The KPP type flame fronts in porous media

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Abstract

In this paper we study the model of pressure driven flames in porous media proposed in (Brailovsky et al. 1997 *Combust. Sci. and Tech.*, **124** 145-165). We show that, under the assumption of first order reaction with linear reaction kinetics (quadratic nonlinearity), the model admits a family of positive traveling wave solutions. Moreover, under the same assumption, we prove that propagation of disturbances in the system is fully determined by the rate of decay of the initial data at infinity. We also give an upper bound of the burning rate in the case of arbitrary chemical kinetics bounded by linear function.

Key words: KPP systems, reactive fronts, porous media

AMS Subject Classifications: 35K57, 34C37, 80A25

1 Introduction

It has long been known that the presence of obstacles may have a profound effect on propagation of flame fronts, sufficiently reducing their propagation velocity. Starting from the early works of Laffitte [13], and Shchelkin [16], obstacle-affected combustion processes have been extensively studied by physicists and engineers. Although significant insights have been obtained through many excellent experimental and theoretical studies, a comprehensive picture of the process is still lacking. Recently, G. Sivashinsky and his collaborators proposed a model describing combustion in inert porous media under condition of high hydraulic resistance [3]. The model reads,

$$\begin{aligned} &T_t - (1-\gamma^{-1})P_t = \bar{\epsilon}T_{xx} + YF(T), \\ &P_t - T_t = P_{xx}, \\ &Y_t = \bar{\epsilon}\mathrm{Le}^{-1}Y_{xx} - \gamma YF(T). \end{aligned} \tag{1.1}$$

Here T, P and Y are the appropriately normalized temperature, pressure and concentration of the deficient reactant, $\gamma > 1$ is the specific heat ratio, $YF(T)$ is the normalized reaction rate, $\bar{\epsilon}$ is the ratio of thermal and pressure diffusivities and Le is a Lewis number. The first and the last equations in (1.1) represent the partially linearized conservation equations for energy and deficient reactant, while the second one follows from the linearized continuity equation, and equations of state and momentum.

The model (1.1) is apparently the simplest model for studying combustion in porous media. Despite of its simplicity, the model (1.1) captures a large variety of major physical effects observed experimentally. In particular, it is capable to describe transition from slowly spreading thermo-diffusive wave to fast pressure-driven fronts [17]. This effect is called the transition from deflagration to detonation, a problem which still remains one of the most challenging issues in combustion theory [20, 19].

There has been a number of studies of the system (1.1), (see recent reviews [5, 17] for references). Nevertheless, most of the results were obtained under the assumption of ignition type non-linearity. In this paper we are mostly concerned about the situation where the reaction kinetics term $F(T)$ is bounded by a linear function.

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We start with a very convenient reformulation of the problem. Following [6], we introduce new functions, u and v, related to P and T by the following linear transformation,

$$
\begin{aligned}
T(t,x) &= hu(t,x) + (1-h)v(t,x), \\
P(t,x) &= \frac{1}{1-\bar{\epsilon}}\left(u(t,x) - \bar{\epsilon}v(t,x)\right),
\end{aligned}
\tag{1.2}
$$

where $\mu = (\sqrt{[(\gamma-\bar{\epsilon})^2 + 4\bar{\epsilon}(\gamma-1)]} + \bar{\epsilon} - \gamma)/2\bar{\epsilon}$ is a positive solution of the following equation,

$$
(1-\mu)(\gamma + \bar{\epsilon}\mu) = 1. \tag{1.3}
$$

In terms of new functions, after rescaling $x \to (1-\mu)^{1/2}x$, we obtain the following system which is equivalent to (1.1),

$$
\begin{aligned}
u_t &= u_{xx} + YF(T), \\
v_t &= \varepsilon v_{xx} + YF(T), \\
Y_t &= \varepsilon((1-\mu)\mathrm{Le})^{-1}Y_{xx} - YF(T), \\
T(t,x) &= hu(t,x) + (1-h)v(t,x),
\end{aligned}
\tag{1.4}
$$

where $\varepsilon = \bar{\epsilon}(1-\mu)^2$ and $h = \frac{\mu}{1-\bar{\epsilon}}$.

Throughout this paper we assume that $\varepsilon \in (0,1)$. This is a natural assumption since the ratio of the thermal and pressure diffusivities is usually small [17]. In this setting one can see that $h \in (0,1)$.

It is important to note, that the system (1.4) can be reduced to the system of two equations in the case when $\mathrm{Le} = \mathrm{Le}^* = (1-\mu)^{-1}$. This situation resembles the situation arising in regular thermo-diffusive combustion when the Lewis number equals unity. Indeed, when $\mathrm{Le} = \mathrm{Le}^*$, by adding the second and the third equations of the system (1.4) one can observe that $Y = 1 - v$ for all times if this condition is satisfied at the time $t = 0$. Thus, in this case the resulting model reads,

$$
\begin{aligned}
u_t &= u_{xx} + (1-v)F(w), \\
v_t &= \varepsilon v_{xx} + (1-v)F(w), \\
w(t,x) &= hu(t,x) + (1-h)v(t,x).
\end{aligned}
\tag{1.5}
$$

Remark 1.1 The system (1.4) and, especially (1.5), resembles systems arising in conventional thermo-diffusive combustion. In fact, if $h = 1$ the system (1.5) becomes formally identical to the model of thermo-diffusive combustion. The major difference is that the nonlinear term F in (1.5) is a function of the linear combination of u and v, whereas, in conventional combustion, F is a function of a single variable. As a matter of fact, this difference provides not only additional technical difficulties for the analysis of the system, it also gives rise to a number of new effects which are impossible in conventional combustion. For example, traveling wave solutions of the system (1.4), in the so-called high activation energy limit, have rather anomalous exponential scaling [8].

In this paper we mostly restrict ourselves to the case of the linear kinetics ($F(w) = w$) with $\mathrm{Le} = \mathrm{Le}^*$. In this case, the system (1.1) reduces to the following system of two reaction diffusion equations:

$$
\begin{aligned}
u_t &= u_{xx} + (1-v)w, \\
v_t &= \varepsilon v_{xx} + (1-v)w, \\
w &= hu + (1-h)v.
\end{aligned}
\tag{1.6}
$$

Remark 1.2 It is important to note that when $h = 0$ the system (1.6) decouples and the second equation becomes the classical KPP-Fisher model [12]. In the case when $h = 1$, the system (1.6) reduces to the model considered by Billingham and Needham in [2]. Thus, the model considered in this paper provides the link between the two models by now considered classical.

The paper is organized as follows. In Section 2 we study traveling wave solutions for (1.6). Section 3 is devoted to the long time behavior of solutions of system (1.6) under assumption of front-like initial data. In Section 4 we obtain an upper bound for the bulk burning rate for the system (1.4) under the assumption that the reaction term $F(w)$ is non-decreasing positive function satisfying $F(w) < w$.

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2 Traveling wave solutions

In this section we study positive traveling wave solutions for the problem (1.6), that is, solutions of the form

$$(u,v)(t,x) = (U,V)(\xi), \quad \xi = x - ct, \tag{2.1}$$

where c is a priori unknown velocity of propagation. Substituting ansatz (2.1) into (1.6) we have

$$\begin{aligned} &U'' + cU' + (1-V)W = 0, \\ &\varepsilon V'' + cV' + (1-V)W = 0, \\ &W = hU + (1-h)V, \end{aligned} \tag{2.2}$$

where the derivatives are taken with respect to ξ. The problem (2.2) is considered with the following boundary-like conditions,

$$(U,V) \to (0,0), \quad \xi \to \infty, \tag{2.3}$$

$$(U,V) \to (1,1), \quad \xi \to -\infty. \tag{2.4}$$

Linearizing (2.2) around $(U,V) = (0,0)$ and substituting $(U,V) = (M,N)\exp(-\lambda\xi)$ we obtain,

$$\begin{aligned} &(\lambda^2 - c\lambda + h)M + (1-h)N = 0, \\ &hM + (\varepsilon\lambda^2 - c\lambda + (1-h))N = 0. \end{aligned} \tag{2.5}$$

Equating the determinant of the system (2.5) to zero, one has,

$$(\lambda^2 - c\lambda + h)(\varepsilon\lambda^2 - c\lambda + (1-h)) = h(1-h). \tag{2.6}$$

Equation (2.6) has two solutions giving relation between velocity c and rate of decay λ (see Figure 2.1),

$$c_{\pm}(\lambda) = \frac{1}{2}\left[\frac{1}{\lambda} + (1+\varepsilon)\lambda \pm \sqrt{\left(\frac{1}{\lambda} + (1-\varepsilon)\lambda\right)^2 - 4(1-\varepsilon)(1-h_\varepsilon)}\right]. \tag{2.7}$$

However, one can easily verify that when $c = c_-$, components M and N of any nontrivial solution of the

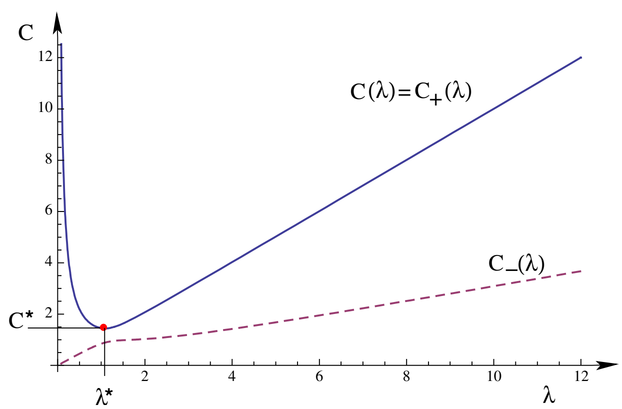

Figure 2.1: Velocity of propagation c vs. rate of decay λ with $\varepsilon = 0.3$ and $h = 0.1$.

system (2.5) must have opposite signs. Thus, for positive solutions of (2.2), the velocity of propagation and the rate of decay are related by the following expression,

$$c(\lambda) = \frac{1}{2}\left[\frac{1}{\lambda} + (1+\varepsilon)\lambda + \sqrt{\left(\frac{1}{\lambda} + (1-\varepsilon)\lambda\right)^2 - 4(1-\varepsilon)(1-h_\varepsilon)}\right]. \tag{2.8}$$

Straightforward computations show that,

$$\frac{d^2c(\lambda)}{d\lambda^2} = \frac{1}{\lambda^3}\left(1+\frac{\left(\frac{1}{\lambda}-(1-\varepsilon)\lambda\right)^3}{\left[\left(\frac{1}{\lambda}-(1-\varepsilon)\lambda\right)^2+4h(1-\varepsilon)\right]^{3/2}}\right)+\frac{2h(1-\varepsilon)\left((1-\varepsilon)^2+\frac{3}{\lambda^4}\right)}{\left[\left(\frac{1}{\lambda}-(1-\varepsilon)\lambda\right)^2+4h(1-\varepsilon)\right]^{3/2}}. \tag{2.9}$$

It is clearly seen from (2.9) that for $h \in (0,1]$ and $\varepsilon \in (0,1)$

$$\frac{d^2c(\lambda)}{d\lambda^2} > 0. \tag{2.10}$$

Thus, $c(\lambda)$ is strictly convex on $\lambda \in (0,\infty)$. Moreover, $\lim_{\lambda\to 0} c(\lambda) = \infty$ and $\lim_{\lambda\to\infty} c(\lambda) = \infty$. Therefore, there exists a unique value $0 < \lambda^* < \infty$, where the minimum of $c(\lambda)$ is attained. We call the minimum value of the expression (2.8) the critical velocity and set $c^* = c(\lambda^*)$. The dependence of critical velocity c^* on the parameters h and ε is shown on Figure 2.2. Similar to the case of single KPP equation, the existence of positive solution for the problem (2.2)(2.3) (2.4) crucially depends on weather $c \geq c^*$ or $c < c^*$.

Theorem 2.1 *For any $c \geq c^*$, the system (2.2) has a unique, up to translation, positive traveling wave solution (front) $U(\xi), V(\xi)$ asymptotically connecting the equilibrium $(U,V) = (1,1)$ as $\xi \to -\infty$ to the equilibrium $(U,V) = (0,0)$ as $\xi \to \infty$. The solution converges to both of equilibria exponentially fast. Moreover,*

$$0 < U(\xi),\ \ V(\xi) < 1, \quad -\frac{1+h}{2c} < U'(\xi),\ \ V'(\xi) < 0 \quad \forall \xi \in \mathbb{R}. \tag{2.11}$$

The system (2.2) has no positive traveling wave solutions for $c < c^*$.

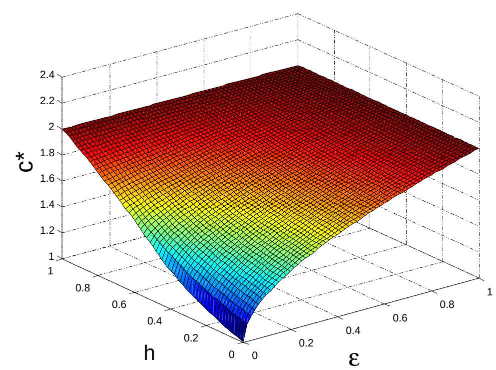

Figure 2.2: Critical velocity c^* as a function of h and ε.

The Theorem 2.1 follows from the three lemmas.

Lemma 2.1 *System (2.2), (2.3), (2.4) has no positive solutions for $c < c^*$.*

Proof. For any $c < c^*$ every solution $\lambda(c)$ of the equation (2.8) is complex. This, in turn, implies that every solution of problem (2.2), if it exists, will be oscillatory when it approaches the equilibrium point $(U,V) = (0,0)$ as $\xi \to \infty$. Since oscillations occur around zero, U and V will take both positive and negative values for sufficiently large ξ. Thus, the statement of the lemma holds. ■

Lemma 2.2 *Let $c \geq c^*$. Assume that the system (2.2) has a bounded non-constant solution satisfying boundary condition (2.4) at $-\infty$. Then,*

$$0 < U(\xi),\ \ V(\xi),\ \ W(\xi) < 1, \quad -\frac{1+h}{2c} < U'(\xi),\ \ V'(\xi),\ \ W'(\xi) < 0 \quad \forall \xi \in \mathbb{R}. \tag{2.12}$$

Proof. For the sake of convenience we rewrite the system (2.2) as follows

$$U'' + cU' + hU + (1-h)V = VW,$$
$$\varepsilon V'' + cV' + hU + (1-h)V = VW. \tag{2.13}$$

Let us show first that if there is a solution of (2.2),(2.4) then it is positive in all the components. We first multiply (2.13) by $e^{\lambda\xi}$ and integrate it from $-\infty$ to ξ. After integration by parts, the system reads,

$$U' + (c-\lambda)U + (\lambda^2 - \lambda c + h)\int_{-\infty}^{\xi} U(y)e^{\lambda(y-\xi)}dy + (1-h)\int_{-\infty}^{\xi} V(y)e^{\lambda(y-\xi)}dy =$$
$$= \int_{-\infty}^{\xi} V(y)W(y)e^{\lambda(y-\xi)}dy, \tag{2.14}$$
$$\varepsilon V' + (c-\varepsilon\lambda)V + h\int_{-\infty}^{\xi} U(y)e^{\lambda(y-\xi)}dy + \big(\varepsilon\lambda^2 - c\lambda + (1-h)\big)\int_{-\infty}^{\xi} V(y)e^{\lambda(y-\xi)}dy =$$
$$= \int_{-\infty}^{\xi} V(y)W(y)e^{\lambda(y-\xi)}dy.$$

Multiplying the first and the second equations of (2.14) by h and by $\lambda^2 - c\lambda + h$ respectively, subtracting the results and taking into account (2.6) we obtain,

$$h\left(U' + (c-\lambda)U\right) - (\lambda^2 - c\lambda + h)(\varepsilon V' + (c-\varepsilon\lambda)V) = \lambda(c-\lambda)\int_{-\infty}^{\xi} V(y)W(y)e^{\lambda(y-\xi)}dy. \tag{2.15}$$

Subtracting the first and the second equations of (2.13) and integrating from $-\infty$ to ξ we have,

$$\varepsilon V' + cV = U' + cU. \tag{2.16}$$

Combining (2.15) and (2.16) one has

$$U' + a_1 U = b_1 V + \int_{-\infty}^{\xi} V(y)W(y)e^{\lambda(y-\xi)}dy,$$
$$V' + a_2 V = b_2 U + \frac{1}{\varepsilon}\int_{-\infty}^{\xi} V(y)W(y)e^{\lambda(y-\xi)}dy, \tag{2.17}$$

where

$$a_1 = \frac{c^2 - c\lambda - h}{c-\lambda}, \quad a_2 = \frac{\varepsilon h + (c-\varepsilon\lambda)(c-\lambda)}{\varepsilon(c-\lambda)}, \quad b_1 = \varepsilon\frac{c\lambda - \lambda^2 - h}{c-\lambda}, \quad \text{and} \quad b_2 = \frac{\lambda h}{\varepsilon(c-\lambda)}. \tag{2.18}$$

It is straightforward to verify using (2.6) that a_1, a_2, b_1, $b_2 > 0$, provided $c \geq c^*$. Multiplying the first and the second equations of the system (2.17) by $e^{a_1\xi}$ and $e^{a_2\xi}$ respectively and integrating from $-\infty$ to ξ we obtain,

$$U(\xi) = b_1\int_{-\infty}^{\xi} V(y)e^{a_1(y-\xi)}dy + \int_{-\infty}^{\xi} e^{a_1(y-\xi)}dy\int_{-\infty}^{y} V(z)W(z)e^{\lambda z - y}dz,$$
$$V(\xi) = b_2\int_{-\infty}^{\xi} U(y)e^{a_2(y-\xi)}dy + \frac{1}{\varepsilon}\int_{-\infty}^{\xi} e^{a_2(y-\xi)}dy\int_{-\infty}^{y} V(z)W(z)e^{\lambda z - y}dz. \tag{2.19}$$

An assumption that U, V and, thus, W approach unity when $\xi \to -\infty$ and continuity of these functions imply that U, V, and W remain positive when ξ is close to $-\infty$. Assume that ξ_0 is a largest number such that $U(\xi), V(\xi) > 0$ for all $\xi \in (-\infty, \xi_0)$. We claim that $\xi_0 = \infty$. Assume that ξ_0 is finite, then $\min(U(\xi_0), V(\xi_0)) = 0$. However, according to (2.19),

$$U(\xi_0) = b_1\int_{-\infty}^{\xi_0} V(y)e^{a_1(y-\xi_0)}dy + \int_{-\infty}^{\xi_0} e^{a_1(y-\xi_0)}dy\int_{-\infty}^{y} V(z)W(z)e^{\lambda(z-y)}dz > 0,$$
$$V(\xi_0) = b_2\int_{-\infty}^{\xi_0} U(y)e^{a_2(y-\xi_0)}dy + \frac{1}{\varepsilon}\int_{-\infty}^{\xi_0} e^{a_2(y-\xi_0)}dy\int_{-\infty}^{y} V(z)W(z)e^{\lambda(z-y)}dz > 0, \tag{2.20}$$

which is a contradiction. Thus U, V and W are positive.

Now, let us show that every bounded non-constant solution of the problem (2.2) satisfies $V(\xi) < 1$ for all $\xi \in \mathbb{R}$.

First, assume that ξ_1 and ξ_2 are two points such that, $V(\xi_1) = V(\xi_2) = 1$. Since $V(\xi)$ is continuous, there exist two points $\xi_m, \xi_M \in [\xi_1, \xi_2]$, where $V(\xi)$ attains its minimum and maximum respectively. In order for V not to be identically one, V must have internal extrema on (ξ_1, ξ_2), that is either $V(\xi_m) < 1$, $V'(\xi_m) = 0$ or $V(\xi_M) > 1$, $V'(\xi_M) = 0$. However, according to the second equation of the system (2.2) we have $V''(\xi_m) = -(1 - V(\xi_m))W(\xi_m) < 0$ and $V''(\xi_M) = -(1 - V(\xi_M))W(\xi_M) > 0$ which is a contradiction. Thus, $V(\xi) = 1$ on $[\xi_1, \xi_2]$. Moreover, since V is $C^1(\mathbb{R})$ we also have $V'(\xi_1) = V'(\xi_2) = 0$. Note that the Cauchy problem,

$$\varepsilon S'' + cS' + (1 - S)Q = 0, \quad S(\xi_1) = 1, \quad S'(\xi_1) = 0, \tag{2.21}$$

considered on an interval $(-\infty, \xi_1]$ has a solution $S(\xi) = 1$ for any constants ε, c and any function $Q(\xi)$, as can be verified by direct substitution. Moreover, this solution is unique provided Q is continuous, which follows from standard result for systems of linear ODE's [11]. By the assumption of the lemma, W is continuous and bounded. Thus, setting $S = V$ and $Q = W$ in (2.21) we have $V(\xi) = 1$ for $(-\infty, \xi_1]$. Similar arguments also give $V(\xi) = 1$ on $[\xi_1, \infty)$. Therefore, we have $V(\xi) = 1$ for all $\xi \in \mathbb{R}$.

Thus, any non-constant solution V may touch or intersect the line $V = 1$ no more than once. Let us show now that this also does not happen. First, multiplying the first and the second equations of the system (2.2) by $e^{c\xi}$ and $e^{c\xi/\varepsilon}$ receptively, and integrating by parts we have,

$$\begin{aligned} U' &= -\int_{-\infty}^{\xi} (1 - V(y))W(y)e^{c(y-\xi)}dy, \\ V' &= -\frac{1}{\varepsilon}\int_{-\infty}^{\xi} (1 - V(y))W(y)e^{c(y-\xi)/\varepsilon}dy. \end{aligned} \tag{2.22}$$

Next, assume that ξ_0 is the only point such that $V(\xi_0) = 1$ we then have either $V(\xi) < 1$ for all $\xi < \xi_0$ or $V(\xi) > 1$ for all $\xi < \xi_0$. However, by the second equation of (2.22) we have $V'(\xi) < 0$ on $(-\infty, \xi_0)$ in the first case and $V' > 0$ on $(-\infty, \xi_0)$ in the second case. Thus we have either $V(\xi) < 1$ and strictly decreasing on $(-\infty, \xi_0)$ or $V(\xi) > 1$ and strictly increasing on $(-\infty, \xi_0)$, which is in obvious contradiction with the assumption $V(\xi_0) = 1$. Thus, we conclude that either $V > 1$ or $V < 1$ for all $\xi \in \mathbb{R}$. If $V > 1$ then by the second equation of (2.22) we have that V is increasing and

$$|V'| > \frac{1-h}{\varepsilon}\int_{-\infty}^{\xi} |1 - V(y)|e^{c(y-\xi)/\varepsilon}dy > k, \tag{2.23}$$

where k is some positive constant, which immediately implies that V is unbounded as $\xi \to \infty$. Therefore, any bounded non-constant solution of (2.2) must have $V(\xi) < 1$ for all $\xi \in \mathbb{R}$.

Since $0 < V < 1$ and $W > 0$, the nonlinear therm $(1 - V)W$ is positive, which implies that any solution of the system (2.2) is monotone. Indeed, since $(1 - V)W > 0$ it is follows immediately from (2.22) that $U', V' < 0$ and, consequently, $W' < 0$. Therefore, $U(\xi), W(\xi) < 1$ for all $\xi \in \mathbb{R}$. Moreover, since

$$(1 - V)W = h(1 - V)U + (1 - h)(1 - V)V < \frac{1+h}{2}, \tag{2.24}$$

we have

$$\begin{aligned} |U'| &= \left|\int_{-\infty}^{\xi} (1 - V(y))W(y)e^{c(y-\xi)}dy\right| < \frac{1+h}{2}\int_{-\infty}^{\xi} e^{c(y-\xi)}dy = \frac{1+h}{2c}, \\ |V'| &= \left|\frac{1}{\varepsilon}\int_{-\infty}^{\xi} (1 - V(y))W(y)e^{c(y-\xi)/\varepsilon}dy\right| < \frac{1+h}{2\varepsilon}\int_{-\infty}^{\xi} e^{c(y-\xi)/\varepsilon}dy = \frac{1+h}{2c}. \end{aligned} \tag{2.25}$$

Combining the first and the second equations of (2.25), we also have

$$|W'| < \frac{1+h}{2c}, \tag{2.26}$$

which completes the proof. ■

Let us now show that any solution of the system (2.2) that converges to the equilibrium $(U,V)=(1,1)$ does so at an exponential rate. Indeed, as a system of first order ODE's, (2.2) reads,

$$\begin{aligned} U' &= U_1, \\ U_1' &= -cU_1-(1-V)(hU+(1-h)V), \\ V' &= V_1, \\ \varepsilon V_1' &= -cV_1-(1-V)(hU+(1-h)V). \end{aligned} \tag{2.27}$$

The system (2.27) has two one-dimensional manifolds of equilibria L_1, L_2 defined as,

$$\begin{aligned} L_1 &= \{U=\alpha,\ U_1=0,\ V=1,\ V_1=0\}, \\ L_2 &= \{U=-\frac{1-h}{h}V,\ U_1=0,\ V=\beta,\ V_1=0\}, \end{aligned} \tag{2.28}$$

where $\alpha,\ \beta\in\mathbb{R}$.

To analyze solutions of (2.27) in the neighborhood of the equilibrium $(U,U_1,V,V_1)=(1,0,1,0)\in L_1$, we linearize (2.27) about any point from L_1 and obtain,

$$\begin{aligned} \tilde{U}' &= \tilde{U}_1, \\ \tilde{U}_1' &= -c\tilde{U}_1+(h\alpha+(1-h))\tilde{V}, \\ \tilde{V}' &= \tilde{V}_1, \\ \varepsilon\tilde{V}_1' &= -c\tilde{V}_1+(h\alpha+(1-h))\tilde{V}. \end{aligned} \tag{2.29}$$

The linearized system (2.29) has the following set of eigenvalues: 0, $-c$ and $(-c\pm\sqrt{c^2+4(h\alpha+1-h)})/2$. At any point of L_1, except $(U,U_1,V,V_1)=((h-1)/h,0,1,0)$, the zero eigenvalue is simple, two of the eigenvalues are negative and one is positive. Therefore, in a neighborhood of $(U,U_1,V,V_1)=(1,0,1,0)$ that does not contain $(U,U_1,V,V_1)=((h-1)/h,0,1,0)$, each point of L_1 has a one dimensional central manifold which is L_1 itself, a two dimensional stable manifold, and a one dimensional unstable manifold. So in such neighborhood, L_1 is normally hyperbolic, and therefore a solution of (2.27) that approaches an equilibrium from that neighborhood does so exponentially fast. In particular, this is true for the point $(U,U_1,V,V_1)=(1,0,1,0)$. Therefore, solutions of (2.2)that converge to the equilibrium (2.4), if they exist, converge exponentially fast.

Using this information, we build a system which is not equivalent to (2.27) in a general sense, but, nevertheless, is equivalent to (2.27) considered with the boundary condition $(U,U_1,V,V_1)\to(1,0,1,0)$ as $\xi\to-\infty$.

We integrate (2.2) from $-\infty$ to ξ and introduce a new variable R,

$$R(\xi)=1-\frac{1}{c}\int_{-\infty}^{\xi}(1-V(s))(hU(s)+(1-h)V(s))\,\mathrm{d}s, \tag{2.30}$$

thus obtaining a system of first order ODE's

$$\begin{aligned} U' &= -cU+cR, \\ \varepsilon V' &= -cV+cR, \\ R' &= -\frac{1}{c}(1-V)(hU+(1-h)V). \end{aligned} \tag{2.31}$$

The three dimensional system (2.31) has only two equilibria $(U,V,R)=(0,0,0)$ and $(U,V,R)=(1,1,1)$. We seek solutions of (2.31) satisfying boundary-like conditions,

$$(U,V,R)\to(0,0,0),\quad \xi\to\infty, \tag{2.32}$$
$$(U,V,R)\to(1,1,1),\quad \xi\to-\infty. \tag{2.33}$$

Any solution of (2.31) satisfying (2.32), (2.33) produces a solution of (2.27) which converges to $(U,U_1,V,V_1)=(1,0,1,0)$ as $\xi\to-\infty$ and to $(U,U_1,V,V_1)=(0,0,0,0)$ as $\xi\to\infty$ and vice versa.

We also note that the results of the Lemma 2.2 imply that if there is a bounded non constant solution of (2.31), (2.33) with $c \geq c^*$ then the following estimates hold,

$$0 < R(\xi) < 1 \qquad \text{and} \qquad R'(\xi) < 0 \qquad \forall \xi \in \mathbb{R}. \tag{2.34}$$

Indeed, since the nonlinear term $(1-V)(hU + (1-h)V)$ is positive we have from (2.30) that R is strictly decreasing. Moreover, since V is bounded and strictly decreasing we have $V'(\xi) \to 0$ as $\xi \to \infty$. Then, by the second equation of (2.31), we have $R(\xi) \to V(\xi)$ as $\xi \to \infty$ which combined with the fact that V is positive yields (2.34).

In a following lemma we show existence and uniqueness of solution for the problem (2.31), (2.32), (2.33) with $c \geq c^*$ which immediately implies existence and uniqueness of the front for the problem (2.2), (2.3), (2.4).

Lemma 2.3 *For any $c \geq c^*$, system (2.31) has a unique orbit satisfying (2.32), (2.33), and hence a unique, up to translation, traveling wave. Moreover, all components of the traveling wave are positive, decreasing, and converge to the corresponding equilibria exponentially fast.*

Proof. Observe first that the linearization of (2.31) about $(U, V, R) = (1, 1, 1)$ reads,

$$\begin{aligned} \tilde{U}' &= -c\tilde{U} + c\tilde{R}, \\ \tilde{V}' &= -\frac{c}{\varepsilon}\tilde{V} + \frac{c}{\varepsilon}\tilde{R}, \\ \tilde{R}' &= \frac{1}{c}\tilde{V}. \end{aligned} \tag{2.35}$$

The system (2.35) has two strictly negative eigenvalues $-c$, $\frac{1}{2}\left(-\frac{c}{\varepsilon} - \sqrt{\frac{c^2}{\varepsilon^2} + \frac{4}{\varepsilon}}\right)$ and one strictly positive eigenvalue $\nu = \frac{1}{2}\left(-\frac{c}{\varepsilon} + \sqrt{\frac{c^2}{\varepsilon^2} + \frac{4}{\varepsilon}}\right)$. Therefore, the equilibrium point $(U, V, R) = (1, 1, 1)$ is hyperbolic. By the Hartman-Grobman theorem [10], the nonlinear system (2.31) in the neighborhood of hyperbolic equilibrium is topologically equivalent to (2.35), that is, the equilibrium $(1, 1, 1)$ of the system (2.31) has a one dimensional unstable manifold $\mathcal{W}^u$. Therefore, there exists a trajectory which follows $\mathcal{W}^u$.

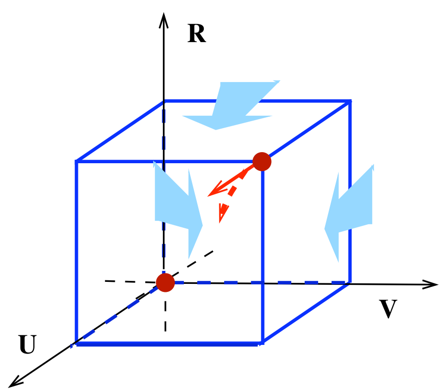

Figure 2.3: A trapping region for the trajectory of (2.31) that starts at $(1, 1, 1)$.

Consider the cube $\{\mathcal{Q} : 0 \leq U,\, V,\, R \leq 1\}$. The trajectory which starts at the equilibrium point $(1, 1, 1)$ leaves this point along the the unstable manifold $\mathcal{W}^u$. Since the eigenvector $(0, -c\nu, -1)$ corresponding to the positive eigenvalue ν of the linear system (2.35) is tangent to $\mathcal{W}^u$ and collinear to the side of $\mathcal{Q}$ with $U = 1$, we conclude that $\mathcal{W}^u$ is tangent to the side of $\mathcal{Q}$ with $U = 1$. It is easy to see that on the all sides of $\mathcal{Q}$ except for the one with $R = 0$ the vector field is directed either inside of $\mathcal{Q}$ or along its faces. In particular, on the side $U = 1$, $0 < V,\ R < 1$, the trajectory is right away pushed inside the cube $\mathcal{Q}$ by the vector field of (2.31). By Lemma 2.2, any trajectory of (2.31) starting out of $(1, 1, 1)$ is confined to $\mathcal{Q}$. Therefore, $\mathcal{Q}$ serves as a trapping region for that trajectory. Since $\mathcal{Q}$ is compact, the ω-limit set of that trajectory belongs to $\mathcal{Q}$. Moreover, since by Lemma 2.2 all three components of the solution of (2.31) are monotone in ξ, the ω-limit set consists of equilibrium points only, and it does not contain $(1,1,1)$. The only possibility for the trajectory is to settle down at the second equilibrium of (2.33), $(0,0,0)$, that also belongs to $\mathcal{Q}$. Since $\mathcal{W}^u$ is one-dimensional, we see that there exists a unique orbit connecting $(1,1,1)$ to $(0,0,0)$.

As follows from (2.35), the solution of (2.31) converges to the equilibrium $(U,V,R)=(1,1,1)$ exponentially fast at the rate ν. In order to obtain the convergence rate of the solution of (2.31) to equilibrium $(U,V,R)=(0,0,0)$, consider lineariztion of (2.31) about this point

$$\begin{aligned} \tilde{U}' &= -c\tilde{U}+c\tilde{R}, \\ \tilde{V}' &= -\frac{c}{\varepsilon}\tilde{V}+\frac{c}{\varepsilon}\tilde{R}, \\ \tilde{R}' &= -\frac{h}{c}\tilde{U}-\frac{1-h}{c}\tilde{V}. \end{aligned} \tag{2.36}$$

The eigenvalues of the linearization are the roots of the algebraic polynomial

$$-\varepsilon\kappa^3 - c(1+\varepsilon)\kappa^2 + (h(1-\varepsilon)-1-c)k - c = 0.$$

It is easy to see that for $c>0$ this polynomial has no purely imaginary eigenvalues, therefore the equilibrium is hyperbolic and the convergence must be exponential. ■

In the following section we will show that the leading edge of the solutions of the problem (1.6) propagate with the speed of corresponding traveling wave constructed in this section.

3 Propagation of supercritical fronts

In this section we study the long time behavior of the system (1.6) under assumption of front-like initial data which decays exponentially as $x\to\infty$. We restrict ourselves to the case when the rate of decay λ is slower then the one corresponding to the critical speed c^*. More precisely, we impose the following restrictions on the initial data,

$$\begin{aligned} &u_0(x)\to 1, \quad v_0(x)\to 1, \quad x\to-\infty, \\ &0\le u_0(x)\le \bar{M}<\infty, \quad 0\le v_0(x)\le 1, \\ &\underline{C}_u e^{-\lambda x}\le u_0(x)\le \bar{C}_u e^{-\lambda x}, \quad \underline{C}_v e^{-\lambda x}\le v_0(x)\le \bar{C}_v e^{-\lambda x}, \quad x\ge 0, \quad \lambda<\lambda^*, \end{aligned} \tag{3.1}$$

where $\bar{M}$, $\underline{C}_i$ and $\bar{C}_i$ are some positive constants. The problem of long time behavior for the single equation with KPP type nonlinearity and with similar restrictions on initial data is well understood by now. In particular, it is known that the solution approaches the traveling wave solution as $t\to\infty$. Moreover, the velocity of propagation is fully determined by the rate of decay of the initial data as $x\to\infty$ [15]. The relation of the velocity of propagation and the rate of decay can be obtained by looking at exponential solution of formal linearization of the equation at zero [15]. Similar results can also be obtained for KPP systems of monotone type [18] and for KPP type systems having variational structure [14]. In a more general setting, convergence to the traveling waves for systems can hardly be obtained due to the absence of maximum principle, or any equivalent tool which allows sharp comparison of solutions. However, one can characterize propagation of the solution by looking at the evolution of the leading edge. This approach proved to be quite useful and was successfully applied to the study of propagation of pulsating fronts in a context of conventional thermo-diffusive combustion. Our result is in a spirit of [9].

Theorem 3.1 *Any solution of the problem (1.6) with initial data satisfying (3.1) with $\lambda<\lambda^*$ propagates at the speed $c(\lambda)$ determined by (2.8) in the following sense: for any $\tilde{c}>c$ and any $x\in\mathbb{R}$ we have $(u,v,w)(t,x+\tilde{c}t)\to 0$ as $t\to\infty$, while for each $x\in\mathbb{R}$, there exists a constant $\mathbf{A}>0$ such that $(u,v,w)(t,x+ct)\ge\mathbf{A}$.*

The proof of the theorem is based on construction of sub and super-solutions for the system (1.6) which propagate with the same speed $c>c^*$. In order to construct these sub and super-solutions, we need the following three elementary lemmas.

Lemma 3.1 *Any solution of the problem (1.6) with $u_0(x)\ge 0$ and $0\le v_0(x)\le 1$ satisfies*

$$u(t,x)\ge 0, \quad 0\le v(t,x)\le 1, \quad w(x,t)\ge 0, \quad \textit{for all } x\in\mathbb{R} \textit{ and } t>0. \tag{3.2}$$

Proof. First, we observe that $v(t,x) = 1$ is a solution of the second equation of the system (1.6). Thus, $v(t,x) = 1$ is a super-solution of this equation provided $0 \le v_0 \le 1$.

Next, convolution of the first and the second equations of the system (1.6) with the corresponding heat kernel and consequent application of Duhamel's formula gives the following system of integral equations

$$\begin{aligned} u(t,x) &= G_1(t) * u_0 + \int_0^t G_1(t-s) * (1 - v(s,\cdot))w(s,\cdot)ds, \\ v(t,x) &= G_\varepsilon(t) * u_0 + \int_0^t G_\varepsilon(t-s) * (1 - v(s,\cdot))w(s,\cdot)ds, \\ w(t,x) &= hu(t,x) + (1-h)v(t,x), \end{aligned} \tag{3.3}$$

where

$$G_\nu(t,x) = \frac{1}{2\sqrt{\pi\nu t}} e^{-\frac{x^2}{4\nu t}} \tag{3.4}$$

is a Green's function of $\partial_t - \nu\partial_x^2$, and $*$ denotes convolution.

Since G_1 and G_ε are positive and $v \le 1$ for all $x \in \mathbb{R}$ and $t > 0$, and using the fact that u_0 and v_0 are non-negative on $\mathbb{R}$ we have $u(t,x), v(t,x) \ge 0$ for $[0,t^*] \times \mathbb{R}$ provided $w(t,x) \ge 0$ on $[0,t^*] \times \mathbb{R}$. Let us show that $w(t,x) \ge 0$ for all $x \in \mathbb{R}$ and $t \in [0,\infty)$.

First, combining the first and the second equations of the system (3.3) we have

$$w(t,x) = q(t,x) + \int_0^t \mathcal{H}(t-s) * (1 - v(s,\cdot))w(s,\cdot)ds, \tag{3.5}$$

where

$$q(t,x) = hG_1(t) * u_0 + (1-h)G_\varepsilon * v_0 \ge 0, \quad \mathcal{H}(t,x) = hG_1(t,x) + (1-h)G_\varepsilon(t,x). \tag{3.6}$$

Therefore,

$$w(t,x) \ge \int_0^t \mathcal{H}(t-s) * (1 - v(s,\cdot))w(s,\cdot)ds. \tag{3.7}$$

Let $P(t) = \inf_{x\in\mathbb{R}} w(t,x)$. Our restrictions on initial data imply $P(0) \ge 0$. Moreover, since $w(t,x)$ is continuous in both variables, $P(t)$ is also continuous. Assume that $t^* < \infty$ (possibly $t^* = 0$) is the largest time such that $P(t) \ge 0$ on $t \in [0,t^*]$. We choose $0 < \delta < \frac{1}{8}$ small enough such that

$$\min_{t^* \le t \le t^*+\delta} P(t) = P(t^* + \delta) < 0.$$

Moreover, since $v(x,t) \ge 0$ on $[0,t^*] \times \mathbb{R}$ we have $v(t,x) \ge -1$ on $[0,t^*+\delta] \times \mathbb{R}$ for small enough δ. Using the fact that $\int_{\mathbb{R}} \mathcal{H}(t,\cdot) = 1$ for all $t > 0$ and the mean value theorem, we deduce from (3.7)

$$\begin{aligned} &w(t^*+\delta,x) \ge \int_0^{t^*+\delta} \mathcal{H}(t^*+\delta-s) * (1 - v(s,\cdot))w(s,\cdot)ds \ge \\ &\int_{t^*}^{t^*+\delta} \mathcal{H}(t^*+\delta-s) * (1 - v(s,\cdot))w(s,\cdot)ds \ge 2P(t^*+\delta)\delta > \frac{1}{4}\inf_{x\in\mathbb{R}} w(t^*+\delta,x). \end{aligned} \tag{3.8}$$

By construction, the inequality (3.8) holds for all $x \in \mathbb{R}$. Continuity of w implies that there exists a point x_0 such that $w(t^*+\delta,x_0) < \frac{1}{2}\inf_{x\in\mathbb{R}} w(t^*+\delta,x)$. That contradicts (3.8). Thus t^* is not finite. ∎

Lemma 3.2 *Let $\bar{u}, \bar{v}, \bar{w}$ be any solution of the system of differential inequalities*

$$\begin{aligned} \bar{u}_t &\ge \bar{u}_{xx} + \bar{w}, \\ \bar{v}_t &\ge \varepsilon v_{xx} + \bar{w}, \\ \bar{w} &= h\bar{u} + (1-h)\bar{v}, \end{aligned} \tag{3.9}$$

with $\bar{u}_0 \ge u_0$, $\bar{v}_0 \ge v_0$. Then $\bar{u}, \bar{v}, \bar{w}$ is a super-solution of the system (1.6). That is $(u,v,w)(t,x) \le (\bar{u},\bar{v},\bar{w})(t,x)$ for all $x \in \mathbb{R}$ and $t > 0$.

Proof. Using the fact that the heat kernel is positive and applying Duhamel's formula to the system (3.9) we have,

$$\int_0^t G(t-s) * (\partial_s \bar{u}(s,\cdot) - \partial_y^2 \bar{u}(s,\cdot))ds \geq \int_0^t G(t-s) * \bar{w}(s,\cdot)ds,$$
$$\int_0^t G(t-s) * (\partial_s \bar{v}_(s,\cdot) - \varepsilon\partial_y^2 \bar{v}(s,\cdot))ds \geq \int_0^t G(t-s) * \bar{w}(s,\cdot)ds. \tag{3.10}$$

Integrating the left hand sides of (3.10) by parts one has,

$$\bar{u}(t,x) - G_1(t) * \bar{u}_0 \geq \int_0^t G_1(t-s) * \bar{w}(s,\cdot)ds,$$
$$\bar{v}(t,x) - G_\varepsilon(t) * \bar{v}_0 \geq \int_0^t G_\varepsilon(t-s) * \bar{w}(s,\cdot)ds. \tag{3.11}$$

Combining the first and the second equations of (3.11) one has,

$$\bar{w}(t,x) \geq \bar{q}(t,x) + \int_0^t \mathcal{H}(t-s) * \bar{w}(s,\cdot)ds, \tag{3.12}$$

where $\bar{q}(t,x) = hG_1(t) * \bar{u}_0 + (1-h)G_\varepsilon * \bar{v}_0$ and $\mathcal{H}(t)$ is defined by (3.6). Next, subtracting (3.5) from (3.11) we obtain

$$\bar{w}(t,x) - w(t,x) \geq \bar{q}(t,x) - q(t,x) + \int_0^t \mathcal{H} * (\bar{w}(s,\cdot) - w(s,\cdot) + v(s,\cdot)w(s,\cdot))ds. \tag{3.13}$$

Since $\bar{q}(t,x) - q(t,x) \geq 0$ and $vw \geq 0$ we have

$$\bar{w}(t,x) - w(t,x) \geq \int_0^t \mathcal{H}(t-s) * (\bar{w}(s,\cdot) - w(s,\cdot))ds. \tag{3.14}$$

Arguing, as in lemma 3.1 and using the fact that $\bar{w}(0,x) \geq w(0,x)$, we have

$$\bar{w}(t,x) - w(t,x) \geq 0, \quad \forall (t,x) \in [0,\infty) \times \mathbb{R}. \tag{3.15}$$

Finally, subtracting (3.3) from (3.11), using (3.15) and the assumption $\bar{u}_0 \geq u$, $\bar{v}_0 \geq v$, we obtain

$$\bar{u}(t,x) - u(t,x) \geq G_1(t) * (\bar{u}_0 - u_0) + \int_0^t G_1(t-s) * (\bar{w}(s,\cdot) - w(s,\cdot))ds \geq 0,$$
$$\bar{v}(t,x) - v(t,x) \geq G_\varepsilon(t) * (\bar{v}_0 - v_0) + \int_0^t G_\varepsilon(t-s) * (\bar{w}(s,\cdot) - w(s,\cdot))ds \geq 0. \tag{3.16}$$

Therefore, $\bar{u}(t,x) \geq u(t,x)$, $\bar{v}(t,x) \geq v(t,x)$ for any $(t,x) \in [0,\infty) \times \mathbb{R}$ which completes the proof. ■

Lemma 3.3 *Let $\underline{u}, \underline{v}, \underline{w}$ be any solution of the system of differential inequalities,*

$$\begin{aligned} \underline{u}_t &\leq \underline{u}_{xx} + \phi \underline{w}, \\ \underline{v}_t &\leq \varepsilon \underline{v}_{xx} + \phi \underline{w}, \\ \underline{w} &= h\underline{u} + (1-h)\underline{v} \end{aligned} \tag{3.17}$$

with $\underline{u}_0 < u_0$, $\underline{v}_0 < v_0$, where

$$\phi(t,x) = \max\{(1 - \bar{v}(t,x)), 0\}. \tag{3.18}$$

Here $\bar{v}(t,x)$ is a super-solution for the function $v(t,x)$ constructed in Lemma 3.2.

Then, $\underline{u}, \underline{v}, \underline{w}$ is a sub-solution of the system (1.6). That is $(u,v,w)(t,x) \geq (\bar{u}, \bar{v}, \bar{w})(t,x)$ for all $x \in \mathbb{R}$ and $t > 0$.

Proof. The proof is almost identical to the proof of Lemma 3.2. First, we apply the Duhamel's formula to the first and the second equations of the system (3.17) which gives,

$$\int_0^t G(t-s)*(\partial_s \underline{u}(s,\cdot)-\partial_y^2\underline{u}(s,\cdot))ds \le \int_0^t G(t-s)*\phi(s,\cdot)\underline{w}(s,\cdot)ds,$$
$$\int_0^t G(t-s)*(\partial_s \underline{v}(s,\cdot)-\varepsilon\partial_y^2\underline{v}(s,\cdot))ds \le \int_0^t G(t-s)*\phi(s,\cdot)\underline{w}(s,\cdot)ds. \tag{3.19}$$

Integrating the left hand sides of (3.19) by parts, we obtain

$$\underline{u}(t,x)-G_1(t)*\underline{u}_0 \le \int_0^t G_1(t-s)*\phi(s,\cdot)\underline{w}(s,\cdot)ds,$$
$$\underline{v}(t,x)-G_\varepsilon(t)*\underline{v}_0 \le \int_0^t G_\varepsilon(t-s)*\phi(s,\cdot)\underline{w}(s,\cdot)ds. \tag{3.20}$$

Combining the first and the second equations of (3.20) one has

$$\underline{w}(t,x)\le \underline{q}(t,x)+\int_0^t \mathcal{H}(t-s)*\phi(s,\cdot)\underline{w}(s,\cdot)ds, \tag{3.21}$$

where $\underline{q}(t,x)=hG_1(t)*\underline{u}_0+(1-h)G_\varepsilon*\underline{v}_0$ and $\mathcal{H}(t)$ is as in (3.6). Next, subtracting (3.21) from (3.5) we obtain

$$w(t,x)-\underline{w}(t,x)\ge q(t,x)-\underline{q}(t,x)+\int_0^t \mathcal{H}(t-s)*((1-v(s,\cdot))w(s,\cdot)-\phi(s,\cdot)\underline{w}(s,\cdot))ds = \tag{3.22}$$
$$q(t,x)-\underline{q}(t,x)+\int_0^t \mathcal{H}(t-s)*(\phi(s,\cdot)(w(s,\cdot)-\underline{w}(s,\cdot))ds+\int_0^t \mathcal{H}(t-s)*(1-v(s,\cdot)-\phi(s,\cdot))w(s,\cdot)ds.$$

Since $q(t,x)\ge \underline{q}(t,x)$, $\max\{(1-v(t,x)),0\}\ge\phi(t,x)\ge 0$, and $w(t,x)\ge 0$ for all $x\in\mathbb{R}$ and $t>0$, we have from (3.22)

$$w(t,x)-\underline{w}(t,x)\ge \int_0^t \mathcal{H}(t-s)*\phi(s,\cdot)(w(s,\cdot)-\underline{w}(s,\cdot))ds. \tag{3.23}$$

Applying the arguments of Lemma 3.1 to inequality (3.23), we deduce that $w(t,x)\ge\underline{w}(t,x)$ for all $(t,x)\in[0,\infty)\times\mathbb{R}$. Consequently applying arguments identical to these of Lemma 3.2, we also obtain $u(t,x)\ge\underline{u}(t,x)$, $v(t,x)\ge\underline{v}(t,x)$, for all $(t,x)\in[0,\infty)\times\mathbb{R}$, which completes the proof. ■

Remark 3.1 It is important to note that Lemmas 3.1, 3.2 and 3.3 essentially use only positivity of the Green's functions and Duhamel's formula. Therefore, these lemmas hold in rather general domains in higher dimensions with more general boundary conditions.

Remark 3.2 The results of Lemmas 3.1 and 3.2 remain true if we replace w with non-decreasing positive function $F(w)\le Kw$ where K is some constant.

Proof of Theorem 3.1 The proof is based on explicit construction of solutions for the systems of inequalities (3.17)and (3.9) that propagate with the same speed c. Thanks to Lemmas 3.2 and 3.3 these solutions are sub and super-solutions for the system (1.6). As a result the solution of the system (1.6) is squeezed between two positive functions propagating with the same speed which proves the statement of the theorem. We also note that the construction of sub-solution is conceptually similar to the one used in [9].

Step 1. Construction of a super-solution for the system (1.6).

Our goal is to construct a solution of the system (3.9), which by Lemma 3.2 is a super-solution of the system (1.6). We are looking for a super-solution of the problem (1.6) in a form

$$\bar{u}(t,x)=\bar{u}(\eta)=\bar{N}e^{-\lambda\eta},$$
$$\bar{v}(t,x)=\bar{v}(\eta)=\bar{M}e^{-\lambda\eta}, \tag{3.24}$$

where $\eta=x-ct+x_0$. Substituting (3.24) into (3.9) we have

$$(\lambda^2-c\lambda+h)\bar{M}+(1-h)\bar{N}\le 0,$$
$$h\bar{M}+(\varepsilon\lambda^2-c\lambda+(1-h))\bar{N}\le 0. \tag{3.25}$$

We now set $\bar{N} = 1$ and $\bar{M} = \frac{c\lambda^2 - \lambda^2 - h}{1-h}$. Thanks to (2.5), $\bar{N}$ and $\bar{M}$ solve (3.25). Therefore, (3.24) is the solution of (3.9). We next choose x_0 large enough so that $\bar{u}(0,x) \geq u_0$ and $\bar{v}(0,x) \geq v_0$ which is always possible under our assumption on the initial data (3.1) that restricts its behavior as $x \to \infty$.

Step 2. Construction of a sub-solution for the system (1.6).

In this step, we explicitly construct a solution of the system of inequalities (3.17), which propagates with the speed c and thanks to Lemma 3.3 is a sub-solution of the problem (1.6).

We are looking for a sub-solution in the form

$$\begin{aligned} \underline{u} &= \underline{M}\left(e^{-\lambda\eta} - m e^{-(\lambda+\delta)\eta}\right), \\ \underline{v} &= \underline{N}\left(e^{-\lambda\eta} - n e^{-(\lambda+\delta)\eta}\right), \end{aligned} \tag{3.26}$$

where $\underline{M}, \underline{N}, m, n > 0$ and $0 < \delta < \lambda$ are some constants to be identified later. Substituting (3.26) into (3.17) and multiplying the result by $e^{\lambda\eta}$ we have

$$\begin{aligned} &\left(\lambda^2 - c\lambda - m\left((\lambda+\delta)^2 - c(\lambda+\delta)\right)e^{-\delta\eta}\right)\underline{M} + \\ &\phi\left(h\underline{M} + (1-h)\underline{N} - \left(hm\underline{M} + (1-h)n\underline{N}\right)e^{-\delta\eta}\right) \geq 0, \\ &\left(\varepsilon\lambda^2 - c\lambda - n\left(\varepsilon(\lambda+\delta)^2 - c(\lambda+\delta)\right)e^{-\delta\eta}\right)\underline{N} + \\ &\phi\left(h\underline{M} + (1-h)\underline{N} - \left(hm\underline{M} + (1-h)n\underline{N}\right)e^{-\delta\eta}\right) \geq 0, \end{aligned} \tag{3.27}$$

where

$$\phi = \max\{(1 - e^{-\lambda\eta}), 0\}. \tag{3.28}$$

First we show that with an appropriate choice of constants $\underline{M}$, $\underline{N}$, m, n, δ the system of inequalities (3.26) holds for all η, that is, for all $(t,x) \in (0,\infty) \times \mathbb{R}$.

We consider the situations $\eta \geq 0$ and $\eta < 0$ separately. First, let us show that (3.27) holds for $\eta \geq 0$. When $\eta \geq 0$ we have $\phi = 1 - e^{-\lambda\eta}$. Substituting this expression into (3.27) we obtain

$$\begin{aligned} &(\lambda^2 - c\lambda + h)\underline{M} + (1-h)\underline{N} - \left[\left((\lambda+\delta)^2 - c(\lambda+\delta) + h\right)m\underline{M} + (1-h)n\underline{N}\right]e^{-\delta\eta} \geq \\ &e^{-\lambda\eta}\left[h\underline{M} + (1-h)\underline{N} - \left(hm\underline{M} + (1-h)n\underline{N}\right)e^{-\delta\eta}\right], \\ &h\underline{M} + \left(\varepsilon\lambda^2 - c\lambda + 1 - h\right)\underline{N} - \left[hm\underline{M} + \left(\varepsilon(\lambda+\delta)^2 - c(\lambda+\delta) - h\right)\underline{N}\right]e^{-\delta\eta} \geq \\ &e^{-\lambda\eta}\left[h\underline{M} + (1-h)\underline{N} - \left(hm\underline{M} + (1-h)n\underline{N}\right)e^{-\delta\eta}\right]. \end{aligned} \tag{3.29}$$

Now, we fix the ratio $\underline{M}$ and $\underline{N}$ in such a way that $\underline{M}$ and $\underline{N}$ solve (2.5). That is,

$$\frac{\underline{M}}{\underline{N}} = \frac{c\lambda - \varepsilon\lambda^2 - (1-h)}{h} = \frac{1-h}{c\lambda - \lambda^2 - h}. \tag{3.30}$$

With this choice of the ratio $\underline{M}/\underline{N}$, the system (3.29) reduces to the following one

$$\begin{aligned} \rho_1(\delta) &\geq \frac{e^{-(\lambda-\delta)\eta}}{m\underline{M}}\left[h\underline{M} + (1-h)\underline{N} - \left(hm\underline{M} + (1-h)n\underline{N}\right)e^{-\delta\eta}\right], \\ \rho_2(\delta) &\geq \frac{e^{-(\lambda-\delta)\eta}}{n\underline{N}}\left[h\underline{M} + (1-h)\underline{N} - \left(hm\underline{M} + (1-h)n\underline{N}\right)e^{-\delta\eta}\right], \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} \rho_1(\delta) &= c(\lambda+\delta) - (\lambda+\delta)^2 - h - (1-h)\frac{n\underline{N}}{m\underline{M}}, \\ \rho_2(\delta) &= c(\lambda+\delta) - \varepsilon(\lambda+\delta)^2 - (1-h) - h\frac{m\underline{M}}{n\underline{N}}. \end{aligned} \tag{3.32}$$

For system (3.31) to hold for all $\eta \geq 0$, it is sufficient to show that $\rho_1 > 0$, $\rho_2 > 0$. Let us show that, with an appropriate choice of n and m, both $\rho_1 > 0$ and $\rho_2 > 0$ for δ sufficiently small. Substituting (3.30) in (3.32), we have

$$\rho_1(\delta) = (c\lambda - \lambda^2 - h)\left(1 - \frac{n}{m}\right) + (c - 2\lambda)\delta - \delta^2,$$
$$\rho_2(\delta) = (c\lambda - \varepsilon\lambda^2 - (1-h))\left(1 - \frac{m}{n}\right) + (c - 2\varepsilon\lambda)\delta - \varepsilon\delta^2. \tag{3.33}$$

Now, we fix the ratio of n and m

$$\frac{n}{m} = 1 - \alpha\delta, \tag{3.34}$$

where

$$\alpha = -\left[\frac{(c-2\lambda)}{c\lambda - \lambda^2 - h} + \frac{1}{2}\frac{dc}{d\lambda}\frac{2c\lambda - (1+\varepsilon)\lambda^2 - 1}{(c\lambda - \lambda^2 - h)(c\lambda - \varepsilon\lambda^2 - (1-h))}\right]. \tag{3.35}$$

Expression (3.32) then takes the form

$$\rho_1(\delta) = [(c\lambda - \lambda^2 - h)\alpha + c - 2\lambda]\delta - \delta^2,$$
$$\rho_2(\delta) = \left[c - 2\varepsilon\lambda - \frac{\alpha}{1-\alpha\delta}(c\lambda - \varepsilon\lambda^2 - (1-h))\right]\delta - \varepsilon\delta^2. \tag{3.36}$$

Thus, we have $\rho_1(0) = \rho_2(0) = 0$. Next we show that $\frac{d\rho_1(0)}{d\delta} > 0$, $\frac{d\rho_2(0)}{d\delta} > 0$. Indeed,

$$\begin{aligned}
\frac{d\rho_1(0)}{d\delta} &= (c\lambda - \lambda^2 - h)\alpha + c - 2\lambda = -\frac{1}{2}\lambda\frac{dc}{d\lambda}\left(\frac{2c\lambda - (1+\varepsilon)\lambda^2 - 1}{c\lambda - \varepsilon\lambda^2 - (1-h)}\right), \\
\frac{d\rho_2(0)}{d\delta} &= c - 2\varepsilon\lambda - \alpha(c\lambda - \varepsilon\lambda^2 - (1-h)) \\
&= c - 2\varepsilon\lambda + \frac{(c-2\lambda)(c\lambda - \varepsilon\lambda^2 - (1-h))}{c\lambda - \lambda^2 - h} + \frac{1}{2}\lambda\frac{dc}{d\lambda}\frac{2c\lambda - (1+\varepsilon)\lambda^2 - 1}{c\lambda - \lambda^2 - h}.
\end{aligned} \tag{3.37}$$

Differentiating (2.6) with respect to λ, we obtain

$$\frac{(c-2\lambda)(c\lambda - \varepsilon\lambda^2 - (1-h))}{c\lambda - \lambda^2 - h} = -(c - 2\varepsilon\lambda) - \lambda\frac{dc}{d\lambda}\left(\frac{2c\lambda - (1+\varepsilon)\lambda^2 - 1}{c\lambda - \lambda^2 - h}\right). \tag{3.38}$$

Substituting (3.38) into (3.37) we have

$$\frac{d\rho_1(0)}{d\delta} = -\frac{1}{2}\lambda\frac{dc}{d\lambda}\left(\frac{2c\lambda - (1+\varepsilon)\lambda^2 - 1}{c\lambda - \varepsilon\lambda^2 - (1-h)}\right),$$
$$\frac{d\rho_2(0)}{d\delta} = -\frac{1}{2}\lambda\frac{dc}{d\lambda}\left(\frac{2c\lambda - (1+\varepsilon)\lambda^2 - 1}{c\lambda - \lambda^2 - h}\right). \tag{3.39}$$

Since $c(\lambda)$ is strictly convex, and $\lambda < \lambda^*$, we have $\frac{dc}{d\lambda} < 0$. Therefore, $\frac{d\rho_1(0)}{d\delta}$, $\frac{d\rho_1(0)}{d\delta} > 0$, which immediately implies that $\rho_1(\delta)$, $\rho_1(\delta) > 0$ for $\delta \in (0, \delta_0)$ with sufficiently small δ_0. Now, we choose δ, m and n in such a way that $m > 1$, $n > 1$,

$$m \geq \frac{1}{\rho_1(\delta)}(h + (1-h)\frac{\underline{N}}{\underline{M}}) = \frac{\lambda}{\rho_1(\delta)}(c - \lambda),$$
$$n \geq \frac{1}{\rho_2(\delta)}\left(1 - h + h\frac{\underline{M}}{\underline{N}}\right) = \frac{\lambda}{\rho_2(\delta)}(c - \varepsilon\lambda), \tag{3.40}$$

and the condition (3.34) is satisfied. This is obviously always possible with large enough m and n and small enough δ. With this choice of parameters the system of inequalities (3.27) is satisfied for all $\eta \geq 0$.

Consider now the situation when $\eta < 0$. In this case, $\phi = 0$, so that system (3.27) becomes

$$\left(\lambda^2 - c\lambda - m\left((\lambda+\delta)^2 - c(\lambda+\delta)\right)e^{-\delta\eta}\right)\underline{M} \geq 0,$$
$$\left(\varepsilon\lambda^2 - c\lambda - n\left(\varepsilon(\lambda+\delta)^2 - c(\lambda+\delta)\right)e^{-\delta\eta}\right)\underline{N} \geq 0. \tag{3.41}$$

Since $\eta < 0$, it is sufficient to verify that the system of inequalities holds at $\eta = 0$. At $\eta = 0$, (3.27) takes the form

$$\begin{aligned} &\lambda^2 - c\lambda - m\left((\lambda+\delta)^2 - c(\lambda+\delta)\right) \geq 0, \\ &\varepsilon\lambda^2 - c\lambda - n\left(\varepsilon(\lambda+\delta)^2 - c(\lambda+\delta)\right) \geq 0. \end{aligned} \tag{3.42}$$

Using (3.39) one can see that

$$\lambda^2 - c\lambda - m\left((\lambda+\delta)^2 - c(\lambda+\delta)\right) = m\rho_1(\delta) + (n-1)(c\lambda - \lambda^2 - h) + (m-1)h > 0, \tag{3.43}$$
$$\varepsilon\lambda^2 - c\lambda - n\left(\varepsilon(\lambda+\delta)^2 - c(\lambda+\delta)\right) = n\rho_2(\delta) + (m-1)\left(c\lambda - \varepsilon\lambda^2 - (1-h)\right) + (n-1)(1-h) > 0.$$

So that the system (3.41) is satisfied at $\eta = 0$ and therefore for all $\eta < 0$. Thus, with our choice of parameters m, n and δ and the ratio of $\underline{M}$ and $\underline{N}$, the system (3.27) holds for all $\eta \in \mathbb{R}$. Therefore, (3.26) is the solution of the system (3.17). We then choose $\underline{M}$ and $\underline{N}$ such that $\underline{u}(0,x) \leq u_0(x)$ and $\underline{v}(0,x) \leq v_0(x)$ and the condition (3.30) is satisfied. This is always possible since $\underline{u}$ and $\underline{v}$ decay faster than u_0 and v_0 when $x \to \infty$. Therefore $\underline{u}$ and $\underline{v}$ defined by (3.26) give a sub-solution for the problem (1.6).

Finally, combining Step 1 and Step 2 we obtain

$$\begin{aligned} &\underline{M}\left(e^{-\lambda(x-ct+x_0)} - me^{-(\lambda+\delta)(x-ct+x_0)}\right) \leq u(t,x) \leq \bar{M}e^{-\lambda(x-ct+x_0)}, \\ &\underline{N}\left(e^{-\lambda(x-ct+x_0)} - ne^{-(\lambda+\delta)(x-ct-x_0)}\right) \leq v(t,x) \leq \bar{N}e^{-\lambda(x-ct+x_0)}, \\ &(h\underline{M} + (1-h)\underline{N})e^{-\lambda(x-ct+x_0)} - (hm\underline{M} + (1-h)n\underline{N})e^{-(\lambda+\delta)(x-ct+x_0)} \\ &\leq w(t,x) \leq (h\bar{M} + (1-h)\bar{N})e^{-\lambda(x-ct+x_0)}. \end{aligned} \tag{3.44}$$

Observe that the functions on the left hand side are positive on an open set. Hence the conclusion of Theorem 3.1 holds. ■

Remark 3.3 In fact, we can say a little more about the solutions of the problem (1.6) with initial data satisfying (3.1). Since $u \geq 0$, we observe that the solution of equation

$$\underline{v}_t = \underline{v}_{xx} + (1-h)\underline{v}(1-\underline{v}) \tag{3.45}$$

with an initial data $\underline{v}_0 = v_0$ is a sub-solution. The equation (3.45) is just the KPP equation. Solutions of this problem are well understood. In particular, it is known that

$$\lim_{t\to\infty} \liminf_{x\in K} \underline{v}(t,x) = 1 \qquad \text{for any given compact } K. \tag{3.46}$$

On the other hand, by Lemma 3.1 we have $v \leq 1$. Therefore, given any compact K we have

$$\lim_{t\to\infty} \liminf_{x\in K} v(t,x) = 1. \tag{3.47}$$

Moreover, using result of Theorem 3.1 we have

$$\lim_{t\to\infty} \liminf_{x\in K} u(t,x) > a > 0, \tag{3.48}$$

where a is some positive constant and

$$\lim_{t\to\infty} \liminf_{x\in K} w(t,x) > (1-h) + ha. \tag{3.49}$$

4 An upper bound of the bulk burning rate

In this section we consider the system (1.5) under the assumption that $F(w)$ is a positive non-decreasing function satisfying,

$$F(w) \leq w. \tag{4.1}$$

Our goal in this section is to study long time behavior of solutions of the system (1.5) when initial data u_0, v_0 is either compactly supported or decay exponentially as $x \to \pm\infty$. Namely, we make the following assumptions on the initial data,

$$\begin{aligned} &0 \le u_0 < \infty, \quad 0 \le v_0 \le 1, \quad \forall x \in \mathbb{R}, \\ &u_0 \le e^{-\lambda_1 |x|}, \quad v_0 \le e^{-\lambda_2 |x|}, \quad |x| \ge a. \end{aligned} \tag{4.2}$$

Here, λ_1, λ_2 and a are some non-negative constants.

In order to measure the velocity of spreading of the reaction front, following [4] we introduce the bulk burning rate as the spatial average of time derivative of the temperature w,

$$B(t) = \frac{1}{2} \int w_t(t, \cdot), \tag{4.3}$$

The long time behavior of solutions of (1.5) is then characterized by the long time average of the of the bulk burning rate defined as,

$$\langle B \rangle_\infty = \lim_{t\to\infty} \langle B(t) \rangle_t, \tag{4.4}$$

where

$$\langle B(t) \rangle_t = \frac{1}{t} \int_0^t B(s) ds. \tag{4.5}$$

The main result of this section concerns an upper bound on $\langle B \rangle_\infty$ and shows that the bulk burning rate for the system (1.5) is bounded by the speed of the corresponding KPP-type traveling wave of the system (1.6).

Theorem 4.1 *Consider the problem (1.5) satisfying (4.1) with an initial data satisfying (4.2).*

If u_0, v_0 are compactly supported or their rate of decay λ is faster then the critical one, that is $\lambda_{min} = \min\{\lambda_1, \lambda_2\} \ge \lambda^*$. Then,

$$\langle B \rangle_\infty \le c^* = c(\lambda^*). \tag{4.6}$$

Here λ and c are as in a section 2 and $c(\lambda)$ is defined by (2.8).

If the rate of decay of the initial data u_0, v_0 is supercritical, that is $\lambda_{min} = \min\{\lambda_1, \lambda_2\} \le \lambda^*$ Then,

$$\langle B \rangle_\infty \le c(\lambda_{min}). \tag{4.7}$$

Proof. Integrating the first and the second equations of the system (1.5), we observe that

$$B(t) = \frac{1}{2} \int w_t(t, \cdot) = \frac{1}{2} \int (1 - v(t, \cdot)) f(w(t, \cdot)) = \frac{1}{2} \int u_t(t, \cdot) = \frac{1}{2} \int v_t(t, \cdot). \tag{4.8}$$

Therefore,

$$\langle B(t) \rangle_t = \frac{1}{2t} \int v(t, \cdot) - v_0(\cdot). \tag{4.9}$$

Thus, if $\bar{v}(t, x)$ is a supersolution of the problem (1.5), we have

$$\langle B(t) \rangle_t \le \frac{1}{2t} \int \bar{v}(t, \cdot). \tag{4.10}$$

Lemma 3.2 and Remark 3.2 imply that any solution of the system (3.9) with $(\bar{u}_0, \bar{v}_0) \ge (u_0, v_0)$ is a supersolution of the system (1.5) with the nonlinearity satisfying (4.1). As can be verified by the direct substitution into (3.9), the pair of functions

$$\bar{u}_r(t, x) = M_+ e^{-\lambda^\dagger (x - ct)}, \qquad \bar{v}_r(t, x) = N_+ e^{-\lambda^\dagger (x - ct)}, \tag{4.11}$$

and

$$\bar{u}_l(t,x) = M_- e^{\lambda^\dagger(x+ct)}, \qquad \bar{v}_l(t,x) = N_- e^{\lambda^\dagger(x+ct)}, \tag{4.12}$$

with $\lambda^\dagger = \lambda^*$ for subcritical initial data and $\lambda^\dagger = \lambda_{min}$ for supercritical initial data, and $\frac{M_+}{N+} = \frac{M_-}{N_-} = \frac{1-h}{c\lambda^\dagger - (\lambda^\dagger)^2 - h}$, are solutions of (3.9). Thus, (4.11) and (4.12) are supersolutions of (1.5) provided N_+ and N_- are sufficiently large. Moreover, since $v(t,x) = 1$ is a supersolution of (1.5), we have

$$\begin{aligned} &v(t,x) \le 1 \quad \text{for} \quad |x| < d + ct, \\ &v(t,x) \le A \min\{e^{-\lambda^\dagger(x-ct)}, e^{\lambda^\dagger(x+ct)}\} \quad \text{for} \quad |x| \ge d + ct, \end{aligned} \tag{4.13}$$

for sufficiently large A and d. Therefore,

$$\begin{aligned} \langle B(t)\rangle_t &\le \frac{1}{2t}\int_{-\infty}^{\infty} v(t,x)dx = \frac{1}{2t}\int_{-\infty}^{-(d+ct)} Ae^{\lambda^\dagger(x+ct)}dx + \frac{1}{2t}\int_{-(d+ct)}^{d+ct} dx + \frac{1}{2t}\int_{d+ct}^{\infty} Ae^{-\lambda^\dagger(x-ct)}dx \\ &\le c + \frac{A}{t\lambda^\dagger} + \frac{d}{t}. \end{aligned} \tag{4.14}$$

Taking the limit of the right hand side of (4.14) we have

$$\langle B\rangle_\infty \le c, \tag{4.15}$$

which completes the proof. ■

Remark 4.1 It is important to note that the result of Theorem 4.1 implies that the propagation of disturbances in the system (1.5) is bounded from above independently from ε. This result is generalization of the result obtained in [7] for the system (1.5) in its singular limit $\varepsilon = 0$.

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