Periodic kinks in reaction - diffusion systems.

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Abstract

In this letter we show that, in nonlinear dissipative homogeneous media, a new effect is possible: the propagation of nonlinear waves (kinks) with time periodic rates. Our example is a reaction - diffusion system consisting of equations of Ginzburg-Landau's type.

1 Introduction

The nonlinear wave propagation plays a key role in physics, biology and chemistry.

Beginning with the famous works [1][2], a number of papers have been focused on traveling wave solutions. For 1D media, these waves (kinks) have the form $u = u(x - vt)$, where v is the velocity. When we deal with parabolic equations (or so-called monotone systems [3]), all the wave solutions have such form.

Not all systems are monotone; the asymptotic behavior of solutions of monotone systems is rather well understood: they converge either to a non-distributed equilibrium, to traveling waves or to a system of waves propagating with different velocities. It is expected that a small perturbation destroying the monotone character could lead to a very complicated behavior, including periodically and even chaotically oscillating fronts. For general non-monotone systems, it has been shown in [4][5], but the models from [4][5] have a complicated form.

The aim of this work is to show that, even for simplest reaction - diffusion systems with cubic nonlinearities (describing homogeneous media), this effect is possible. Namely, the kinks exist, but their forms are complicated. They move with time periodic velocities within long time intervals.

The model under consideration is

$$\frac{\partial u_i}{\partial t} = \frac{\partial^2 u_i}{\partial x^2} + 2(u_i - u_i^3) + \epsilon f_i(\mathbf{u}), \qquad (1)$$

where $x \in \mathbf{R}$, the vector $\mathbf{u} = (u_1, u_2, ...u_n)$ is an order parameter, ϵ is a small number and functions f_i are smooth and bounded.

First, let us describe a general approach to these systems [4][5].

2 Investigation of system (1)

For small ϵ and special initial data, system (1) can be reduced to some ordinary differential equations (ODE) [4][5].

We shall seek the solution of (1) in the following form

$$u_i(x,t) = \tanh(x - q_i(t)) + w_i(x,t). \tag{2}$$

Suppose initial data $u_i(x,0)$ have the form (2), where functions $w_i(x,t)$ at the initial moment $(t=0)$ satisfy

$$\sup |w_i(x,t)| < C\epsilon, \quad C > 0 \tag{3}$$

where C is some constant.

Briefly, the method can be described as follows. We make the substitution $\mathbf{u} \to (\mathbf{w}, \mathbf{q})$, $\mathbf{w} = \mathbf{w}(x,t)$, $\mathbf{q}(\mathbf{t}) = (q_1(t), q_2(t), ..., q_n(t))$ where parameters $q_i(t)$ are defined by

$$\int_{-\infty}^{\infty} \cosh^{-2}(x - q_i(t))\mathbf{w}(x,t)dx = 0, \quad i = 1, 2, ...n. \tag{4}$$

This condition has a transparent interpretation. We approximate the solution $u_i(x,t,\epsilon)$ with a maximal accuracy in L_2 - norm by the unperturbed fronts $\tanh(x - q_i)$ (which give the exact solution for $\epsilon = 0$). One can show, for small w_i, this substitution $\mathbf{u} \to (\mathbf{q}, \mathbf{w})$ is correctly defined and allows us to find q_i. The function w_i describes a small distortion of i - th front. These distortions occur due to the perturbations ϵf_i. Notice the functions $\cosh^{-2}$ are the Goldstone modes generated by the kink translations. Thus, from the physical point of view, eqs. (4) mean that from the corrections w_i we exclude some basic contributions connected with translation motions. (This is a classical idea; see for example [6],[7]).

Physically, the solution $\mathbf{u} = (u_1, u_2, ..., u_n)$ can be considered as a connected state of the kinks. Each kink has the corresponding coordinate $q_i(t)$ and the velocity $v_i = \frac{dq_i}{dt}$.

There holds **Proposition**.

1) There exists a constant C_1 such that, for sufficiently small ϵ, one has $|w_i(x,t)| < C_1\epsilon$ (this means that the front distortions remain small).

2) Functions q_i satisfy

$$\frac{dq_i}{dt} = \epsilon\tilde{h}_i(\mathbf{w}, \mathbf{q}, \epsilon), \qquad \tilde{h}_i = \frac{3}{4}h_i(\mathbf{q}) + \gamma_i(\mathbf{w}, \mathbf{q}, \epsilon), \tag{5}$$

where $|\gamma_i| < C_\gamma\epsilon$ and

$$h_i = -\int_{-\infty}^{\infty} f_i(\tanh(x-q_1), \tanh(x-q_2), ..., \tanh(x-q_n)) \cosh^{-2}(x-q_i)dx. \tag{6}$$

This assertion is sufficiently standard and can be proved by classical methods [7, Ch.5]. Similar results were obtained in [4],[5].

As a consequence of the translation invariance, system (5) can be simplified. In fact, it is easy to see that the functions h_i depend only on the coordinate differences $q_i - q_j$.

Define new (relative) coordinates $Q_i = q_i - q_n, \;\; i = 1, ...n-1$. Then system (5) can be rewritten (we remove small corrections γ_i and use a rescaling time $\tau = 3\epsilon t/4$)

$$\frac{dQ_i}{d\tau} = H_i(\mathbf{Q}), \quad \frac{dq_n}{d\tau} = h_n(\mathbf{Q}), \tag{7}$$

where $H_i = h_i - h_n$, $\mathbf{Q} = (Q_1, Q_2, ...Q_{n-1})$ and $i = 1, ..., n-1$.

The variables Q_i specify relative positions of kinks in the connected state of the kinks. The time evolution of the mass center of the kinks system is defined by $q_c = n^{-1}(q_1 + q_2 + ... + q_n)$.

Below we shall show, that even for linear functions f_i, the coordinates Q_i and q_c can periodically oscillate.

3 The periodic motion of nonlinear waves

Let us consider system (1) with $n = 3$ and set

$$f_1 = -u_3 + \mu, \quad f_2 = u_3 + \mu, \quad f_3 = u_1 - u_2 + \mu, \tag{8}$$

where μ is a parameter.

The asymptotic solution has the form (2). Following sect. 2, let us introduce the relative coordinates $Q_1 = q_1 - q_3, Q_2 = q_2 - q_3$.

Then equations (7) lead to

$$\frac{dQ_1}{d\tau} = F(Q_2), \quad \frac{dQ_2}{d\tau} = -F(Q_1) \tag{9}$$

where the function F is defined by

$$F(Q) = \frac{2}{\tanh Q} - \frac{2Q}{\sinh^2 Q}. \tag{10}$$

This system is a Hamiltonian conservative system, with energy

$$E = \frac{Q_1}{\tanh Q_1} + \frac{Q_2}{\tanh Q_2}. \tag{11}$$

Clearly, all the solutions of (9) are time periodic. For the original coordinates q_i we obtain

$$q_1 = -Q_2 - 2\mu\tau, \quad q_2 = -Q_1 - 2\mu\tau, \quad q_3 = -Q_1 - Q_2 - 2\mu\tau. \tag{12}$$

Therefore, the motion of the center mass of this kink system is periodic and is defined by

$$q_c(t) = \frac{1}{3}(q_1 + q_2 + q_3) = -\frac{2}{3}(Q_1 + Q_2) - \mu\frac{3}{2}\epsilon t \tag{13}$$

Finally, the basic physical results are as follows. The nonlinear wave is a kink system (the connected state of kinks). The mass center of this system propagates with a time periodic speed V. Moreover, all kinks periodically oscillate at this mass center. The kink center mass average velocity is constant and is equal to $\bar{V} = \frac{3}{2}\epsilon\mu$. When $\mu = 0$, we observe the standing connected states of the periodically oscillating kinks. In the case $Q_1 = Q_2 = 0$ we obtain a stable traveling front (oscillations vanish).

4 Results of computer simulations. Conclusion

These results have been checked by computer simulations. Actually, the periodic oscillations occur, however (depending on ϵ) they slowly damp and for large times we observe a standing wave. The periodic regime exists within time intervals of the order ϵ^{-2}. For large ϵ, the kink fronts are strongly distorted, the periodic motion vanishes and the standing wave appears at once.

These results can be analytically explained, if we take into account the small dissipative terms $\epsilon\gamma_i(q,t)$ removed in (9).

In fact, the original system (1) is dissipative, thus *exact* system (5) has the same property. On the other hand, truncated systems (6) and (9) are conservative. Therefore, one concludes that the dissipation in the kink evolution is induced by the small corrections $\epsilon\gamma_i$.

These dissipative contributions break the periodic motion. Since they are very small, with comparison to principal terms (9), actually system (6) (describing the kink motion) is a nonlinear oscillator with a small lubrication. Clearly, it generates the slowly damping periodic oscillations.

It is interesting to compare this effect with the well known breather occurring in sine -Gordon equation [8]. The breather is a localized and immobile connected state of two kinks (solitons). In contrast to the breather, our waves can move as a whole with the periodic rates. Another important difference is that our solution is structurally stable: it survives under small perturbations (for example, one can replace $2(u-u^3)$ to any bistable nonlinearity), whereas the breather is unstable and vanishes even under small perturbations of the sine - Gordon equation (It has been well understood in the last few years, see for example [8]). At last, the third difference is that the breathers exist eternally, whereas these periodic kinks survive for a long (but finite) time.

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