"Can Disordered Sphere Packings Ever Be Maximally Dense?"

Salvatore Torquato
Department of Chemistry, Princeton Center for Theoretical Science,
and Program in Applied & Computational Mathematics
Princeton University, Princeton, NJ
Institute for Advanced Study, School of Natural Sciences, Princeton, NJ
http://cherrypit.princeton.edu

Packing problems, such as how densely nonoverlapping particles fill d-dimensional Euclidean space $\mathbb{R}^d$, are ancient and still provide fascinating challenges for scientists and mathematicians [1,2]. Bernal has remarked that "heaps" (particle packings) were the first things that were ever measured in the form of basketfuls of grain for the purpose of trading or of collection of taxes. While maximally dense packings are intimately related to classical ground states of matter, disordered sphere packings have been employed to model glassy states of matter. There has been a resurgence of interest in maximally dense sphere packings in high-dimensional Euclidean spaces. Interestingly, the optimal ways of sending digital signals over noisy channels correspond to the densest sphere packings in high-dimensional spaces.

Remarkably, no one has been able to provide exponential improvement on a 100-year- old lower bound on the maximal packing density due to Minkowski in d-dimensional Euclidean space $\mathbb{R}^d$. The asymptotic behavior of this bound in any dimension is controlled by 2^{-d}. Using an optimization procedure and a conjecture concerning the existence of disordered sphere packings, we obtain a conjectural lower bound on the density whose asymptotic behavior is controlled by $2^{-(0.7786...)d}$, thus providing the putative exponential improvement of Minkowski's bound [3]. A more recent investigation reveals that this exponential improvement is robust over a wide class of optimized functions [5]. Our work suggests that disordered (rather than ordered) sphere packings may be the densest for sufficiently large d, implying not only the existence of disordered ground states for some continuous potentials but their preponderance among all other possibilities.

1. J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups* (Springer-Verlag, New York, 1993).

2. S. Torquato, *Random Heterogeneous Materials: Microstructure and Macroscopic Properties*, (Springer-Verlag, New York, 2002).

3. H. Minkowski, "Diskontinuitätsbereich für arithmetische Äquivalenz," J. reine angew. Math. **129**, 220 (1905).

4. S. Torquato and F. H. Stillinger, "New Conjectural Lower Bounds on the Optimal Density of Sphere Packings," Experimental Math. **15**, 307 (2006).

5. A. Scardicchio, F. H. Stillinger and S. Torquato, "Estimates of the Optimal Density of Sphere Packings in High Dimensions," J. Math. Phys. **49**, 043301 (2008).