Random signl analysis I (ECE673)
Solution assignment 8

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1) A random walk process is a sum process defined as

$$X[n] = \sum_{i=0}^{n} U[i],$$

where $U[n]$ is an IID random process with $U[n]$ described by the PMF

$$p_U[k] = \begin{cases} 1/2 & k=-1 \\ 1/2 & k=1 \end{cases}.$$

(i) Generate a few realizations of the process using MATLAB for, e.g., $n = 0, ..., 99$ (attach at least one with your solution). At a first glance, looking at the realization, do you expect the process to be stationary?
(ii) Evaluate the mean sequence $\mu_X[n] = E[X[n]]$ and the covariance sequence $c_X[n, n+k]$. Based on these results, is the process stationary? Is it wide sense stationary (WSS)?
(iii) Now consider the process $Y[n] = U^2[n]$. Is this transformed process IID?.

Solution:
(i) The code that generates and plots a single realization of the random process $X[n]$ for $n = 0, 1, ..., 99$ is as follows (notice that in MATLAB vectors cannot be indexed starting from zero):

```
N=100;
u=sign(rand(N,1)-0.5);
for n=1:N
 x(n)=sum(u(1:n));
end
plot([0:N-1],x);
```

The figure below shows two realizations of the random walk $X[n]$. From the observation of a few limited realizations of a process, nothing can be concluded on the statistical properties of the process itself. However, the process seems to present larger values for larger n, which suggests that the process is non-stationary.
(ii) The mean sequence $\mu_X[n]$ reads

$$\mu_X[n] = E[X[n]] = E\left[\sum_{i=0}^{n} U[i]\right] = \sum_{i=0}^{n} E[U[n]] = 0$$

whereas the covariance sequence is

$$\begin{aligned} c_X[n, n+k] &= E[x[n]x[n+k]] = E\left[\sum_{i=0}^{n} U[i] \sum_{j=0}^{n+k} U[j]\right] = \\ &= E\left[\sum_{i=0}^{n}\sum_{j=0}^{n+k} U[i]U[j]\right] = E\left[\sum_{i=0}^{n} U^2[i]\right] = \sum_{i=0}^{n} E[U^2[i]] = \\ &= (n+1)\cdot var[U[i]], \end{aligned}$$

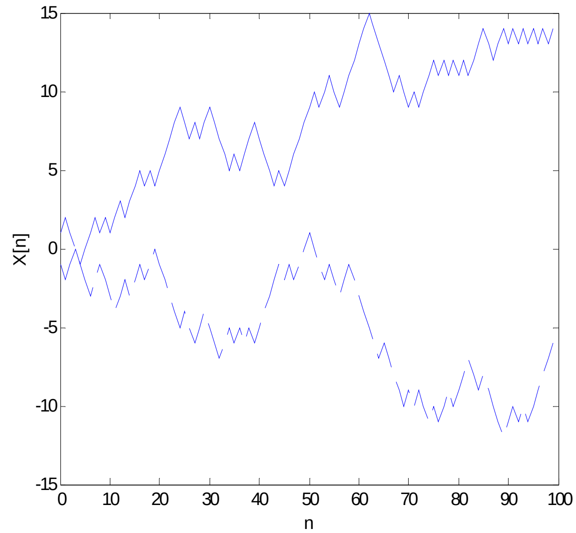

Figure 1:

where we have used the uncorrelation of the random variables $U[i]$. The variance of $U[i]$ is

$$var[U[i]] = E[U^2[i]] = 1^2 \cdot 1/2 + (-1)^2 \cdot 1/2 = 1,$$

therefore we have

$$c_X[n, n+k] = (n+1) \cdot var[U[i]] = n+1,$$

which is NOT independent of n. We then conclude that the random walk is not a stationary process. In particular notice that the variance of $X[n]$ increases linearly with n:

$$var[X[n]] = c_X[n, n] = n+1,$$

thus the process is not only non-stationary but it is also not WSS.
(iii) Since transformations of independent random variables are still independent, $Y[n] = U^2[n]$ is an IID random process.

2) Consider the random process defined as

$$X[n] = 2U[n] - 4U[n-1],$$

where $U[n]$ is a white noise with zero mean and variance $\sigma^2 = 1$.
(i) Is this process WSS? If so, evalutate the auto-correlation sequence.
(ii) Generate a realization of 1000 samples of $X[n]$ by using MATLAB.
Solution:
(i) In order to check if the process is WSS, we need to verify the following conditions on the mean and covariance sequences

$$\begin{aligned} \mu_X[n] &= \mu \\ c_X[n, n+k] &= c_X[k] \end{aligned}$$

or equivalently

$$\begin{aligned} \mu_X[n] &= \mu \\ E[X[n]X[n+k]] &= r_X[k]. \end{aligned}$$

Let us calculate these moments for the random process at hand

$$\begin{aligned} \mu_X[n] &= E[2U[n] - 4U[n-1]] = 0 \\ E[X[n]X[n+k]] &= E[(2U[n] - 4U[n-1])(2U[n+k] - 4U[n+k-1])] = \\ &= 4E[U[n]U[n+k]] + 16E[U[n-1]U[n+k-1]] - \\ &\quad -8E[U[n]U[n+k-1]] - 8E[U[n-1]U[n+k]] \\ &= \begin{cases} 20 & k = 0 \\ -8 & k = \pm 1 \\ 0 & elsewhere \end{cases} = 20\delta[k] - 8\delta[k-1] - 8\delta[k+1]. \end{aligned}$$

It can be concluded that the process is WSS (it is a specific kind of MA process) and we have mean sequence and correlation function as follows:

$$\begin{aligned} \mu_X[n] &= 0 \\ r_X[k] &= 20\delta[k] - 8\delta[k-1] - 8\delta[k+1]. \end{aligned}$$

(ii) A realization of the process can be obtained by the following MATLAB code:

```
N=1000;
u=randn(N,1);
for n=1:N
 if n==1 x(n)=2*u(n);
 else
x(n)=2*u(n)-4*u(n-1);
end
end
```