An upper bound of the bulk burning rate in porous media combustion.

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Abstract

The long time behavior of the system of degenerate reaction-diffusion equations describing detonation in porous media is considered. An upper bound of the bulk burning rate is found.

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1. Introduction

The presence of obstacles may have a profound effect on the process of gaseous detonation sufficiently reducing its propagation velocity compared to associated Chapman-Jouguet value [10], [13]. When the level of resistance caused by the presence of obstacles is sufficiently high, the conventional detonation may convert into a subsonic shock-less wave, driven by drag induced diffusion of pressure [1]. This regime which is called a subsonic detonation is the main concern of this study.

The physically and mathematically simplest system for studying subsonic detonation is a combustion in inert porous media [11]. In this case the distortion introduced by porous media matrix can be ignored and resistance of the media to the gas flow is so strong that one may neglect the inertial effects and use Darcy's law as the momentum equation. In such a high drag limit the shocks are ruled out and pressure non-uniformities are equalized not by acoustic waves but through diffusion of pressure. For many gas-porous medium systems the gas pressure diffusivity exceeds its thermal diffusivity by several orders of magnitude thereby emerging as the principal transport agency controlling the reaction spread.

To single out the impact of momentum loss the effective features of the reactive gas-porous medium system will be assumed to be controlled exclusively by its gaseous phase subjected to the resistance of the porous medium matrix. The thermal and molecular diffusivities will be regarded as negligibly small (compared to the pressure diffusivity) which indeed holds for many real world systems. As an additional simplification the small-heat-release (SHR) approximation [12] will be employed where variations of temperature, pressure, density and gas velocity are regarded as small and, hence, the nonlinear effects are ignored everywhere but in the reaction rate term, generally highly sensitive even to minor temperature changes. In the non-dimensional formulation the resulting model reads [1],

$$\gamma\Theta_t - (\gamma-1)\Pi_t = \Omega(\Psi,\Theta), \tag{1}$$

$$\Psi_t = -\Omega(\Psi,\Theta), \qquad x \in \mathbb{R},\ t \geq 0. \tag{2}$$

$$\Pi_t - \Theta_t = \Pi_{xx}. \tag{3}$$

Here Π,Θ,Ψ are the appropriately scaled pressure, temperature and reactant concentration; $\gamma > 1$ is the specific heat ratio, Ω is the scaled reaction rate

$$\Omega(\Theta,\Psi) = \Psi E(\Theta). \tag{4}$$

Nonlinearity $E(\Theta)$ is assumed to be of the Arrhenius type with cut-off [15], that is $E(\Theta)$ is positive nondecreasing bounded function for $\Theta > \Theta^*$ and

$$E(\Theta) = 0 \quad \text{for} \quad \Theta < \Theta^*, \tag{5}$$

where Θ^* is the ignition temperature. This type of nonlinearity is one of the most common in the combustion theory[15],[14]. An example of $E(\Theta)$ which is used in [1],[2] is as follows

$$E(\Theta) = \begin{cases} \exp\left(\frac{Z(\Theta-1)}{\sigma+(1-\sigma)\Theta}\right) & \text{if} \quad \Theta \geq \Theta^* \\ 0 & \text{if} \quad \Theta < \Theta^* \end{cases} \tag{6}$$

where $\sigma = T_0/T_b$, T_0, T_b are temperatures of fresh and burned gas and Z is the Zeldovich number. Another important for applications example is the step function ($E(\Theta) = 1$ for $\Theta > \Theta^*$) [3].

Eqs.(1),(2) represent the partially linearized conservation equations for energy and deficient reactant, while Eq.(3) is a linearized continuity equation.

Initially pressure and concentration are constant while a temperature is localized function of the space coordinate. In non-dimensional quantities these conditions are as follows:

$$\Pi = 0, \quad \Theta = \Theta_0(x) \geq 0, \quad \Psi = 1. \tag{7}$$

The initial conditions (7) physically correspond to initiation of the detonation by localized temperature evaluation (hot spot).

In order to measure the characteristic velocity of reactions spread it is convenient to introduce the bulk burning rate [4]

$$V(t) = \frac{1}{2}\int \Theta_t(\cdot,t) = -\frac{1}{2}\int \Psi_t(\cdot,t) = \frac{1}{2}\int \Omega(\Psi(\cdot,t),\Theta(\cdot,t)) \tag{8}$$

and its time average

$$< V >_t = \frac{1}{t}\int_0^t V(t')dt'. \tag{9}$$

The long-time propagation velocity is described by [8]

$$< V >_\infty = \lim_{t\to\infty} sup < V >_t . \tag{10}$$

The numerical simulation of the system (1)-(3) with initial condition (7) performed in [1] show that as time goes to infinity the solution approaches two asymptotically independent traveling waves propagating in opposite directions. Thus putting factor 1/2 in (8) we equalize $< V >_\infty$ to the speed of the traveling front in this particular case.

There have been a number of works on the model (1)-(3). In [3], [6] the existence and uniqueness of traveling wave solution was proved and sufficient conditions for the lock of the propagation were established. In [7], [5] the velocity of traveling front in a high activation energy limit have been calculated and structure of the detonation wave was described. Numerical simulation performed in [2] reveled galloping and spinning modes occurring in the model due to losing of stability of the traveling fronts. In particular it has been observed numerically [2], [7] the presence of the traveling fonts oscillating with the very high magnitude. This observation leads to the natural question of the boundedness of the propagation speed or more general of the bulk burning rate. The resolution of this issue is the main motivation for this study.

2. Upper bound for detonation velocity

In this section we show that if an initial temperature $\Theta_0(x)$ is compactly supported then the average bulk burning rate defined by (10) is bounded.

Let us note first that the system (1)-(3) can be significantly simplified. Indeed combining (1) and (2), integrating with respect to time and taking into account initial conditions we have

$$\Theta(x,t) = \Theta_0(x) + \gamma^{-1}(1-\Psi(x,t)) + (1-\gamma^{-1})\Pi(x,t) \quad \forall x, \ \forall t \tag{11}$$

The combination of (1) and (3) then yields,

$$\Pi_t - \gamma\Pi_{xx} = \Omega(\Psi,\Theta). \tag{12}$$

Rescaling the space coordinate ($x \to x/\sqrt{\gamma}$) we finally have the following system,

$$\Pi_t = \Pi_{xx} + \Omega(\Theta,\Psi), \tag{13}$$

$$\Psi_t = -\Omega(\Theta,\Psi), \tag{14}$$

$$\Theta = \Theta_0(x) + \gamma^{-1}(1-\Psi) + (1-\gamma^{-1})\Pi \tag{15}$$

with initial conditions (7).

Thus the special structure of the system (1)-(3) allows its reduction to the problem (13)-(15), considered below.

Theorem 1. *Let* $\Theta_0(x) \geq 0$ *and set*

$$I = \{x \in R : \Theta_0(x) > 0\}, \quad l = meas\ I < \infty \tag{16}$$

Then,

$$< V >_\infty \leq 2\sqrt{\frac{(1-\gamma^{-1})\bar{E}}{\Theta^*}} \tag{17}$$

where

$$\bar{E} = \max_s E(s). \tag{18}$$

For proof of this theorem two lemmas are needed.

Lemma 1. *Let* $\Theta_0(x) \geq 0$ *be as in theorem 1 and set*

$$t^* = \frac{\Theta^* - \epsilon}{\bar{E}(1-\gamma^{-1})}. \tag{19}$$

where ϵ *is arbitrary small number.*
Then,

$$\Psi(x,t) \geq \exp(-\bar{E}t), \quad \Omega\left(\Psi(x,t),\Theta(x,t)\right) \leq \bar{E} \quad \forall x, \ \forall t \tag{20}$$

$$\Psi(x,t) = 1, \quad \Omega(\Psi(x,t), \Theta(x,t)) = 0 \quad \forall x \notin I \ \forall t \leq t^* \tag{21}$$

and

$$\Pi(x,t) \leq \bar{E} \int_0^t \int_I G(x-y, t-\tau) dy d\tau \quad \forall x, \ \forall t \leq t^* \tag{22}$$

with

$$G(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right). \tag{23}$$

Proof. First note that solution of Eq.(14) can be written as

$$\Psi(x,t) = \exp(-\int_0^t E(\Theta(x,t'))dt'), \tag{24}$$

whence it follows that

$$\Psi(x,t) \leq 1, \quad \forall x, \ \forall t \tag{25}$$

and

$$\Omega(\Psi, \Theta) = \Psi E(\Theta) \leq E(\Theta) \leq \bar{E} \quad \forall x, \ \forall t. \tag{26}$$

Next observe that due to (26) and maximum principle

$$\Pi \leq \bar{E} t \quad \forall x, \ \forall t. \tag{27}$$

Therefore

$$\Theta(x,t) \leq \Theta_0(x) + \gamma^{-1}(1-\Psi) + \Theta^* - \epsilon \quad \forall x, \ \forall t \leq t^*. \tag{28}$$

Since for any $x_0 \notin I$, $\Theta_0(x) = 0$

$$\Psi(x_0, t) = 1, \quad \forall t \leq t^* \tag{29}$$

because $\Psi = 1$ is a solution of (14). Here the condition (5) and uniqueness of solution are used. Thus,

$$\Omega(\Psi(x_0,t), \Theta(x_0,t)) = 0, \quad \forall t \leq t^*. \tag{30}$$

On the other hand, as a consequence of (26),(27)

$$\Psi(x,t) \geq \exp(-\bar{E}t), \quad \Omega(x,t) \leq \bar{E}, \quad \forall x, \ \forall t, \tag{31}$$

which proves (20), (21).

Let us prove (22). Define a function

$$F(x) = \begin{cases} \bar{E} & \text{if} \quad x \in I \\ 0 & \text{if} \quad x \notin I \end{cases} \tag{32}$$

and solve equation

$$\phi_t = \phi_{xx} + F(x) \tag{33}$$

with initial condition $\phi(x,0)=0$. The solution of (33) is as follows

$$\phi(x,t)=\int_0^t\int_{-\infty}^{\infty}G(x-y,t-\tau)F(y)dyd\tau=\bar{E}\int_0^t\int_I G(x-y,t-\tau)dyd\tau, \tag{34}$$

where heat kernel $G(x,t)$ is as in (23).

Since $F\geq\Omega$ for all x and $t\in(0,t^*)$ we have

$$\Pi(x,t)\leq\phi(x,t)\quad\forall x,\forall t\leq t^*. \tag{35}$$

□

Lemma 2. *Let $\Theta_0(x)\geq 0$ be as in theorem 1.*
Then

$$\Psi(x,t)=1\quad \textit{if}\quad |x|>m+ct\quad\forall t \tag{36}$$

with some constant $m>0$ and

$$c=2\sqrt{\frac{(1+\epsilon)(1-\gamma^{-1})\bar{E}}{\Theta^*-\epsilon}} \tag{37}$$

Proof. Consider a system of inequalities

$$u_t\geq u_{xx}+D(w), \tag{38}$$

$$v_t\leq -vD(w), \tag{39}$$

$$w=\Theta_0+\gamma^{-1}(1-v)+(1-\gamma^{-1})u, \tag{40}$$

where $D(s)$ is defined as follows

$$D(w)=\begin{cases}0 & \text{if}\quad w<\Theta^*-\epsilon\\ (1+\epsilon)\bar{E} & \text{if}\quad w\geq\Theta^*-\epsilon.\end{cases} \tag{41}$$

Among solutions of system (38)-(40) we are interesting in that possessing the following properties

$$u(x,t)\geq\phi(x,t)=\bar{E}\int_0^t\int_I G(x-y,t-\tau)dyd\tau,\quad\forall x,\ \forall t\leq t^* \tag{42}$$

$$v(x,t)\leq\begin{cases}\exp(-\bar{E}t) & \text{if}\quad x\in I,\ t\leq t^*\\ 1 & \text{if}\quad x\notin I,\ t\leq t^*\end{cases} \tag{43}$$

Let us show that there are solutions of this kind of the system (38), (39). It is easy to check by direct substitution that

$$u_1(x,t)=\frac{\Theta^*-\epsilon}{1-\gamma^{-1}}\exp(-\frac{c}{2}(x-a_1-ct)) \tag{44}$$

$$v_1(x,t)=\begin{cases}\exp(-\frac{(1+\epsilon)\bar{E}}{c}(ct-x+a_1)) & \text{if}\quad x<a_1+ct\\ 1 & \text{if}\quad x\geq a_1+ct\end{cases} \tag{45}$$

and

$$u_2 = \frac{\Theta^* - \epsilon}{1-\gamma^{-1}} \exp(\frac{c}{2}(x + a_2 + ct)) \tag{46}$$

$$v_2(x,t) = \begin{cases} \exp(-\frac{(1+\epsilon)\bar{E}}{c}(x + ct + a_2) & \text{if} \quad x > -(ct + a_2) \\ 1 & \text{if} \quad x \leq -(ct + a_2) \end{cases} \tag{47}$$

are solutions of (38),(39). Moreover these solutions satisfy properties (42),(43) for appropriately chosen a_i. Indeed let $I = (-b_2, b_1)$, then for any $a_i > b_i$ we have $v_i \leq \exp(-\bar{E}t)$ for all $x \in (-a_2, a_1)$ thus condition (43) is fulfilled. Similarly by choosing a_i large enough we can obtain $u_i \geq \phi$ because ϕ is bounded and tends to zero as $x \to \pm\infty$ faster then u_i.

The solution of the system (38),(39),(40) reads

$$u(x,t) \geq \int_{-\infty}^{\infty} G(x-y,t)u_0(y)dy + \int_0^t \int_{-\infty}^{\infty} G(x-y,t-\tau)D(w(y,\tau))dyd\tau, \tag{48}$$

$$v(x,t) \leq \exp(-\int_0^t D(w(x,\tau))d\tau) \tag{49}$$

and

$$w(x,t) \geq \Theta_0(x) + \gamma^{-1}(1 - \exp(-\int_0^t D(w(x,\tau))d\tau)+$$

$$(1-\gamma^{-1})(\int_{-\infty}^{\infty} G(x-y,t)u_0(y)dy + \int_0^t \int_{-\infty}^{\infty} G(x-y,t-\tau)D(w(y,\tau))dyd\tau), \tag{50}$$

while solution of system (13)-(15) can be written as

$$\Pi(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-y,t-\tau)\Psi(y,\tau)E(\Theta(y,\tau))dyd\tau, \tag{51}$$

$$\Psi(x,t) = \exp(-\int_0^t E(\Theta(x,\tau))d\tau), \tag{52}$$

$$\Theta(x,t) = \Theta_0(x) + \gamma^{-1}(1 - \exp(-\int_0^t E(\Theta(x,\tau))d\tau)+$$

$$(1-\gamma^{-1})\int_0^t \int_{-\infty}^{\infty} G(x-y,t-\tau)\Psi(y,\tau)E(\Theta(y,\tau))dyd\tau. \tag{53}$$

Therefore

$$w(x,t) - \Theta(x,t) \geq \gamma^{-1}(\exp(-\int_0^t E(\Theta(x,\tau))d\tau) - \exp(-\int_0^t D(w(x,\tau))d\tau))+$$

$$(1-\gamma^{-1})\int_{-\infty}^{\infty} G(x-y,t)u_0(y)dy+$$

$$(1-\gamma^{-1})\int_0^t \int_{-\infty}^{\infty} G(x-y,t-\tau)(D(w(y,\tau)) - \Psi(y,\tau)E(\Theta(y,\tau)))dyd\tau. \tag{54}$$

The last expression is true for all solutions. Suppose we consider a solution w satisfying (42),(43), then by lemma 1 $w \geq \Theta$ for $t \leq t^*$. Suppose that $T^* \geq t^*$ is a finite time when $w(x,t) \leq \Theta(x,t)$ then

$$\gamma^{-1}(\exp(-\int_0^{T^*} E(w(x,\tau))d\tau) - \exp(-\int_0^{T^*} D(w(x,\tau))d\tau))+$$

$$(1-\gamma^{-1})(\int_{-\infty}^{\infty} G(x-y,t)u_0(y)dy+$$

$$\int_0^{T^*}\int_{-\infty}^{\infty} G(x-y,t-\tau)(D(w(y,\tau)) - \Psi(y,\tau)E(w(y,\tau)))dyd\tau) \leq 0 \tag{55}$$

at least at one point $x = x_0$ but it is impossible because $D(\Theta) \geq E(\Theta)$. Thus, T^* is not finite.

As a consequence of it we can write that

$$(w,u,1-v)(x,t) \geq (\Theta,\Pi,1-\Psi)(x,t) \quad \forall x, \forall t \tag{56}$$

if

$$(w,u,1-v)(x,t) \geq (\Theta,\Pi,1-\Psi)(x,t) \quad \forall x, \forall t \leq t^* \tag{57}$$

Since the above-mentioned solutions u_1, v_1 and u_2, v_2 satisfy (57) they can be used to estimate Ψ from bellow and we write

$$\Psi \geq \max(v_1, v_2) \tag{58}$$

Observing that

$$v_1(x,t) = 1 \quad \text{for} \quad x > a_1 + ct \tag{59}$$

$$v_2(x,t) = 1 \quad \text{for} \quad x < -(a_2 + ct) \tag{60}$$

and setting $m = a_1 + a_2$ we have (36). □

Proof of Theorem 1. By lemma 2 we know that $\Psi \geq \tilde{\Psi}$ where

$$\tilde{\Psi}(x,t) = \begin{cases} 0 & \text{if} \quad |x| \leq m + ct \\ 1 & \text{if} \quad |x| > m + ct \end{cases} \tag{61}$$

Then, integrating Eq.(14) we have

$$< V >_t = -\frac{1}{2t}\int_0^t \int_{-\infty}^{\infty} \Psi_t(x,t)dxdt = \frac{1}{2t}\int_{-\infty}^{\infty}(1-\Psi(x,t))dx \leq$$

$$\frac{1}{2t}\int_{-\infty}^{\infty}(1-\tilde{\Psi}(x,t))dx = c + \frac{m}{t}. \tag{62}$$

Passing to the limit as $t \to \infty$ in (62) we obtain (17). □

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