1. $f'''(x_0) = \dfrac{f(x_3) - 3f(x_2) + 3f(x_1) - f(x_0)}{h^3}$

Error $= O(h)$

$P_3(x) = f(x_0) + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) f[x_0, x_1, x_2] +$
$(x-x_0)(x-x_1)(x-x_2) + f[x_0, x_1, x_2, x_3]$

$E_3(x) = (x-x_0)(x-x_1)(x-x_2)(x-x_3) f[x_0, x_1, x_2, x_3, x]$

$P_3'''(x) = 6 f[x_0, x_1, x_2, x_3]$

$f[x_0, x_1, x_2, x_3] = \dfrac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \dfrac{\dfrac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}}{x_3 - x_0} -$

$\dfrac{\dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}}{x_3 - x_0} = \dfrac{\dfrac{\dfrac{f(x_3) - f(x_2)}{x_3 - x_2}}{x_2 - x_0}}{x_3 - x_0} - \dfrac{\dfrac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}}{x_2 - x_0}}{x_3 - x_0} -$

$\dfrac{\dfrac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}}{x_2 - x_0}}{x_3 - x_0} - \dfrac{\dfrac{\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}}{x_3 - x_0} = \dfrac{f(x_3) - f(x_2)}{6h^3} - \dfrac{f(x_2) - f(x_1)}{6h^3} -$

$\dfrac{f(x_2) - f(x_1)}{6h^3} - \dfrac{f(x_1) - f(x_0)}{6h^3}$

$\Rightarrow 6f[x_0, x_1, x_2, x_3] = \dfrac{f(x_3) - 3f(x_2) + 3f(x_1) - f(x_0)}{h^3}$

$E_3'''(x_1) = \left((x-x_0) * (x-x_1)(x-x_2)(x-x_3) \right)''' f[x_0, x_1, x_2, x_3, x]$

$E_3(x_1) = 18x_0 - 6x_1 - 6x_2 - 6x_3\ f[x_0, x_1, x_2, x_3, x] = -36h\, f[x_0, x_1, \ldots, x]$

$= \dfrac{-36h}{4!} f^{(4)}(\xi_n) = -\tfrac{3}{2} h f^{(4)}(\xi_n)$

2- $A^{-1} = \begin{bmatrix} -1/3 & 1/3 & 1/3 \\ 1/3 & -4/3 & 2/3 \\ 1/3 & 2/3 & -1/3 \end{bmatrix}$

3- $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}$ $\det(A) = 23 \neq 0$ A^{-1} exists $\Rightarrow$ the system has a uinque solution.

Condition for convergence is

$$\max_{1 \le i \le n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| \frac{a_{ij}}{a_{ii}} \right| < 1 \Rightarrow \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| < |a_{ii}| \quad 1 \le i \le n$$

A is diagonally dominant since is satisfies

$$\sum_{\substack{j=1 \\ j \neq i}}^{3} |a_{ij}| < |a_{ii}| \quad \text{for } 1 \le i \le 3$$

therefore the method converges.

4. $\frac{d^2y}{dt^2} = -0.1\left(\frac{dy}{dt}\right)^2 - 0.6y$ $\quad Y_1(x) = Y(x) \quad Y_2(x) = Y'(x)$

$\Rightarrow Y''(x) = Y_2'(x) = -0.1\left(Y'(x)\right)^2 - 0.6Y(x)$

$\Rightarrow Y_2'(x) = -0.1\left(Y_2(x)\right)^2 - 0.6Y_1(x)$, $Y_2(X=0) = Y'(X=0) = 0$

$Y_1'(x) = Y_2(x)$, $Y_1(X=0) = Y(X=0) = 1$

$\Rightarrow Y_1'(X) = Y_2(X)$, $Y_1(0) = 1$

$Y_2'(X) = -0.1\left(Y_2(x)\right)^2 - 0.6\,Y_1(x)$, $Y_2(0) = 0$

$$Y_1'(x) = Y_2(x) \qquad Y_1(0) = 1$$

$$Y_2'(x) = 0.1(Y_2(x))^2 - 0.6Y_1(x) \qquad Y_2(0) = 0$$

$$y_1(x=0.05) = y_1(x=0) + (0.05)(y_2(0)) = 1$$

$$y_2(x=0.05) = [0.1(y_2(x=0))^2 - 0.6(y_1(x=0))]0.05 + y_2(x=0) =$$

$$[0.1(0) - 0.6(1)]0.05 + 0 = -(0.6)(0.05) =$$

$$-0.035$$

Do the same for $y_1(0.1)$ and $y_2(0.1)$

5- $Y'(x) = Y(x)$ and $Y(0) = 1$

$$y_{n+1} = y_n + h y_n$$

$$\Rightarrow y_{n+1} = (1+h)y_n \qquad y_0 = 1$$

$$\Rightarrow y_n = (1+h)y_{n-1} = (1+h)^2 y_{n-2} = \cdots = (1+h)^n y_0 = (1+h)^n$$

$n \geq 0$

$$y_n = (1+h)^{\frac{x_n - x_0}{h}} = \left[(1+h)^{1/h}\right]^{x_n} \quad \text{since } x_0 = 0$$

a) $$y_n = \left((1+h)^{\frac{1}{h}}\right)^{x_n}$$

b) $$\lim_{h \to 0} (1+h)^{1/h} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$\Rightarrow \lim_{h \to \infty} y_n = e^{x_n} = e^x$$ which is the solution to $y' = y$ $y(0) = 1$

$y_n(x_n) = e^{x_n \ln(1+h)^{1/h}} = e^{x_n \frac{\ln(1+h)}{h}}$ Recall that $x = e^{\ln x}$

$$\Rightarrow y_n(x_n) = e^{x_n \frac{\ln(1+h)}{h}}$$

$\ln(1+h) \approx h - \frac{h^2}{2}$ Taylor Approximation

$$\Rightarrow y_n(x_n) = e^{x_n \frac{(h - h^2/2)}{h}} + O(h^2) = e^{x_n (1 - \frac{h}{2})} + O(h^2)$$

$$Y(x_n) - y_n(x_n) = e^{x_n} - e^{x_n(1 - h/2)} + O(h^2)$$

$$Y(x_n) - y_n(x_n) = e^{x_n}\left(1 - e^{(-\frac{x_n h}{2})}\right) + O(h^2)$$

$$e^{-x_n h/2} \approx 1 - \frac{x_n h}{2}$$

$$\Rightarrow Y(x_n) - y_n(x_n) = e^{x_n}\left(1 - \left(1 - \frac{x_n h}{2}\right)\right) + O(h^2) =$$

$$\frac{x_n e^{x_n}}{2} h + O(h^2) \qquad h \ll 1$$

6- Recall $e_{n+1} = \left[1 + h \frac{\partial f(x_n, y_n)}{\partial y}\right] e_n + \frac{h^2}{2} Y''(\xi_n)$

here $\frac{\partial f(x,y)}{\partial y} = -1$ since $Y'(x) = -Y(x)$

$\Rightarrow f(x,y) = -Y(x)$

and since $\frac{\partial f}{\partial y} = -1$

$Y'(x) = -Y(x)$ with $Y(0)=1$

$\Rightarrow Y(x) = e^{-x} \Rightarrow Y''(\xi_n) = e^{-\xi_n}$

$$\Rightarrow \quad e_{n+1} = [1-h]\, e_n + \frac{h^2}{2} e^{-\xi_n}$$

Note that $e_0 = 0$

$$\max_{x_0 \leq x \leq b} |Y''(x)| = 1$$

$$\text{Max} \left| \frac{\partial f(x,y)}{\partial y} \right| = 1$$

$$\Rightarrow \quad |e_{n+1}| \leq |e_n| + \frac{1}{2} h^2 \qquad \Rightarrow \left| 1 + h \frac{\partial f(x,y)}{\partial y} \right| \leq 1$$

$$\Rightarrow \quad |e_n| \leq \frac{n}{2} h^2 = \frac{x_n h}{2} \quad \Rightarrow$$

$$\underline{|e_n| \leq \frac{x_n h}{2}}$$

Compare to true error:

$$\left| e^{-x_n} - y_n(x_n) \right|$$

8- $Y'(x) = -Y(x) + 2\cos(x) \quad Y(0)=1$

$f(x,y) = -Y(x) + 2\cos(x) \quad X_0 = 0 \quad Y_0 = 1$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_n + h,\ y_n + h f(x_n, y_n)\right)\right] \quad n \geq 0$$

$h = 0.1 \quad y_0 = 1$

$$y_1 = y(x=0.1) = y_0 + \frac{0.1}{2}\left[f(0,1) + f(0+0.1,\ 1+0.1 f(0,1))\right] =$$

$$f(0,1) = -y_0 + 2\cos(0) = -1 + 2\cos(0) = 1$$

$$f(0.1,\ 1+0.1 f(0,1)) = f(0.1,\ 1+(0.1)(1)) = f(0.1,\ 1.1) = -1.1 + 2\cos(0.1)$$

$$\Rightarrow y_1 = 1 + \frac{0.1}{2}\left[1 + (-1.1) + 2\cos(0.1)\right] = 1 + \frac{0.1}{2}\left[-0.1 + 2\cos(0.1)\right]$$

7- $y_{n+1} = y_n + h f\left(\frac{x_n + x_{n+1}}{2},\ \frac{y_n + y_{n+1}}{2}\right)$

Apply $y' = \lambda y \quad y(0) = 1 \Rightarrow$

$$y_{n+1} = y_n + h\lambda\left(\frac{y_n + y_{n+1}}{2}\right)$$

$$y_{n+1} = \frac{2+h\lambda}{2-h\lambda}\, y_n \qquad n \geq 0$$

$$y_n = \left(\frac{2+h\lambda}{2-h\lambda}\right)^n \qquad n \geq 0$$

if $\mathrm{Re}(h\lambda) < 1 \Rightarrow \left|\frac{2+h\lambda}{2-h\lambda}\right| < 1$ Method is absolutely stable

9(B) $y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}) \right]$

consider $y' = \lambda y$ $Re(\lambda) < 0$ and $y(0) = 1$

$$\Rightarrow \quad y_{n+1} = y_n + \frac{h}{2}\left[\lambda y_n + \lambda y_{n+1} \right]$$

$$y_{n+1} - \frac{h}{2}\lambda y_{n+1} = y_n + \frac{h}{2}\lambda y_n$$

$$y_{n+1} = \frac{\left(1 + \frac{h\lambda}{2}\right)}{\left(1 - \frac{h\lambda}{2}\right)} y_n$$

$$y_n = \left(\frac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}}\right) y_{n-1} = \cdots \left(\frac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}}\right)^n y_0$$

$$y_0 = 1 \Rightarrow y_n = \left(\frac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}}\right)^n$$

since $Re(\lambda) < 0$ $\Rightarrow$ $\left| \frac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}} \right| < 1$

the method is absolutely stable.