CHAOS

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Abstract

In class we will: (a) introduce typical systems with chaos, (b) consider in detail a driven non-linear pendulum, and (c) discuss the typical features of chaos. The *Mathematica* realization is in a separate file "chaos.m" (posted on the Web)

Note: These notes may hard to read without additional class explanations. There still might be some small typos in this version.

I. INTRODUCTION

There are several large groups of problems with chaos:

- Hamiltonian systems in mechanics (no friction, conserved total energy) with several degrees of freedom. Examples: double pendulum from one of the previous classes, but with a large amplitude, or the Henon-Heiles potential $x^2y - y^3/3$ (a problem from astronomy).
- periodically driven physical systems, usually with small friction, such as the non-linear pendulum which we discussed for small amplitudes, and which we will study below
- "toy" models which are described by several ordinary differential equations (not necessarily from physics). Best example - the Lorentz model in atmospheric science, for which chaos was first discovered.

II. THE EQUATION

Consider an equation

$$\ddot{x} + c\dot{x} + \sin(x) = F\cos(\omega t) \tag{1}$$

A. Analytical domains

First, when eq. (1) can be solved analytically? [Analytical solution just means that here is *no* chaos.]

a) No driving.

Eventually energy will be lost, and for small x it becomes just a linear pendulum with dissipation, which can be solved exactly. Phase portrait is a spiral which goes towards the origin, as in Fig. 1. Can be used to test the numerical scheme (which is an issue since an incredible precision might be required - see below).

Small project (optional) Do that.

b) small driving.

For small F , x is also small and there is a weak non-linearity. Can be solved via expansion in amplitude. [It is harder if driving frequency is closer to natural, which leads to a non-linear resonance, as we discussed in class, but otherwise rather straightforward].

Large project (optional). Do the analytics and compare with numerics.

c) Large driving.

For large F the $\sin(x) \leq 1$ can be neglected in eq. (1). This leads to forced "oscillations" with no spring ($\omega_0 = 0$) with a known solution. For small c

$$x \to \left(F/\omega^2\right)\cos(\omega t)$$

The sin of the above can be included to get the next iteration.

Project (optional) do that and compare with numerics

III. CHAOS VS. NON-CHAOS

One of the way to identify chaos is by looking at the phase trajectory. For non-chaotic motion it will be regular looking [though, possibly a very complex one - see Fig. 2(a)]. In case of chaos the phase trajectory will be without repetitions [Fig. 2(b)]. Eventually, for large time it will completely fill out the corresponding domain of phase space (not shown in the figure).

A. The "butterfly effect"

One of the important differences between stable and chaotic systems is the behavior of initially close phase trajectories. Do they stay close, or do they diverge? At small times the divergence (convergence) is often exponential, $\epsilon \exp(\lambda t)$ with λ known as the Lapunov exponent (although more frequently discussed in Hamiltonian systems). For $\lambda < 0$ the system is (locally) stable, otherwise there might be chaos. This is no all the story, however, since we are often interested in stability over a long time interval (at least larger than $2\pi/\omega$. The figure Fig. 3 tells the story.

IV. THE PATH TO CHAOS

From our study of the non-linear problem with small amplitude, it could seem that higher amplitudes just add more Fourier harmonics into the expansion of the solution, which otherwise remains periodic with the same period $2\pi/\omega$ as the driving. What we did not do, however, is the study of stability of such solutions. That would mean that we take a solution, $x(t)$ and look for a small correction $\delta(t)$ (say, we artificially gave a slight push to the pendulum during the swing, and watch what will happen). Expanding eq.(1) for small δ (and considering that $x(t)$ is a solution, so many terms will be zero), one obtains

$$\ddot{\delta} + c\dot{\delta} + \delta \cos[x(t)] = 0$$

Note that since $x(t)$ is periodic, this is exactly an equation for parametric resonance which we studied before. There can be (and there will be) instabilities. This means that the solution does not have to be a periodic function with the same period - the period can get two, three, etc. times larger (see below). This is not yet chaos, the solution is regular but just very complex. With further increase of amplitude, however, this increase of period will happen more and more easily, and at some modest amplitude, the period gets infinite. This is chaos - the solution never repeats itself, not even closely.

A. The Poincare cross section

The main problem with chaotic (or near-chaotic) systems is that there is too much information to handle. One could see that from the full phase portrait. Poincare suggests to make a "snapshot" every driving period. (In addition, all values of x are brought to the $-\pi < x < \pi$ interval, since x is an angle and there is no physical difference between x and $x + 2\pi$). After every period one gets one point on the (x, p) plane. These points may or may not coalesce with each other. If the motion is periodic with the same period $2\pi/\omega$, there will be just one point on the plane - see Fig. 5(a). For a larger amplitude the period doubles - see Fig. 5 for $F = 0.7$, and there are still more points for $F = 0.8$, i.e. a still larger period which is an integer multiple of $2\pi/\omega$. This are all regular motions (although very complex for larger amplitudes). Chaos means that points do not coalesce with one another, but rather after a very large number of periods will form a continuous curve, often of very weird shape (sometimes called *attractor*) see the last of Figs. 5.

What one achieves using the Poincare cross section? - It is mostly the reduction of dimension. For regular motion one has a zero-dimensional picture of separate points (and the number of points gives further information about the period), while for chaos one has a one-dimensional curve.

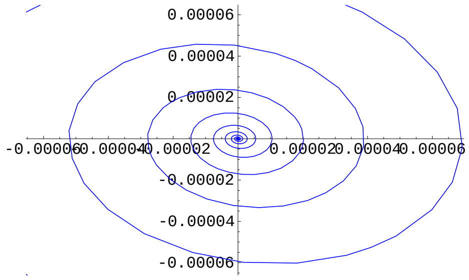

FIG. 1: Phase portrait of a non-driven pendulum.

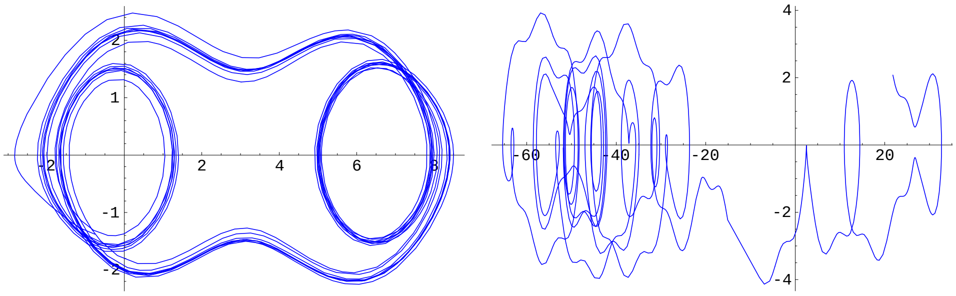

FIG. 2: Phase portrait of a driven pendulum for a stable and for a chaotic regimes

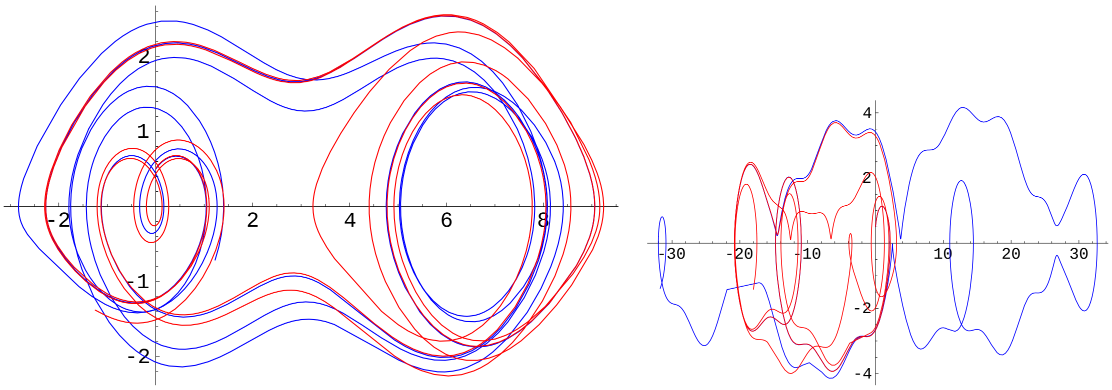

FIG. 3: Stability of phase trajectories in a stable regime, and their divergence for chaos, the "butterfly effect".

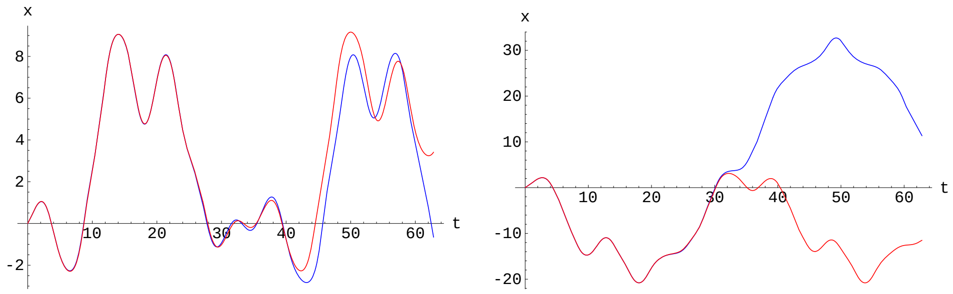

FIG. 4: Same as in previous figure, but for $x(t)$.

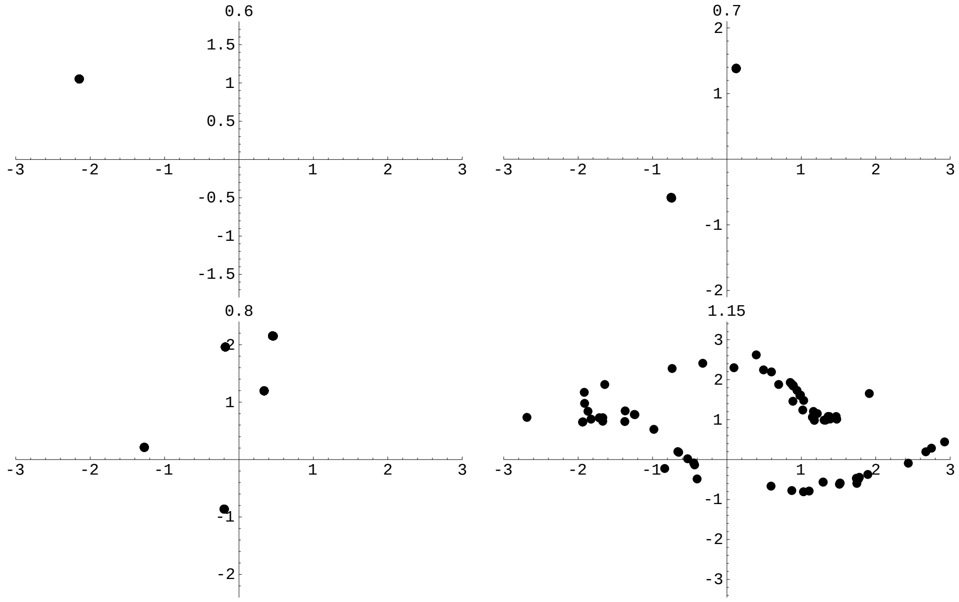

FIG. 5: Poincare cross-sections for stable regimes, $F = .6, .7, .8$ and for chaos, $F = 1.15$ (in all cases $c = 0.2$)