondecreasing function. Taking the derivative of the last equation in (13) and combining it with the second equation in (13) we obtain

$$\Pi'' + c\gamma^{-1}\Pi' = -\gamma^{-1}\Omega(\Psi, \Theta),$$

which can be rewritten as follows

$$(\Pi' \exp(c\gamma^{-1}\eta))' = -\gamma^{-1}\Omega(\Psi, \Theta)\exp(c\gamma^{-1}\eta) \leq 0.$$

Hence $\Pi' \exp(c\gamma^{-1}\eta)$ is nonincreasing. Therefore Π is a nonincreasing function and thus by the first equation in (13) function Θ is a nonincreasing function as well. These observations allow us to write

$$\Psi > 1 - \Pi, \qquad \Pi > \Theta. \tag{14}$$

Next, suppose that a is the smallest number such that $\Theta < \beta$. The system (13) for $\eta \geq a$ then becomes,

$$\Theta = (1-\gamma^{-1})\Pi + \gamma^{-1}(1-\Psi), \qquad c\Psi' = 0, \qquad \Pi' = c\gamma^{-1}(1 - \Pi - \Psi). \tag{15}$$

Taking into account the boundary conditions (8) at $\eta \to \infty$ we have

$$\begin{aligned} \Theta(\eta) &= \beta \exp\left(-c\gamma^{-1}(\eta - a)\right), \\ \Pi(\eta) &= \frac{\beta}{1-\gamma^{-1}} \exp\left(-c\gamma^{-1}(\eta - a)\right), \\ \Psi(\eta) &= 1, \end{aligned} \tag{16}$$

for all $\eta \geq a$. Note that due to the monotonicity of Π $\beta < (1 - \gamma^{-1})$.

Let b be the largest number so that $\Psi = 0$. For any $\eta < b$ we then have the same system (15) for Θ, Ψ, Π. Using boundary conditions (8) at $\eta \to -\infty$, we finally have

$$\Theta(\eta) = 1, \quad \Pi(\eta) = 1, \quad \Psi(\eta) = 0, \tag{17}$$

for all $\eta \leq b$.

Now we can consider Ψ as a function of Π for $\eta \in (b, a)$ and rewrite (13) as a dynamical system on a plane,

$$\frac{d\Psi}{d\Pi} = \frac{\gamma\Omega(\Psi, \Theta)}{c^2(1 - \Pi - \Psi)} \tag{18}$$

with Θ defined by the first expression in (13) and the boundary conditions following from (16) at $\eta = a$ and (17) at $\eta = b$,

$$\Psi\left(\frac{\beta}{1-\gamma^{-1}}\right) = 1, \qquad \Psi(1) = 0. \tag{19}$$

□

Theorem 1. *Suppose that the function $\Omega(\tau, s)$ satisfies a Lipschitz condition with respect to τ on $[0,1]$ and with respect to s on $[\beta, 1]$. Assume also that $\beta < (1-\gamma^{-1})$. Then there is at least one value of $c > 0$ for which there exists a solution of the problem (13) satisfying conclusions of Proposition 1.*

Proof. We begin with the problem (18) (19). For any fixed $c > 0$ there exists a unique integral curve which begins at the point $(\beta/(1-\gamma^{-1}), 1)$ and $d\Psi/d\Pi < 0$. Consider the vector field defined by (18). Let $c \searrow 0$. The integral curve $\Psi(\Pi)$ then tends to the vertical line $\Pi = \beta/(1-\gamma^{-1})$. Therefore, by continuity arguments, for c small enough the integral curve will cross the line $\Psi = 1 - \Pi$ at some point $\Pi < 1$ and thus will not reach the point $(1, 0)$. Similarly, if $c \to \infty$ the integral curve tends to the line $\Psi = 1$ and, for c large enough, the corresponding integral curve will cross the line $\Psi = 0$ at some point $\Pi > 1$ and will not reach the point $(1, 0)$. Hence there exists at least one value $c = \tilde{c}$ such that the corresponding solution of (18) $\tilde{\Psi}(\Pi)$ satisfies both conditions (19).

This completes the proof of existence for the problem (18) (19). Finally we find $\Psi(\eta)$, $\Pi(\eta)$ and $\Theta(\eta)$ as solutions of the system (13) and therefore Theorem 1 is proved. □

To prove the uniqueness of such a solution we need to impose some additional restrictions on the reaction rate Ω.

Theorem 2. *Suppose that* $\Omega(\tau, s)$ *satisfies the assumptions of Theorem 1 and in addition*

$$\Omega(\tau, s)/\tau \quad \textit{is a nonincreasing function of} \quad \tau. \tag{20}$$

There is then a unique value of c for which there exists a solution of the problem (13) satisfying conclusions of Proposition 1.

Proof. As in the proof of Theorem 1 we begin with (18) (19). Let us first rewrite (18) in new coordinates (Θ, Π). Using the first expression in (13) for Θ, we have

$$\frac{d\Theta}{d\Pi} = (1-\gamma^{-1}) + \frac{\Omega(1-\gamma\Theta+(\gamma-1)\Pi, \Theta)}{\gamma c^2(\Pi-\Theta)} \tag{21}$$

with the boundary conditions

$$\Theta\left(\frac{\beta}{1-\gamma^{-1}}\right) = \beta, \qquad \Theta(1) = 1. \tag{22}$$

Now suppose that by contradiction there exist two velocities c_1 and c_2 (say $c_2 > c_1$) such that both integral curves $\Theta = \Theta_1(\Pi)$, $\Theta = \Theta_2(\Pi)$ connect the points $(\beta/(1-\gamma^{-1}), \beta)$ and $(1, 1)$. We claim that

$$\Theta_1(\Pi) > \Theta_2(\Pi) \tag{23}$$

for all $\Pi \in (\beta/(1-\gamma^{-1}), 1)$.

Introduce a new function

$$H(\Pi, \Theta) = \frac{\Omega(1-\gamma\Theta+(\gamma-1)\Pi, \Theta)}{\gamma(\Pi-\Theta)}.$$

Fix some small $\epsilon > 0$. Then,

$$H(\Pi, \Theta_1) - H(\Pi, \Theta_2) > -C(\Theta_1 - \Theta_2)$$

for all $\Pi \in (\beta/(1-\gamma^{-1}), 1-\epsilon)$ and some constant C. Here we use the Lipschitz condition on Ω and the boundedness of $1/(\Pi - \Theta_i)$, $i = 1, 2$ in the chosen interval. Thus we obtain

$$\frac{d}{d\Pi}(\Theta_1 - \Theta_2) = \left(\frac{1}{c_1^2} - \frac{1}{c_2^2}\right) H(\Pi, \Theta_1) + \frac{1}{c_2^2}\left(H(\Pi, \Theta_1) - H(\Pi, \Theta_2)\right) > -C(\Theta_1 - \Theta_2),$$

which yields (23).

Now let Π_1, Π_2 be such that $\Theta_1(\Pi_1) = \Theta_2(\Pi_2) = \Theta_*$ where $\Theta_* \in (\beta, 1)$. It follows from (23) that $\Pi_2 > \Pi_1$. Hence using (21) and condition (20) we obtain

$$\frac{d\Theta_1}{d\Pi}_{|\Pi_1} > \frac{d\Theta_2}{d\Pi}_{|\Pi_2}. \tag{24}$$

This inequality holds for any value of $\Theta_* \in (\beta, 1)$ which contradicts conditions (22). The solution of the problem (18) (19) is therefore unique. Finally we find $\Psi(\eta)$, $\Pi(\eta)$ and $\Theta(\eta)$ as solutions of the system (13) and therefore Theorem 2 is proved. □

3 Initial data without ignition

In this section we present a sufficient condition on the initial data Θ_0 such that ignition does not occur. More precisely we define a class of initial data Θ_0 for the problem (1)-(3) such that: $\Pi(x,t) \to 0$, $\Psi(x,t) \to \Psi_\infty(x) \leq 1$, $\Theta(x) \to \Theta_\infty(x) \geq \Theta_0(x)$ as $t \to \infty$.

As a first step let us note that the system (1)-(3) can be significantly simplified. Indeed combing (1) and (2), integrating with respect to time and taking into account initial conditions (6), we have,

$$\Theta(x,t) = \left(1 - \gamma^{-1}\right) \Pi(x,t) + \gamma^{-1} \left(1 - \Psi(x,t)\right) + \Theta_0(x) \qquad \forall x, \forall t.$$

The combination of (1) and (3) then yields,

$$\Pi_t - \gamma \Pi_{xx} = \Omega(\Psi, \Theta).$$

Rescaling the space coordinate ($x \to x/\sqrt{\gamma}$) we finally have the following equivalent system,

$$\begin{aligned} \Theta = (1 - \gamma^{-1})\Pi + \gamma^{-1}(1 - \Psi) + \Theta_0, \\ \Psi_t = -\Omega(\Psi, \Theta), \qquad (P) \\ \Pi_t - \Pi_{xx} = \Omega(\Psi, \Theta). \end{aligned}$$

with initial conditions,

$$\Psi(x,0) = 1, \qquad \Pi(x,0) = 0, \qquad \Theta(x,0) = \Theta_0(x), \tag{25}$$

where $\Theta_0(x) \geq 0$ is some given function.

Thus, the special structure of the system (1)-(3) allows its reduction to the problem (P), considered below.

Theorem 3. *Let $\Theta_0(x) \geq 0$ and set*

$$I = \{x \in R;\ \Theta_0(x) > 0\}. \tag{26}$$

Suppose

$$l = \text{meas } I \le l_0 = \frac{\beta\sqrt{2\pi}}{3(1-\gamma^{-1})\sqrt{M}} \quad \text{with} \quad M = \max_s \Omega(1,s). \tag{27}$$

Then

$$\Pi(x,t) \to 0 \text{ as } t \to \infty, \text{ uniformly in } x \in R, \tag{28}$$

$$\Psi(x,t) \to \Psi_\infty(x) \le 1 \text{ as } t \to \infty, \tag{29}$$

$$\Theta(x,t) \to \Theta_\infty(x) \ge \Theta_0(x) \text{ as } t \to \infty \tag{30}$$

and

$$\Psi_\infty(x) = 0,\ \Theta_\infty(x) = \gamma^{-1} + \Theta_0(x) \quad \text{if} \quad \Theta_0(x) > \beta,$$

$$\Psi_\infty(x) = 1, \qquad \Theta_\infty(x) = 0 \quad \text{if} \quad \Theta_0(x) = 0.$$

Moreover

$$\lim_{t\to\infty} \Omega\left(\Psi(x,t), \Theta(x,t)\right) = 0\ \forall x \in R. \tag{31}$$

Let us start from some properties of the solution of the system (P) that hold for any nonnegative initial data Θ_0,

$$\Pi(x,t), \Psi(x,t), \Theta(x,t) \ge 0 \qquad \forall x, \quad \forall t, \tag{32}$$

$$\Psi(x,t) \text{ is nonincreasing in } t \text{ for all } x, \tag{33}$$

$$\int_0^\infty \Omega\left(\Psi(x,s), \Theta(x,s)\right) ds \le 1 \qquad \forall x. \tag{34}$$

For the proof of Theorem 3 three lemmas are needed.

Lemma 1. *Assume that* Π *satisfies*

$$\Pi_t - \Pi_{xx} = F(x,t), \qquad \Pi(0,x) = 0, \tag{35}$$

$$0 \le F(x,t) \le M \qquad \forall x,\ \forall t, \tag{36}$$

$$\text{Supp } F \subset J \times [0,\infty) \quad \text{with} \quad \text{meas } J < \infty, \tag{37}$$

$$\int_0^\infty F(x,s) ds \le 1 \qquad \forall x. \tag{38}$$

Then

$$\Pi \le \frac{3\sqrt{M}}{\sqrt{2\pi}} \text{meas } J \qquad \forall x,\ \forall t. \tag{39}$$

The proof of this lemma is presented in [6].

Lemma 2. *Suppose* (Π, Ψ, Θ) *is a solution of the problem* (P). *Assume also that for some* $T > 0$

$$\Pi(x,t) \le \frac{\beta}{1-\gamma^{-1}} \qquad \forall x,\ \forall t \le T \tag{40}$$

and set

$$I = \{x : \Theta_0(x) > 0\}, \qquad l = \text{meas } I. \tag{41}$$

Then

$$\Pi \le \frac{3\sqrt{M}}{\sqrt{2\pi}} l \qquad \forall x,\ \forall t \le T. \tag{42}$$

Proof. Since for any $x_0 \notin I$ $\Theta_0(x_0) = 0$, then

$$\Psi(x_0, t) = 1 \qquad \forall t \le T$$

because $\Psi = 1$ is a solution of

$$\Psi_t = -\Omega(\Psi, \Theta).$$

Here the condition (40) and uniqueness of the solution are used. Thus,

$$\Omega\left(\Psi(x_0,t), \Theta(x_0,t)\right) = 0 \qquad \forall t \le T.$$

Therefore, with $F(x,t) = \Omega\left(\Psi(x,t), \Theta(x,t)\right)$ we have

$$\text{Supp } F \subset I \times [0, T].$$

Using the result of Lemma 1 on the time interval $(0, T)$, we finally have

$$\Pi \le \frac{3\sqrt{M}}{\sqrt{2\pi}} l \qquad \forall x, \forall t \le T.$$

□

We now deduce a consequence of Lemma 2.

Lemma 3. *Suppose* (Π, Ψ, Θ) *is a solution of the problem* (P)*and*

$$l = \text{meas}\{x \in R : \Theta_0 > 0\} < \frac{\beta\sqrt{2\pi}}{3(1-\gamma^{-1})\sqrt{M}} \tag{43}$$

then

$$\Pi(x,t) \le \frac{\beta}{1-\gamma^{-1}} \qquad \forall x, \forall t. \tag{44}$$

Proof. Suppose T^* is the maximum time such that

$$\Pi(x,t) \le \frac{\beta}{1-\gamma^{-1}} \qquad \forall x,\ \forall t \le T^*.$$

We claim that $T^* = \infty$. Otherwise, if $T^* \leq \infty$, by Lemma 2 we would have

$$\Pi(x,t) \leq \frac{3\sqrt{M}}{\sqrt{2\pi}} l \qquad \forall x, \ \forall t \leq T^*.$$

On the other hand,

$$\frac{3\sqrt{M}}{\sqrt{2\pi}} l < \frac{\beta}{1-\gamma^{-1}}$$

and thus T^* is not the maximum. □

Proof of Theorem 3. By Lemma 3 and the assumption $l \leq l_0$ we have

$$\Pi(x,t) \leq \frac{\beta}{1-\gamma^{-1}} \quad \forall x, \forall t.$$

Setting $F(x,t) = \Omega\left(\Psi(x,t), \Theta(x,t)\right)$, as in the proof of Lemma 2, we obtain Supp $F \subset I \times (0,\infty)$.

We first prove that

$$\lim_{t\to\infty} \Pi(x,t) = 0 \text{ uniformly in } x.$$

Using the Green function of the heat equation we have

$$\Pi(x,t) \leq \int_0^t \int_R \frac{F(y,\tau)}{\sqrt{t-\tau}} d\tau dy.$$

Set

$$\phi(\tau) = \int_R F(y,\tau) dy.$$

Note that $\phi \in L^\infty(0,\infty)$ since F bounded and Supp$F \subset I \times (0,\infty)$ with $l = \text{meas } I \leq \infty$. Also by (34) $\phi \in L^1(0,\infty)$.

For $\delta < t/2$ we write

$$\Pi(x,t) \leq \frac{1}{\sqrt{2\pi}} \left(\int_0^{t/2} \frac{\phi(\tau)d\tau}{\sqrt{t-\tau}} + \int_{t/2}^{t-\delta} \ldots + \int_{t-\delta}^{t} \ldots \right) \leq$$

$$\frac{1}{\sqrt{2\pi}} \left(\sqrt{\frac{2}{t}} \parallel \phi \parallel_{L_1} + \frac{1}{\sqrt{\delta}} \int_{t/2}^\infty \phi(\tau) d\tau + 2 \parallel \phi \parallel_{L^\infty} \sqrt{\delta} \right).$$

Given $\epsilon \geq 0$ we first choose δ such that $\parallel \phi \parallel_{L^\infty} \sqrt{\delta} \leq \epsilon$. We then fix $t_0 > 2\delta$ so that

$$\frac{1}{\sqrt{t_0}} \parallel \phi \parallel_{L^1} + \frac{1}{\sqrt{\delta}} \int_{t/2}^\infty \phi(s) ds \leq \epsilon.$$

We then have

$$\Pi \leq 2\epsilon \qquad \forall x \in R, \ \forall t \geq t_0.$$

This completes the proof of (28).

Let us now describe the qualitative behavior of the solution. The function $\Psi(x,t) \geq 0$ is nonincreasing in time, hence $\lim_{t\to\infty} \Psi(x,t)$ does exist. Therefore $\lim_{t\to\infty} \Omega$ exists and equal to zero. Moreover for all x outside I we have $\lim_{t\to\infty} \Psi(x,t) = 1$ and $\lim_{t\to\infty} \Theta(x,t) = 0$ (as in the proof of Lemma 2).

On the other hand, $\lim_{t\to\infty} \Psi(x,t) = 0$ and $\lim_{t\to\infty} \Theta(x,t) = \gamma^{-1} + \Theta_0(x)$ at least for $x \in I$ where $\Theta_0(x) > \beta$. For other points inside I the situation is a little more complicated. There are two possible situations: $\Psi_\infty(x) = 0$, $\Theta_\infty > \beta$ or $\Psi_\infty(x) \leq 1$, $\Theta_0(x) \leq \Theta_\infty < \beta$. In both cases the reaction rate $\Omega(\Psi_\infty, \Theta_\infty) = 0$. □

4 Successful initiation

In this section we present a family of initial data $\Theta_0(x)$ in order that the solution of the problem (P) meets the following property,

$$\lim_{t\to\infty} \Theta(x,t), \Pi(x,t) > 0 \quad \forall x. \tag{45}$$

Theorem 4. *Let $\Theta_0 \geq 0$ be such that*

$$\lim_{x\to\infty} \Theta_0(x) = 0, \qquad \lim_{x\to-\infty} \Theta_0(x) = \chi \qquad \chi > \beta. \tag{46}$$

Then

$$\lim_{t\to\infty} \Pi(x,t) > \frac{1}{2}, \qquad \lim_{t\to\infty} \Theta(x,t) > \frac{1-\gamma^{-1}}{2} \qquad \forall x. \tag{47}$$

Proof. As a first step consider the behavior of the solution for the problem (P) when $x \to \pm\infty$. Let us introduce the following functions

$$\Pi_\pm(t) = \lim_{x\to\pm\infty} \Pi(x,t), \qquad \Psi_\pm(t) = \lim_{x\to\pm\infty} \Psi(x,t), \qquad \Theta_\pm(t) = \lim_{x\to\pm\infty} \Theta(x,t).$$

The behavior of $\Pi_\pm$,$\Psi_\pm$ are clearly governed by the system,

$$\frac{\partial \Pi_\pm}{\partial t} = \Omega(\Psi_\pm, \Theta_\pm), \qquad \frac{\partial \Psi_\pm}{\partial t} = -\Omega(\Psi_\pm, \Theta_\pm) \tag{48}$$

with the initial conditions

$$\Pi_\pm(0) = 0, \qquad \Psi_\pm(0) = 1, \qquad \Theta_+(0) = 0, \qquad \Theta_-(0) = \chi. \tag{49}$$

Combining equations (48) and integrating with respect to time we have

$$\Pi_-(t) + \Psi_-(t) = 1 \qquad \forall t. \tag{50}$$

The assumption (46) implies that $\Omega(\Psi_-(t), \Theta_-(t)) \mid_{t=0} = \Omega(1,\chi) \neq 0$. Thus the function Π_- increases and Ψ_- decreases up $t = t^*$ (possibly infinity) , where $\Omega = 0$, $\Psi_- = 0$. Using (50) we deduce

$$\Pi_-(t) = 1, \qquad \Psi_-(t) = 0 \quad \forall t > t^*.$$

Given $\epsilon > 0$, choose T such that,

$$\Pi_-(T) \geq 1 - \epsilon,$$

then choose x_0 (close to $-\infty$) so that,

$$\Pi(x, T) \geq 1 - 2\epsilon \qquad \forall x < x_0.$$

Next solve the heat equation,

$$u_t - u_{xx} = 0$$

with the initial data,

$$u(x, 0) = 1 - 2\epsilon \quad \text{at} \quad x < x_0,$$

$$u(x, 0) = 0 \qquad \text{otherwise.}$$

By the maximum principle $\Pi(x, t + T) \geq u(x, t)$ and, as a consequence, $\Theta(x, t + T) \geq (1 - \gamma^{-1})u(x, t)$ for any x. On the other hand

$$\lim_{t \to \infty} u(x, t) = \frac{1}{2}(1 - 2\epsilon)$$

as can be seen from the explicit integral representation of $u(x, t)$, which proves (47). □

Remark. As can be seen from the explicit representation for $u(x, t)$ for any $k > 0$

$$\lim_{t \to \infty} \Theta(k\sqrt{t}, t) > 0, \quad \lim_{t \to \infty} \Pi(k\sqrt{t}, t) > 0$$

These inequalities mean that for the initial data (46) the reaction propagates with a speed at least of the order of $1/\sqrt{t}$.

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