"Small-World" Effects of Shadowing in Pulse-Coupled Distributed Time Synchronization

Osvaldo Simeone, *Member, IEEE*, Umberto Spagnolini, *Senior, IEEE*, and Yeheskel Bar-Ness, *Life Fellow, IEEE*

***Abstract*— Performance of pulse-coupled distributed synchronization in wireless ad hoc networks is known to depend critically on the topology of the corresponding connectivity graph. This paper evaluates the impact of log-normal shadowing on the convergence of distributed discrete-time phase locked loops (PLLs). It is argued that the beneficial effect of shadowing on distributed synchronization can be seen as an instance of the "small-world" phenomenon. Evidence of this is further provided by evaluating average path length and clustering coefficient of the wireless network through simulations.**

***Index Terms*— Distributed synchronization, small-world networks, shadowing.**

I. Introduction

DISTRIBUTED synchronization based on pulse-coupled oscillators is currently being investigated for wireless networks as a promising alternative to the traditional packet-based approach [1] [2]. Its convergence properties have been shown to depend critically on the topology of the connectivity graph describing the underlying wireless network [3] [4], which in turns hinges on the considered propagation scenario.

Conventional models for ad hoc or sensor networks are based on random geometric graphs [5] (or equivalently on Boolean models [6]), wherein nodes are assumed to be randomly located in a given area (possibly infinite), and any two nodes are considered connected if their relative distance is smaller than a given transmission radius. Although useful abstractions for analysis, these models neglect the inherent randomness of wireless connections due to fading and shadowing. Recently, there have been a few attempts to consider more realistic scenarios where connection between any two randomly located nodes is not deterministically fixed by the network geometry, but occurs with a given probability which is a function of the relative distance. In particular, references [7] [8] [9] studied connectivity and coverage of the network in the presence of slow fading due to log-normal shadowing[1]. Moreover, it was briefly stated in [7] [9] that shadowing effectively turns wireless networks into "small-world" networks [10] by breaking a few close connections and creating a small number of long-range links.

Manuscript received October 25, 2006. The associate editor coordinating the review of this letter and approving it for publication was Dr. Javier Gozálvez.

O Simeone and Y. Bar-Ness are with the CWCSPR, ECE Department, New Jersey Institute of Technology, University Heights, Newark, NJ 07102 (email: osvaldo.simeone@njit.edu, barness@yegal.njit.edu).

U. Spagnolini is with the Dip. di Elettronica e Informazione, Politecnico di Milano, P.zza L. da Vinci, 32 I-20133 Milano, Italy (email: spagnoli@elet.polimi.it).

Digital Object Identifier 10.1109/LCOMM.2007.061737.

[1]The framework studied in [7] [8] [9] can be seen as an instance of the random connection model studied in continuum percolation [6].

In this paper, we investigate the impact of log-normal shadowing on distributed synchronization by considering the system of distributed discrete-time Phase Locked Loops (PLLs) proposed in [4]. It is argued that the beneficial effects of shadowing on the performance of distributed synchronization can be interpreted as a manifestation of the "small-world" effect on distributed processing discussed in [11] [14]. This conclusion is corroborated by the evaluation of average path length and clustering coefficient [10] of the graph underlying the wireless network through simulation.

II. Shadowing and Small-World ad hoc Networks

In this section, we briefly discuss the model considered in [7] [8] [9] of wireless network with log-normal random connections (Sec. II-A). Then, we introduce the basic concepts and measures defining a small world network (Sec. II-B). Finally, Sec. II-C provides a numerical example to show the impact of shadowing on the topology of a wireless system and the relationship with small-world networks.

A. Modelling shadowing

Consider K nodes randomly located in a given area. Focusing on any two nodes at a relative distance d, we define as $P(d)$ the power received over the distance d. The two nodes are considered connected if $P(d) > P_0$, where P_0 measures the sensitivity of the receiver. In random geometric graph models [5], that neglect channel randomness, $P(d)$ is deterministic. A typical choice is $P(d) \propto 1/d^\alpha$, where the path loss exponent is $\alpha = 2 \div 4$. The previous condition then reads $d < r$, with r usually referred to as the transmission radius. More realistic models account for the random nature of propagation, and model power $P(d)$ as a random variable. In this case, the connectivity event $P(d) > P_0$ occurs with a given probability that depends on d [6]. A typical and relevant case is that of log-normal shadowing, that models slow fading, with $P(d) = 10^{\frac{\nu}{10}}/d^\alpha$, where random variable ν is Gaussian with zero average and variance σ^2 (see [7] [8] [9]). Notice that for $\sigma^2 \rightarrow \infty$, the model reduces to a random graph [12] while for $\sigma^2 \rightarrow 0$ it boils down to a random geometric graph [5].

B. Small-world networks

Small-world networks are characterized by the fact that, despite the large size, there exists a short chain of paths between any two nodes [10] [11]. A measure of the extent to which a network resembles a small world is the *average path length*, that is the average distance (in terms of hops) between any two nodes. Another important parameter is the *clustering coefficient*, that measures to what extent the network

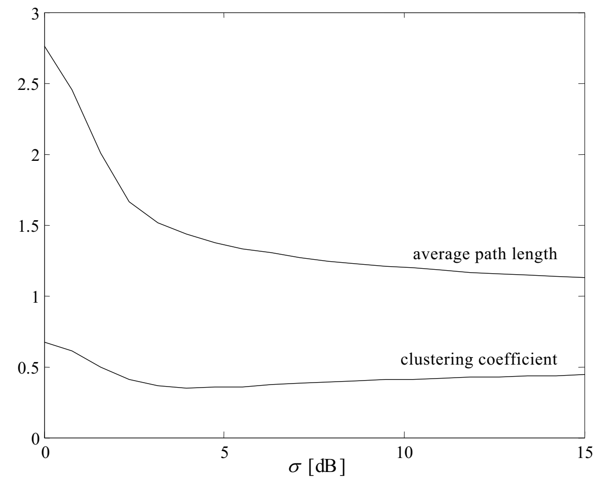

Fig. 1. Average path length and clustering coefficient of a wireless network with $K = 100$ randomly located nodes and log-normal shadowing versus the standard deviation of shadowing σ.

is localized in clusters. Consider any node i, having a given number k_i of edges to k_i other nodes. If the all the k_i neighbors of node i were connected among themselves in a cluster, then there would be $k_i(k_i - 1)/2$ edges among them. The ratio between the actual number of edges E_i among the k_i neighbors of node i and $k_i(k_i-1)/2$ is the so called clustering coefficient of node i: $C_i = 2E_i/(k_i(k_i - 1))$. The clustering coefficient of the network is the evaluated by averaging over all the nodes.

Small-world networks are usually characterized by both high clustering coefficient and small average path length. Such a scenario is made possible by the occurrence of a few long-range connections that link different clusters [10] [11]. As remarked in [7] [9] and further discussed below, log-normal shadowing makes this phenomenon possible. A related numerical investigation based on an artificial model where links are broken and rewired at random following [11] was reported in [13].

C. A numerical example

Due to the long tails of the log-normal distribution, while a few short-range connections are destroyed by shadowing, a small number of long-range connections are formed. Therefore, by increasing the standard deviation of shadowing σ, it is expected that the network increasingly resembles a small world network. This is illustrated with the following numerical example, and through investigation of a distributed synchronization algorithm in the next section. Consider $K = 100$ nodes randomly located in a square with unit area and the connection model described in Sec. II-A with log-normal shadowing. The threshold is selected as $P_0 = 4$. Fig. 1 shows the average path length and clustering coefficient of the network, averaged over a large number of Monte Carlo iterations of random locations and shadowing, as a function of the shadowing standard deviation σ. As expected, the average path length decreases with increasing σ and, moreover, it is enough to have a shadowing standard deviation of around $\sigma = 3\text{-}4dB$ in order to get the most relevant average path length reduction. On the other hand, the average clustering does not vary as significantly as the average path length (similarly to [11] [13]) and slightly increases for increasing σ due to the formation of larger-scale clusters.

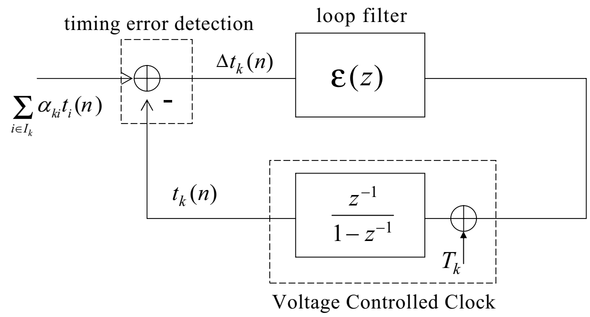

Fig. 2. Discrete-time PLL run by each node for the example in Sec. III.

III. The Impact of Shadowing on Distributed Time Synchronization

As discussed in [11] [14] and references therein, small-world networks enhance the capacity of the participating nodes to perform distributed tasks, such as broadcasting a given information or attain consensus starting from a disorganized state. In this Section, we show that the same small-world effect is observed while studying distributed processing in wireless networks in the presence of log-normal shadowing. For this, we consider the system of distributed discrete-time PLLs for time synchronization studied in [4].

1) Distributed discrete-time PLLs: We consider a network of K clocks with different free oscillation frequencies $\{1/T_k\}_{k=1}^K$, that communicate over the wireless network at hand. The clocks are defined by discrete-time functions $t_k(n)$, that, in case of isolated nodes, evolve as $t_k(n) = nT_k + \theta_k(n)$, where index $n = 1, 2, ...$ runs over the periods of the clock and $0 \leq \theta_k(n) < T_k$ is the instantaneous phase. Two synchronization conditions are of interest. We say the K clocks are *frequency* synchronized if $t_k(n+1) - t_k(n) = T$ for each k and for sufficiently large n, where $1/T$ is the common frequency. A more strict condition requires full *frequency and phase* synchronization, i.e., $t_1(n) = \cdots = t_K(n)$ for n sufficiently large.

Towards the goal of achieving synchronization, clocks are coupled through the transmission by each node, say the kth, of a pulse at each time $t_k(n)$. The topology of the network determines the power received by any kth node from the ith as $P(d_{ki})$, where $d_{ki} = d_{ik}$ is the distance between the nodes. Only pulses received with sufficient power ($P(d_{ki}) > P_0$) are detected. Based on the detected pulses, any kth clock updates its instantaneous phase $\theta_k(n)$. This operation is performed according to the discrete-time PLL shown in Fig. 2. A timing error detector estimates a convex weighted sum of the time differences between the local clock and the others, $t_i(n) - t_k(n)$ for $i \in \mathcal{I}_k = \{i \neq k \colon P(d_{ki}) > P_0\}$, at the nth period[2]:

$$\Delta t_k(n+1) = \sum_{i \in \mathcal{I}_k} \alpha_{ki} \cdot (t_i(n) - t_k(n)), \qquad (1)$$

with $\alpha_{ki} \geq 0$ and $\sum_{i \in \mathcal{I}_k} \alpha_{ki} = 1$. This measure is fed to a loop filter $\varepsilon(z)$, whose output $\Delta\theta_k(n+1)$ drives the local

[2]Propagation delays can be shown to contribute to frequency mismatch among nodes.

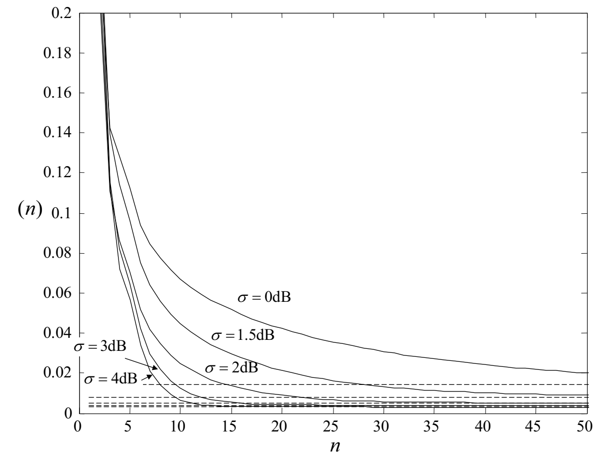

Fig. 3. Standard deviation of the clocks for the distributed system of discrete-time PLLs versus time n for different values of the standard deviation of shadowing σ. Dashed lines correspond to the analytical result (4) ($\varepsilon_0 = 0.6, \mu = 0$).

Voltage Control Clock (VCC)

$$t_k(n+1) = t_k(n) + \Delta\theta_k(n+1) + T_k. \qquad (2)$$

The number of poles in the loop filter $\varepsilon(z)$ determines the order of the loops as for conventional analog PLLs. Following the algorithm in [2], we select the weighting coefficients α_{ki} as $\alpha_{ki} = P(d_{ki})/\sum_{i\in\mathcal{I}_k} P(d_{ki})$.

2) Convergence: Reference [4] studies the convergence of the system of discrete-time PLLs (2) for second-order loops with filters $\varepsilon(z) = \varepsilon_0/(1-\mu z^{-1})$. Let us denote a possible common value for the frequency of all nodes as $1/T$ (to be determined), i.e., $t_k(n) - t_k(n-1) = T$ for sufficiently large n, so that the clock of the kth sensor can be written (for large n) as

$$t_k(n) = nT + \tau_k(n), \qquad (3)$$

where $\tau_k(n)$ denotes the relative phase with respect to the common frequency. Moreover, let us define the vector $\tau(n) = [\tau_1(n) \ \cdots \tau_K(n)]^T$ and denote as $\mathbf{L}$ the Laplacian of the network (i.e., $[\mathbf{L}]_{ii} = 1$ and $[\mathbf{L}]_{ij} = -\alpha_{ij},\ i \neq j$) and $\mathbf{A} = \mathbf{I} - \varepsilon_0\mathbf{L}$. It is shown in [4] that, if the gain ε_0 and the pole μ are sufficiently small and the graph of the network is strongly connected[3], then the system (2) synchronizes the clocks of the K nodes to the common period $T = \mathbf{v}^T\mathbf{T}$, where $\mathbf{T} = [T_1 \cdots T_K]^T$ and $\mathbf{v}$ is the normalized left eigenvector of matrix $\mathbf{A}$ corresponding to the eigenvalue $\lambda(\mathbf{A}) = 1$ ($\mathbf{A}^T\mathbf{v} = \mathbf{v}$ with $\mathbf{1}^T\mathbf{v} = 1$). However, under the same assumptions, the timing phases $\tau(n)$ remain generally mismatched and given for $n \to \infty$ by

$$\tau(n) \to \tau^* = \mathbf{1}\cdot\eta + (1-\mu)\frac{\mathbf{L}^\dagger}{\varepsilon}\mathbf{\Delta T}, \qquad (4)$$

with $(\cdot)^\dagger$ denoting the pseudoinverse, η a given constant and the kth element of vector $\mathbf{\Delta T}$ being $[\mathbf{\Delta T}]_k = T_k - T$. Notice that [4] also shows that, in absence of frequency mismatch among the clocks ($\mathbf{\Delta T} = \mathbf{0}$), the network achieves full frequency and phase synchronization to the value $\tau(n) \to \tau^* = \mathbf{1}\cdot\mathbf{v}^T\tau(0)$.

[3] A graph is said to be strongly connected if there exists at least a path that links every pair of nodes.

A. A numerical example on distributed time synchronization

In this section, we evaluate the convergence of the synchronization algorithm discussed above on the random network with log-normal shadowing employed in Sec. II-C. We evaluate the standard deviation $\xi(n)$ of the clocks, where $\xi^2(n) = 1/K\cdot\sum_{k=1}^{K}(t_k(n) - 1/K\sum_{k=1}^{K} t_k(n))^2$, versus time n, averaged over random location of nodes and shadowing. The initial phases $\tau(0) = \mathbf{t}(0)$ are selected randomly in the set $(0,1)$ (and $\mathbf{t}(-1) = \mathbf{0}$), while the local free-oscillation frequencies are selected independently in the set $1 \pm 1\%$. The dashed lines in Fig. 3 correspond to the asymptotic result (4). It can be seen that increasing the amount of shadowing in the model (i.e., the variance σ^2) improves both the convergence speed and the asymptotic phase error of the system of distributed PLLs. Moreover, as expected from the discussion in Sec. II-C, it is enough to have a standard deviation of around 3-$4dB$ to harness the most relevant advantages of shadowing.

IV. CONCLUDING REMARKS

This letter focused on the effect of log-normal shadowing on wireless ad hoc/ sensor networks in terms of "small-world" properties of the associated connectivity graphs. It has been shown, via numerical simulations, that a shadowing standard deviation of around 3-$4dB$ is enough to reduce significantly the average path length of the network. The beneficial impact of this phenomenon on a distributed synchronization algorithm has also been validated, thus confirming the notion that small-world networks enhance the capacity of participating nodes to perform distributed tasks.

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