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On regularizing the strongly nonlinear model for two-dimensional internal waves

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HIGHLIGHTS

- A model is derived for 2D strongly nonlinear internal waves in a two-layer system.
- The model is regularized to remove ill-posedness due to shear instability.
- The dynamics of vorticity described by the regularized model is discussed.
- The model is further extended to include the effects of bottom topography.
- Some asymptotic limits including weakly 2D and weakly nonlinear ones are discussed.

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ABSTRACT

To study the evolution of two-dimensional large amplitude internal waves in a two-layer system with variable bottom topography, a new asymptotic model is derived. The model can be obtained from the original Euler equations for weakly rotational flows under the long-wave approximation, without making any smallness assumption on the wave amplitude, and it is asymptotically equivalent to the strongly nonlinear model proposed by Choi and Camassa (1999) [3]. This new set of equations extends the regularized model for one-dimensional waves proposed by Choi et al. (2009) [30], known to be free from shear instability for a wide range of physical parameters. The two-dimensional generalization exhibits new terms in the equations, related to rotational effects of the flow, and possesses a conservation law for the vertical vorticity. Furthermore, it is proved that if this vorticity is initially zero everywhere in space, then it will remain so for all time. This property – in clear contrast with the original strongly nonlinear model formulated in terms of depth-averaged velocity fields – allows us to simplify the model by focusing on the case when the velocity fields involved by large amplitude waves are irrotational. Weakly two-dimensional and weakly nonlinear limits are then discussed. Finally, after investigating the shear stability of the regularized model for flat bottom, the effect of slowly-varying bottom topography is included in the model.

1. Introduction

Large amplitude internal solitary waves, excited typically by the interaction of tidal currents with bottom topography, have been observed frequently in coastal oceans through in situ measurements and satellite images. The importance of this geophysical phenomenon has been increasingly appreciated as it is believed to be responsible for a significant fraction of the mixing that must exist to maintain the observed ocean circulation.

Weakly nonlinear models have been extensively used to study internal waves and, among them, the (uni-directional) KdV model has stood for many years as the "canonical" equation for the evolution of these waves. Internal solitary waves often have amplitudes comparable to the thickness of the well-mixed upper layer and, therefore, the validity of weakly nonlinear models is expected to be rather limited [1]. Nevertheless, these simplified models have provided valuable information that helped one characterize such waves, and have paved the way for more elaborate higher-order nonlinear models that allow a more accurate description.

Among higher-order nonlinear models, we single out the strongly nonlinear model first proposed by Miyata [2], and later by Choi and Camassa [3]. The model not only has a rich mathematical structure, endowed with several "physical" conservation

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laws and a Hamiltonian structure for traveling-wave solutions, but also describes remarkably well the large amplitude solitary wave profiles observed in laboratory experiments and predicted by numerical solutions of the Euler equations [3,4]. The model has been further generalized to include the effects of a top free surface [5,6], varying bottom topography [7], multiple layers [8,9], background shear [10], and linear stratification [11].

Unfortunately, the strongly nonlinear two-layer system, although linearly stable around the rest state, suffers from shear instability due to a velocity jump (even if arbitrarily small) across the deformed interface. This is also true for the original Euler equations in the absence of surface tension between the two fluids (see e.g. [12]) although their critical wave number for instability is different from that for the strongly nonlinear model, as shown in Jo and Choi [13]. This compromises the applicability of these models to solve time-dependent internal wave problems (see [13–15]). This instability is not a mere theoretical artifact of a two-layer approximation and is also present for continuously stratified fluids with a relatively sharp transition region. In agreement with field observations [16] and laboratory experiments [17], as they propagate, large amplitude internal waves induce a shear, which may trigger Kelvin–Helmholtz (KH) instability characterized by developing billow roll-ups, often leading to turbulence and wave breaking.

This stability problem has attracted considerable attention and has been carefully examined by means of direct numerical simulations (see e.g. [18–22]) and analytical works, such as the studies by Kataoka [23], providing evidence that only convective instability occurs for internal waves, and Camassa and Viotti [24] on the response of these waves to upstream disturbances. It should be mentioned that even in the non-dispersive limit of shallow-water equations, characterizing the nonlinear stability of the flow is not a trivial matter. These are mixed-type problems with both hyperbolic and elliptic regions, which, quoting Ovsyannikov [25], *impugns (but does not disprove!) the correctness of the Cauchy problem with arbitrary initial data at* $t = 0$ (see also [26–29]).

We should remark that internal solitary waves of finite amplitude subject to shear instability often propagate without losing their original shapes except some local disturbances shed downstream. Therefore, to remove the unrealistic KH instability that is present for any arbitrary small wave amplitudes in the strongly nonlinear model, some regularization is necessary. Low-pass numerical filters have been successfully employed by Jo and Choi [15]. However, this strategy is not systematic and may have issues with more general time-dependent problems, given that the choice of the cutoff wavenumber is arbitrary. Recently, Choi et al. [30] have shown that regularization can be achieved by simply re-writing the system in terms of the horizontal velocities evaluated at the top and bottom boundaries, instead of the depth-averaged velocities (see also [31]). The one-dimensional model proposed by these authors is asymptotically equivalent to the original strongly nonlinear model and, as it has been shown, changes the dispersive behavior of short waves without altering that of long waves.

The same idea is used in this paper to propose a regularized model for two-dimensional large amplitude internal waves propagating over variable bottom topography. Two-dimensional effects may be of great importance in real applications since important spreading effects can occur [32], as well as oblique wave interactions, diffraction and refraction, in response to bathymetry, islands, or other geological features [33].

The paper is organized as follows. After presenting the model derivation in Section 2 for the flat bottom case from first principles, under the long-wave approximation and a weak rotational assumption on the original three-dimensional flow, we study the vorticity dynamics for the new model and reveal the property that the flow preserves a zero (scalar) vorticity condition. This allows us to simplify the model by focusing on the case when the internal wave motions start from rest. Weakly two-dimensional and weakly nonlinear limits are then discussed and it is shown, in particular, that the model contains the well-known KP equation (see the work by Kadomtsev and Petviashvili [34]). Finally, after investigating in Section 3 the shear stability of the regularized model for flat bottom, the model is then further generalized to include topographic effects in Section 4.

2. A regularized strongly nonlinear model

2.1. The original two-dimensional strongly nonlinear model

The Cartesian coordinates $(\boldsymbol{x}, z) = (x, y, z)$ are introduced with origin at the interface of two fluids of different constant densities, ρ_1 for the upper fluid and ρ_2 for the lower fluid, with $\rho_1 < \rho_2$ for stable stratification. The velocity components $(\boldsymbol{u}_i, w_i) = (u_i, v_i, w_i)$ and the pressure p_i for inviscid and incompressible fluids satisfy the Euler equations:

$$\nabla \cdot \boldsymbol{u}_i + w_{i,z} = 0, \tag{2.1}$$

$$\boldsymbol{u}_{i,t} + (\boldsymbol{u}_i \cdot \nabla)\, \boldsymbol{u}_i + w_i\, \boldsymbol{u}_{i,z} = -\nabla p_i/\rho_i, \tag{2.2}$$

$$w_{i,t} + \boldsymbol{u}_i \cdot \nabla w_i + w_i\, w_{i,z} = -p_{i,z}/\rho_i - g, \tag{2.3}$$

where g is the gravitational acceleration, $\nabla = (\partial/\partial x, \partial/\partial y)$, and subscripts with respect to coordinates, or time, stand for partial differentiation. These equations apply to both upper and lower fluids for $i = 1$ and 2, respectively, and are complemented by appropriate boundary conditions. At the interface $z = \zeta(\boldsymbol{x}, t)$, the following kinematic and dynamic boundary conditions apply

$$\zeta_t + \boldsymbol{u}_i \cdot \nabla\zeta = w_i, \qquad p_1 = p_2, \quad \text{at } z = \zeta(\boldsymbol{x}, t).$$

At the upper and lower rigid boundaries, the kinematic boundary conditions require

$$w_1(\boldsymbol{x}, h_1, t) = 0, \qquad w_2(\boldsymbol{x}, -h_2, t) = 0, \tag{2.4}$$

where h_1 and h_2 are the undisturbed thicknesses of each fluid layer, and $h_2 = O(h_1)$ will be assumed.

By introducing the depth-averaged velocities $\overline{\boldsymbol{u}}_i$ defined by

$$\overline{\boldsymbol{u}}_1 = \frac{1}{\eta_1}\int_{\zeta}^{h_1} \boldsymbol{u}_1(\boldsymbol{x}, z, t)\, dz, \qquad \overline{\boldsymbol{u}}_2 = \frac{1}{\eta_2}\int_{-h_2}^{\zeta} \boldsymbol{u}_2(\boldsymbol{x}, z, t)\, dz,$$

where $\eta_1 = h_1 - \zeta(\boldsymbol{x}, t)$, and $\eta_2 = h_2 + \zeta(\boldsymbol{x}, t)$ are the layer thicknesses, and assuming the small long-wave parameter $\epsilon = h_1/\lambda \ll 1$, Choi and Camassa [5] derived – under the assumption that the leading order of the horizontal components of the velocity fields are z-independent – the strongly nonlinear model

$$\eta_{i,t} + \nabla \cdot (\eta_i\, \overline{\boldsymbol{u}}_i) = 0, \tag{2.5}$$

$$\overline{\boldsymbol{u}}_{i,t} + (\overline{\boldsymbol{u}}_i \cdot \nabla)\, \overline{\boldsymbol{u}}_i + g\nabla\zeta = -\frac{1}{\rho_i}\nabla P + \frac{1}{\eta_i}\nabla\left(\frac{1}{3}\eta_i{}^3\, \overline{G}_i\right), \tag{2.6}$$

which approximates the Euler equations with errors of $O(\epsilon^4)$. In (2.6), $P(\boldsymbol{x}, t)$ is the pressure at the interface, and the system (2.5)–(2.6) can be thought of as the two-layer shallow water equations modified to include nonlinear dispersive effects accounted for here by

$$\overline{G}_i = \nabla \cdot \overline{\boldsymbol{u}}_{i,t} + \overline{\boldsymbol{u}}_i \cdot \nabla\big(\nabla \cdot \overline{\boldsymbol{u}}_i\big) - \big(\nabla \cdot \overline{\boldsymbol{u}}_i\big)^2. \tag{2.7}$$

If the upper layer is neglected and P is regarded as the external pressure applied to a free surface, (2.5)–(2.6) are precisely the Green–Naghdi equations for a homogeneous layer (see the works by Su and Gardner [35], and Green and Naghdi [36]).

The model is the natural extension of the one-dimensional strongly nonlinear model derived by Miyata [2], and Choi and Camassa [3]. Unfortunately, this model has been known to suffer

from wave-induced shear instability (see e.g. [13,14]). To overcome this difficulty and render the model more suitable for general time-dependent problems, we follow the strategy employed in [30] to obtain a regularized model. The key step to regularize the strongly nonlinear model (2.5)–(2.6), formulated in terms of the depth-averaged velocity field, is to rewrite the model (in an asymptotically equivalent way) in terms of the horizontal components evaluated at certain preferred vertical levels. As suggested by Nguyen and Dias [31] and Choi et al. [30], we use the levels placed at the top and bottom boundaries.

It is worth noting that if the top rigid lid is replaced by a free surface, this same choice of vertical levels would lead to a model linearly unstable, even in the absence of background shear (cf. [37]).

2.2. Derivation of a regularized model

Unlike the one-dimensional case, it is not straightforward to establish a relationship between the two velocity fields, $\overline{\boldsymbol{u}}_i$ and $\hat{\boldsymbol{u}}_i \equiv \boldsymbol{u}_i(\boldsymbol{x}, \pm h_i, t)$, even under the long-wave approximation. To find an explicit relation between the two velocities, a stronger restriction on the vorticity of the original three-dimensional velocity field $(\boldsymbol{u}_i, w_i)$ needs to be assumed, as discussed below.

As a consequence of the long-wave approximation [3,5], any component $f = (u_i, v_i, w_i)$ of the original three-dimensional velocity field can be expanded as

$$f(\boldsymbol{x}, z, t) = f^{(0)} + \epsilon^2 f^{(1)} + O(\epsilon^4).$$

Under the condition of weak horizontal vorticity, more specifically, being $O(\epsilon^4)$, the leading-order horizontal components of the original three-dimensional velocity field are z-independent (i.e., $\boldsymbol{u}_{i,z}^{(0)} = 0$), which is consistent with the derivation of (2.5)–(2.6). Then, by vertically integrating the continuity equation for each layer and imposing boundary conditions (2.4), the leading-order vertical velocity w_i can be obtained as

$$w_i^{(0)} = (\pm h_i - z)\nabla \cdot \boldsymbol{u}_i^{(0)}, \tag{2.8}$$

where

$$\boldsymbol{u}_i^{(0)} = \boldsymbol{u}_i^{(0)}(\boldsymbol{x}, t). \tag{2.9}$$

In (2.8), the positive (or negative) sign has to be taken for the upper (or lower) layer and this notation will be adopted hereafter. Since the horizontal vorticity is assumed to be $O(\epsilon^4)$, the horizontal vorticity vanishes at $O(\epsilon^2)$ and, therefore, we can relate the second-order horizontal velocity components with the leading-order vertical velocity component as

$$u_{i,z}^{(1)} = w_{i,x}^{(0)}, \qquad v_{i,z}^{(1)} = w_{i,y}^{(0)}, \tag{2.10}$$

from which the second-order horizontal velocity can be obtained as

$$\boldsymbol{u}_i^{(1)}(\boldsymbol{x}, z, t) = \boldsymbol{u}_i^{(1)}(\boldsymbol{x}, z = \pm h_i, t) - \frac{1}{2}(\pm h_i - z)^2 \nabla(\nabla \cdot \boldsymbol{u}_i^{(0)}). \tag{2.11}$$

Then, from (2.9) and (2.11), we can finally relate $\overline{\boldsymbol{u}}_i$ with $\hat{\boldsymbol{u}}_i$ (in dimensional variables) through

$$\overline{\boldsymbol{u}}_i = \hat{\boldsymbol{u}}_i - \frac{1}{6}\eta_i^2 \nabla(\nabla \cdot \hat{\boldsymbol{u}}_i) + O(\epsilon^4). \tag{2.12}$$

It should be pointed out that, if we assume the horizontal vorticity is $O(\epsilon^2)$ instead of $O(\epsilon^4)$, Eq. (2.10) does not hold; therefore, (2.12) is no longer valid and no simple relationship between $\overline{\boldsymbol{u}}_i$ with $\hat{\boldsymbol{u}}_i$ can be found. It is also of interest to estimate an order of magnitude of the vertical vorticity consistent with our assumption about the horizontal vorticity. It is shown in the Appendix that the magnitude of the vertical vorticity should be $O(\epsilon^2)$ for the regularized model derived from (2.12) to be valid, opposed to the original strongly nonlinear model (2.5)–(2.6) where the vertical vorticity can be $O(1)$.

Here and hereafter, for convenience, we will write $\boldsymbol{u}_i$ to denote $\hat{\boldsymbol{u}}_i$, unless clearly stated otherwise in the text. By substituting (2.12) into the original system given by (2.5)–(2.6) and neglecting any terms of $O(\epsilon^4)$ or higher, we obtain a regularized model for two-dimensional strongly nonlinear internal waves:

$$\eta_{i,t} + \nabla \cdot \left[\eta_i \left(\boldsymbol{u}_i - \frac{1}{6}\eta_i^2 \nabla(\nabla \cdot \boldsymbol{u}_i)\right)\right] = 0, \tag{2.13}$$

$$\boldsymbol{u}_{i,t} + (\boldsymbol{u}_i \cdot \nabla)\boldsymbol{u}_i + g\nabla\zeta = -\frac{1}{\rho_i}\nabla P + \nabla\left(\frac{1}{2}\eta_i{}^2 G_i\right) + \boldsymbol{f}_i. \tag{2.14}$$

In the expression above, G_i are given by (2.7) with $\overline{\boldsymbol{u}}_i$ replaced by $\boldsymbol{u}_i$, and $\boldsymbol{f}_i$ are defined by

$$\boldsymbol{f}_i = \frac{1}{6}\eta_i{}^2\Big[\Big(\nabla(\nabla \cdot \boldsymbol{u}_i) \cdot \nabla\Big)\boldsymbol{u}_i + (\boldsymbol{u}_i \cdot \nabla)\nabla(\nabla \cdot \boldsymbol{u}_i) - \nabla\{\boldsymbol{u}_i \cdot \nabla(\nabla \cdot \boldsymbol{u}_i)\}\Big].$$

Notice that $\boldsymbol{f}_i$ on the right-hand side of (2.14) disappears in the one-dimensional case, where it vanishes trivially, and Eqs. (2.13)–(2.14) reduce to the regularized system obtained in [30].

2.3. A regularized strongly nonlinear model for irrotational flows

Another case of interest where $\boldsymbol{f}_i$ vanishes trivially is when the two-dimensional velocity field represented by $\boldsymbol{u}_i$ is irrotational, meaning that curl $\boldsymbol{u}_i$=0. This is a straightforward remark by noticing that $\boldsymbol{f}_i$ can be rewritten as

$$\boldsymbol{f}_i = \frac{1}{6}\eta_i^2 \,\mathrm{curl}\,\boldsymbol{u}_i \times \nabla(\nabla \cdot \boldsymbol{u}_i).$$

Nevertheless it remains to be proved that this irrotational assumption is compatible with the vorticity dynamics governed by (2.14).

By introducing the vorticity $\boldsymbol{\omega}_i = \mathrm{curl}\,\boldsymbol{u}_i$, we can rewrite (2.14) into the form

$$\boldsymbol{u}_{i,t} + \nabla\left(\frac{1}{2}|\boldsymbol{u}_i|^2 + g\zeta + \frac{1}{\rho_i}P - \frac{1}{2}\eta_i^2 G_i\right) + \boldsymbol{\omega}_i \times \left(\boldsymbol{u}_i - \frac{1}{6}\eta_i^2 \nabla(\nabla \cdot \boldsymbol{u}_i)\right) = 0,$$

leading to

$$\boldsymbol{\omega}_{i,t} + \mathrm{curl}\left[\boldsymbol{\omega}_i \times \left(\boldsymbol{u}_i - \frac{1}{6}\eta_i^2 \nabla(\nabla \cdot \boldsymbol{u}_i)\right)\right] = 0. \tag{2.15}$$

Notice that, since $\boldsymbol{u}_i$ represent two-dimensional velocity fields, the only non-trivial component of $\boldsymbol{\omega}_i$ is the vertical one. When denoting this scalar vorticity by ω_i, (2.15) is given, in conservative form, by

$$\omega_{i,t} + \nabla \cdot \left[\omega_i \left(\boldsymbol{u}_i - \frac{1}{6}\eta_i^2 \nabla(\nabla \cdot \boldsymbol{u}_i)\right)\right] = 0. \tag{2.16}$$

Assuming that $\boldsymbol{u}_i$ vanish sufficiently rapidly as $|\boldsymbol{x}| \to \infty$, Eq. (2.16) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \omega_i \,\mathrm{d}\boldsymbol{x} = 0.$$

As a consequence, if $\omega_i = 0$ at a certain instant of time t_0, then $\int \omega_i \,\mathrm{d}\boldsymbol{x} = 0$ for all time. This argument, however, is not sufficient to establish that the vorticity itself remains zero for all time. To

prove this assertion, it is convenient to consider the following equation for the potential vorticity

$$D_i\left(\omega_i/\eta_i\right) = \frac{1}{6}\eta_i^2\,\nabla(\nabla\cdot\boldsymbol{u}_i)\cdot\nabla\left(\omega_i/\eta_i\right), \tag{2.17}$$

with D_i defined as the material derivative $D_i = \partial/\partial t + \boldsymbol{u}_i\cdot\nabla$. Assuming enough regularity to ensure the existence and unicity of solution of this ordinary differential equation along particle trajectories, we can conclude from (2.17) that the two-dimensional flow described by the regularized model will remain irrotational for all time if ω_i are initially zero everywhere in space.

This is in clear contrast with the original two-dimensional strongly nonlinear model given by (2.5)–(2.6) since the irrotational assumption for $\overline{\boldsymbol{u}}_i$ is not compatible with the dynamics of the system. As shown by Miles and Salmon [38] for the single layer case (see also [39]), the original system conserves the modified potential vorticity defined by $(\overline{\omega}_i + \overline{\omega}_i^*)/\eta_i$ so that

$$D_i\left[(\overline{\omega}_i + \overline{\omega}_i^*)/\eta_i\right] = 0, \tag{2.18}$$

where $\overline{\omega}_i^* = J(D_i\eta_i, \eta_i)/3$ and $J(f,g) = \partial(f,g)/\partial(x,y)$ is the Jacobian operator. Therefore, even if the vorticity $\overline{\omega}_i$ defined by $(\nabla\times\overline{\boldsymbol{u}}_i)_z$ is initially zero, it does not remain zero – although it stays weak of $O(\epsilon^2)$ – as the wave field evolves.

Since we are interested in large amplitude internal wave motions starting from rest, we assume hereafter that the original three-dimensional flow field is irrotational. Then, the corresponding two-dimensional flow field resulting from the regularized model is also assumed irrotational.

Under the irrotational assumption for $\boldsymbol{u}_i$, Eqs. (2.13)–(2.14) can be simplified to

$$\eta_{i,t} + \nabla\cdot\left[\eta_i\left(\boldsymbol{u}_i - \frac{1}{6}\eta_i^2\,\nabla^2\boldsymbol{u}_i\right)\right] = 0, \tag{2.19}$$

$$\boldsymbol{u}_{i,t} + (\boldsymbol{u}_i\cdot\nabla)\,\boldsymbol{u}_i + g\,\nabla\zeta = -\frac{1}{\rho_i}\nabla P + \nabla\left(\frac{1}{2}{\eta_i}^2 G_i\right), \tag{2.20}$$

with G_i being now given by

$$G_i = \nabla\cdot\boldsymbol{u}_{i,t} + \boldsymbol{u}_i\cdot\nabla^2\boldsymbol{u}_i - \left(\nabla\cdot\boldsymbol{u}_i\right)^2,$$

where the irrotationality condition has been used to write $\nabla(\nabla\cdot\boldsymbol{u}_i) = \nabla^2\boldsymbol{u}_i$. Eqs. (2.19) and (2.20), for $i = 1, 2$, form a complete set of conservation laws for ζ, $\boldsymbol{u}_1 = (u_1, v_1)$, $\boldsymbol{u}_2 = (u_2, v_2)$, and P.

In fact, under the irrotational assumption for the original three-dimensional velocity field, a relationship between $\overline{\boldsymbol{u}}_i$ and $\hat{\boldsymbol{u}}_i$, as the one given by (2.12), can be easily found from the velocity potential. From Section 13.11 in Whitham's book [12], the velocity potential ϕ_i can be expanded, under the long-wave approximation, as

$$\phi_i(\boldsymbol{x}, z, t) = \Phi_i(\boldsymbol{x}, t) - \frac{1}{2}(\pm h_i - z)^2\,\nabla^2\Phi_i + O(\epsilon^4),$$

where $\Phi_i(\boldsymbol{x}, t)$ are the velocity potentials evaluated at the top and bottom boundaries for $i = 1$ and 2, respectively. Then, the depth-averaged velocity $\overline{\boldsymbol{u}}_i$ can be obtained as

$$\overline{\boldsymbol{u}}_i = \nabla\Phi_i - \frac{1}{6}{\eta_i}^2\,\nabla^2(\nabla\Phi_i) + O(\epsilon^4),$$

while the velocities evaluated at the top and bottom boundaries ($\boldsymbol{u}_i = \nabla\phi_i$ at $z = \pm h_i$) are given by $\boldsymbol{u}_i = \nabla\Phi_i$. Therefore, the relationship between $\overline{\boldsymbol{u}}_i$ and $\boldsymbol{u}_i$ can be written as

$$\overline{\boldsymbol{u}}_i = \boldsymbol{u}_i - \frac{1}{6}\eta_i^2\,\nabla^2\boldsymbol{u}_i + O(\epsilon^4),$$

which is precisely (2.12) with $\nabla(\nabla\cdot\boldsymbol{u}_i)$ replaced by $\nabla^2\boldsymbol{u}_i$. After integrating once, (2.19)–(2.20) can be rewritten, in terms of ζ and Φ_i, as

$$\eta_{i,t} + \nabla\cdot\left[\eta_i\left(\nabla\Phi_i - \frac{1}{6}\eta_i^2\,\nabla^2\nabla\Phi_i\right)\right] = 0,$$

$$\Phi_{i,t} + \frac{1}{2}\nabla\Phi_i\cdot\nabla\Phi_i + g\zeta = -P/\rho_i + \frac{1}{2}{\eta_i}^2\left[\nabla^2\Phi_{i,t} + \nabla\Phi_i\cdot\nabla^2\nabla\Phi_i - \left(\nabla^2\Phi_i\right)^2\right].$$

It is shown, for the flat bottom case, that if the original three-dimensional flow is irrotational with the velocity potentials $\phi_i(x, y, z, t)$, it is also the case for the two-dimensional flow described by the present reduced model. As expected, the potentials Φ_i and ϕ_i are related in the simplest way as $\Phi_i = \phi_i(x, y, \pm h_i, t)$. However, when including the topographic effects, the reduced model does not preserve this property even if the three-dimensional flow is originally irrotational. The system can no longer be written in terms of the velocity potential, having thus a more similar structure to (2.19)–(2.20), as discussed in Section 4.

2.4. Weakly two-dimensional waves

If we assume a weak dependence of physical variables on the transverse direction (or the y-direction) such that

$$\zeta_y/\zeta_x = O(\epsilon), \qquad \boldsymbol{u}_{iy}/\boldsymbol{u}_{ix} = O(\epsilon),$$
$$P_y/P_x = O(\epsilon), \qquad v_i/u_i = O(\epsilon),$$

the regularized model under the irrotational assumption given by (2.19)–(2.20) can be reduced to the weakly two-dimensional (2D) model:

$$\eta_{i,t} + (\eta_i u_i)_x = -(\eta_i v_i)_y + \left(\frac{1}{6}\eta_i^3\,u_{i,xx}\right)_x, \tag{2.21}$$

$$u_{i,t} + u_i u_{i,x} + g\,\zeta_x + \frac{1}{\rho_i}P_x = -v_i u_{i,y} + \left[\frac{1}{2}{\eta_i}^2\left(u_{i,xt} + u_i u_{i,xx} - u_{i,x}^2\right)\right]_x, \tag{2.22}$$

$$v_{i,t} + u_i v_{i,x} + g\,\zeta_y + \frac{1}{\rho_i}P_y = -v_i v_{i,y} + \left[\frac{1}{2}{\eta_i}^2\left(u_{i,xt} + u_i u_{i,xx} - u_{i,x}^2\right)\right]_y, \tag{2.23}$$

where we have moved terms of $O(\epsilon^2)$ to the right-hand sides and have neglected terms of $O(\epsilon^4)$. Since the terms with v_i in (2.21)–(2.22) are $O(\epsilon^2)$, the leading-order contribution for v_i from (2.23) is asymptotically important and, therefore, the right-hand side of (2.23) could be neglected with preserving the same order of approximation. As long as a wave field of interest is consistent with the weakly 2D assumption, we should remark that (2.21)–(2.23) are much more convenient for numerical computations than the fully 2D model given by (2.19)–(2.20). Otherwise, the regularized fully 2D system could be solved numerically by adopting an iterative scheme similar to that suggested by Choi et al. [40].

2.5. Weakly nonlinear waves

For weakly nonlinear waves, when assuming that $\zeta/h_1 = O(\epsilon^2)$, $|\boldsymbol{u}_i|/(gh_1)^{1/2} = O(\epsilon^2)$, and $h_2/h_1 = O(1)$, the regularized model given by (2.19)–(2.20) can be further approximated to the Boussinesq-type system

$$\eta_{i,t} + \nabla\cdot(\eta_i\boldsymbol{u}_i) = \frac{1}{6}h_i^3\,\nabla\cdot\nabla^2\boldsymbol{u}_i, \tag{2.24}$$

$$\boldsymbol{u}_{i,t} + \boldsymbol{u}_i\cdot\nabla\boldsymbol{u}_i + g\,\nabla\zeta = -\frac{1}{\rho_i}\nabla P + \frac{1}{2}{h_i}^2\,\nabla^2\boldsymbol{u}_{i,t}, \tag{2.25}$$

where again we have used the fact that $\nabla(\nabla \cdot \boldsymbol{u}_i) = \nabla^2 \boldsymbol{u}_i$, and neglected terms of $O(\epsilon^4)$. This system generalizes the one-dimensional model of Nguyen and Dias [31], who first introduced the velocities at the top and bottom boundaries to write the evolution equations for a two-layer system. The resulting equations should also be related to the three-parameter family of weakly nonlinear systems proposed by Bona et al. (see Section 3.1.3 in [41]).

Under the weakly two-dimensional assumption used to derive (2.21)–(2.22), the regularized weakly nonlinear model (2.24)–(2.25) can be reduced to

$$\eta_{i,xt} + (\eta_i u_i)_{xx} = \frac{1}{6} h_i^3 u_{i,xxxx} - h_i u_{i,yy}, \tag{2.26}$$

$$u_{i,t} + u_i u_{i,x} + g\,\zeta_x = -\frac{1}{\rho_i} P_x + \frac{1}{2} {h_i}^2 u_{i,xxt}, \tag{2.27}$$

where we have used the irrotationality condition $u_{i,y} = v_{i,x}$ to write (2.26). Notice that (2.26)–(2.27) can also be obtained directly from (2.21)–(2.23) by imposing the weakly nonlinear assumption.

For uni-directional waves, (2.26)–(2.27) can be further reduced to the well-known KP equation [34], which was first introduced in the context of internal waves by Ablowitz and Segur [42]:

$$\left(\zeta_t + c_0 \zeta_x + c_1 \zeta\,\zeta_x + c_2 \zeta_{xxx}\right)_x + \frac{c_0}{2} \zeta_{yy} = 0. \tag{2.28}$$

In (2.28), c_0 stands for the linear long wave speed:

$${c_0}^2 = \frac{g h_1 h_2 (\rho_2 - \rho_1)}{\rho_1 h_2 + \rho_2 h_1},$$

and the coefficients of nonlinearity and dispersion, c_1 and c_2, respectively, are prescribed by

$$c_1 = -\frac{3c_0}{2} \frac{\rho_1 h_2^2 - \rho_2 h_1^2}{\rho_1 h_1 h_2^2 + \rho_2 h_1^2 h_2}, \qquad c_2 = \frac{c_0}{6} \frac{\rho_1 h_1^2 h_2 + \rho_2 h_1 h_2^2}{\rho_1 h_2 + \rho_2 h_1}.$$

3. Local instability near the maximal displacement

For one-dimensional waves, the system can be reduced to that of Choi et al. [30], where it has been shown through local stability analysis that the dispersive behavior of (2.19)–(2.20) stabilizes the system, even in the presence of background shear, provided that the velocity jump ΔU across the interface complies with

$$\Delta U^2 \leqslant \frac{g(\rho_2 - \rho_1)(\rho_2 h_1 + \rho_1 h_2)}{3\rho_1 \rho_2}. \tag{3.1}$$

A disturbance of an arbitrary wave number k is then found neutrally stable. This stability criterion has the advantage of being easily interpreted in terms of the amplitude of propagating internal solitary waves since these induce a shear (see [3]) given by

$$\Delta U = \frac{ca(h_1 + h_2)}{(h_1 - a)(h_2 + a)}, \qquad \frac{c^2}{{c_0}^2} = \frac{(h_1 - a)(h_2 + a)}{h_1 h_2 - ({c_0}^2/g)a}. \tag{3.2}$$

From (3.1)–(3.2), it can be shown that the critical wave amplitude below which internal solitary waves are stable is close to the maximum wave amplitude (front wave solution) for a wide range of parameters relevant for real oceanic applications (see Fig. 1 in [30]).

We seek a similar result for the extended two-dimensional model (2.19)–(2.20). The system is linearized about constant states $\eta_1 = h_1 - a$, $\eta_2 = h_2 + a$, $\boldsymbol{u}_i = \boldsymbol{U}_i \equiv (U_i, V_i)$, and $P = P_0$. Let

$$\zeta \to a + \zeta', \qquad \boldsymbol{u}_i \to \boldsymbol{U}_i + \boldsymbol{u}_i', \qquad P \to P_0 + P', \tag{3.3}$$

where primed variables denote infinitesimal perturbations. By introducing (3.3) in (2.19)–(2.20) and neglecting terms of second order in each one of these variables, we obtain the linearized system for the perturbations:

$$\mp\zeta_t + h_i \nabla \cdot \boldsymbol{u}_i - \frac{1}{6} \eta_i^3\, \nabla \cdot \left(\nabla^2 \boldsymbol{u}_i\right) \mp \nabla\zeta \cdot \boldsymbol{U}_i = 0,$$

$$\boldsymbol{u}_{i,t} + (\boldsymbol{U}_i \cdot \nabla)\boldsymbol{u}_i + g\nabla\zeta + \frac{1}{\rho_i}\nabla P = \frac{1}{2} h_i^2\, \nabla \left[\nabla \cdot \boldsymbol{u}_{i,t} + \boldsymbol{U}_i \cdot \left(\nabla^2 \boldsymbol{u}_i\right)\right],$$

where the prime notation has been dropped, and the minus (plus) sign is exceptionally chosen for $i = 1$ ($i = 2$). Notice that we are considering the case when $a = 0$. The result, however, can easily be generalized to the case when $a \neq 0$.

We look for particular solutions proportional to $\exp(i\boldsymbol{k} \cdot \boldsymbol{x} - i\omega t)$ with vector wave number $\boldsymbol{k} = (k, l)$ and wave frequency ω. Moreover, since the propagation of internal solitary waves is mostly uni-directional, we will assume without loss of generality that the shear is induced only in the x-direction, so that $V_1 = V_2 = 0$. The dispersion relation is then given by the quadratic equation

$$A\,\omega^2 - 2kB\,\omega + C = 0, \tag{3.4}$$

where

$$A = 3\,(\rho_1 h_2 \alpha_1 \beta_2 + \rho_2 h_1 \alpha_2 \beta_1),$$
$$B = 3\,(\rho_1 h_2 \alpha_1 \beta_2\, U_1 + \rho_2 h_1 \alpha_2 \beta_1\, U_2),$$
$$C = 3k^2\,(\rho_1 h_2 \alpha_1 \beta_2\, U_1^2 + \rho_2 h_1 \alpha_2 \beta_1\, U_2^2) - g(\rho_2 - \rho_1) h_1 h_2\, |\boldsymbol{k}|^2 \beta_1 \beta_2,$$
$$\alpha_i = 2 + |\boldsymbol{k}|^2 h_i^2, \qquad \beta_i = 6 + |\boldsymbol{k}|^2 h_i^2.$$

For ω to be real, the discriminant of (3.4) has to be non-negative so that Δ satisfies the following inequality

$$\Delta = 12\, h_1 h_2 \beta_1 \beta_2 \Big[-3\rho_1 \rho_2\, k^2 \alpha_1 \alpha_2 (U_2 - U_1)^2 + g(\rho_2 - \rho_1)\, |\boldsymbol{k}|^2 (\rho_1 h_2 \alpha_1 \beta_2 + \rho_2 h_1 \alpha_2 \beta_1)\Big] \geqslant 0.$$

Hence, this stability criterion can then be written as

$$(U_2 - U_1)^2 \leqslant \frac{1}{3} g(\rho_2 - \rho_1)\, \frac{|\boldsymbol{k}|^2}{k^2} \left(\frac{h_1}{\rho_1} \frac{\beta_1}{\alpha_1} + \frac{h_2}{\rho_2} \frac{\beta_2}{\alpha_2}\right), \tag{3.5}$$

resulting that the flow is linearly stable as long as the shear does not exceed a certain critical value. Since we have $|\boldsymbol{k}|^2 \geqslant k^2$, the whole expression of the right-hand side of (3.5) is bounded from below by

$$\frac{g(\rho_2 - \rho_1)}{3} \left(\frac{h_1}{\rho_1} \frac{\beta_1}{\alpha_1} + \frac{h_2}{\rho_2} \frac{\beta_2}{\alpha_2}\right).$$

This is attained precisely when $|\boldsymbol{k}|^2 = k^2$, i.e., when only one-directional perturbations in the x-direction are present. This implies that two-dimensional perturbations are more stable than one-dimensional ones. We also remark, from (3.1), that the flow is always stable for any arbitrary disturbance if the shear does not exceed $[g(\rho_2 - \rho_1)(\rho_2 h_1 + \rho_1 h_2)/3\rho_1 \rho_2]^{1/2}$ since $\beta_i/\alpha_i > 1$.

4. Effect of bottom topography

The effect of bottom topography can be easily included in the long wave model as long as the characteristic length of bottom variation is as large as the characteristic wavelength. Since the regularized long wave model becomes more stable as the depth ratio of the lower layer thickness to the upper layer thickness decreases [30], the evolution of internal solitary waves propagating in water of decreasing depth can be studied effectively with the

regularized model written in terms of the lower-layer velocity evaluated at the bottom surface.

When the lower layer ($i = 2$) is bounded by slowly-varying bottom topography with $|\nabla h_2| = O(|\nabla\zeta|) = O(\epsilon)$, the original strongly nonlinear model (2.5)–(2.6) for the lower layer $i = 2$ is modified (see Appendix[1] in [5]) to

$$\eta_{2,t} + \nabla\cdot(\eta_2\,\overline{\boldsymbol{u}}_2) = 0, \qquad \eta_2 = h_2(\boldsymbol{x}) + \zeta, \tag{4.1}$$

$$\begin{aligned}\overline{\boldsymbol{u}}_{2,t} + \overline{\boldsymbol{u}}_2\cdot\nabla\overline{\boldsymbol{u}}_2 + g\nabla\zeta &= -\frac{1}{\rho_2}\nabla P + \frac{1}{\eta_2}\nabla\left(\frac{1}{3}{\eta_2}^3\,\overline{G}_2 + \frac{1}{2}{\eta_2}^2\,\overline{F}_2\right)\\ &\quad - \left(\frac{1}{2}\eta_2\,\overline{G}_2 + \overline{F}_2\right)\nabla h_2,\end{aligned} \tag{4.2}$$

where G_2 is defined in (2.7) and $\overline{F}_2$ is given by

$$\overline{F}_2 \equiv {\overline{D}_2}^2 h_2 = \overline{\boldsymbol{u}}_{2,t}\cdot\nabla h_2 + \left(\overline{\boldsymbol{u}}_2\cdot\nabla\right)^2 h_2, \tag{4.3}$$

with $\overline{D}_2 = \partial/\partial t + \overline{\boldsymbol{u}}_2\cdot\nabla$.

Assuming the three-dimensional velocity field is irrotational, the velocity potential for the lower layer ϕ_2 satisfying the Laplace equation and the bottom boundary condition $\phi_{2,z} = -\nabla\phi_2\cdot\nabla h_2$ at $z = -h_2$ can be expanded as

$$\begin{aligned}\phi_2(\boldsymbol{x}, z, t) &= \Phi_2(\boldsymbol{x}, t) - \frac{1}{2}(z + h_2)^2\,\nabla^2\Phi_2\\ &\quad - (z + h_2)\,\nabla\Phi_2\cdot\nabla h_2 + O(\epsilon^4),\end{aligned}$$

where $\Phi_2(\boldsymbol{x}, t) = \phi_2(\boldsymbol{x}, z = -h_2, t)$.

Then, the depth-averaged horizontal velocity $\overline{\boldsymbol{u}}_2$ and the horizontal velocity at the bottom $\boldsymbol{u}_2$ can be expressed as:

$$\begin{aligned}\overline{\boldsymbol{u}}_2 &= \nabla\Phi_2 - \frac{{\eta_2}^2}{6}\nabla^2(\nabla\Phi_2)\\ &\quad - \frac{\eta_2}{2}\Big[\nabla^2\Phi_2\,\nabla h_2 + \nabla(\nabla\Phi_2\cdot\nabla h_2)\Big]\\ &\quad - (\nabla\Phi_2\cdot\nabla h_2)\nabla h_2 + O(\epsilon^4),\end{aligned}$$

$$\boldsymbol{u}_2 = \nabla\Phi_2 - (\nabla\Phi_2\cdot\nabla h_2)\nabla h_2 + O(\epsilon^4), \tag{4.4}$$

from which the relationship between $\overline{\boldsymbol{u}}_2$ and $\boldsymbol{u}_2$ can be found as

$$\begin{aligned}\overline{\boldsymbol{u}}_2 &= \boldsymbol{u}_2 - \frac{{\eta_2}^2}{6}\nabla^2\boldsymbol{u}_2 - \frac{\eta_2}{2}\Big[(\nabla\cdot\boldsymbol{u}_2)\,\nabla h_2 + \nabla(\boldsymbol{u}_2\cdot\nabla h_2)\Big]\\ &\quad + O(\epsilon^4).\end{aligned} \tag{4.5}$$

From (4.4), it can be seen that unlike the flat bottom case, the horizontal velocity evaluated at the bottom is rotational even though the original three-dimensional flow is irrotational. The vertical vorticity persists relatively weak, or $O(\epsilon^2)$:

$$\nabla\times\boldsymbol{u}_2 = -\nabla(\nabla\Phi_2\cdot\nabla h_2)\times\nabla h_2 + O(\epsilon^4). \tag{4.6}$$

By substituting (4.5) into (4.1)–(4.2) with $\nabla\times\boldsymbol{u}_2 = O(\epsilon^2)$ from (4.6) and neglecting all terms smaller than $O(\epsilon^2)$, the evolution equations for the lower layer can be found, in terms of η_2 and $\boldsymbol{u}_2$, as

$$\begin{aligned}\eta_{2,t} + \nabla\cdot\Bigg[\eta_2\Bigg(\boldsymbol{u}_2 - \frac{{\eta_2}^2}{6}\nabla^2\boldsymbol{u}_2 - \frac{\eta_2}{2}(\nabla\cdot\boldsymbol{u}_2)\,\nabla h_2\\ - \frac{\eta_2}{2}\nabla(\boldsymbol{u}_2\cdot\nabla h_2)\Bigg)\Bigg] = 0,\end{aligned} \tag{4.7}$$

$$\begin{aligned}\boldsymbol{u}_{2,t} + \boldsymbol{u}_2\cdot\nabla\boldsymbol{u}_2 + g\,\nabla\zeta &= -\nabla P/\rho_2 + \nabla\left(\frac{{\eta_2}^2}{2}G_2 + \eta_2 F_2\right)\\ &\quad - F_2\nabla h_2,\end{aligned} \tag{4.8}$$

where $\nabla\times\boldsymbol{u}_2 = O(\epsilon^2)$ has been used, and G_2 and F_2 are defined in (2.7) and (4.3), respectively, with replacing $\overline{\boldsymbol{u}}_2$ by $\boldsymbol{u}_2$. When combined with the evolution equations for the upper layer given by (2.19)–(2.20) with $i = 1$, (4.7)–(4.8) form a complete set of equations for the case of uneven bottom.

We remark that it may be convenient to rewrite (4.8) in the following form:

$$\begin{aligned}&\left[\boldsymbol{u}_2 + \Big(\boldsymbol{u}_2\cdot\nabla h_2\Big)\nabla h_2 - \nabla\left(\frac{{\eta_2}^2}{2}\,\nabla\cdot\boldsymbol{u}_2 + \eta_2\,\boldsymbol{u}_2\cdot\nabla h_2\right)\right]_t\\ &\quad + \nabla\left(\frac{1}{2}\,\boldsymbol{u}_2\cdot\boldsymbol{u}_2 + g\zeta + P/\rho_2\right)\\ &= \nabla\left[\nabla\cdot\left\{\frac{{\eta_2}^2}{2}\,(\nabla\cdot\boldsymbol{u}_2)\boldsymbol{u}_2 + \eta_2(\boldsymbol{u}_2\cdot\nabla h_2)\boldsymbol{u}_2\right\}\right]\\ &\quad - \Big[(\boldsymbol{u}_2\cdot\nabla)^2 h_2\Big]\nabla h_2.\end{aligned} \tag{4.9}$$

While the momentum equation is given in conservative form for the flat bottom case ($\nabla h_2 = F_2 = 0$), an uneven bottom prevents this property to hold because of the last term on the right-hand side of (4.9). However, it is interesting to notice that, for one-dimensional waves, such a feature is preserved since the last term simplifies to $\left[(u_2\,h_{2,x})^2/2\right]_x$. Then, we are able to obtain an extra conserved quantity for the lower layer given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left[\left(1 + {h_{2,x}}^2\right)u_2\right]\mathrm{d}x = 0.$$

On the other hand, for weakly nonlinear waves, the system for the lower layer with bottom variation can be simplified to

$$\begin{aligned}\eta_{2,t} + \nabla\cdot\Bigg[\eta_2\boldsymbol{u}_2 - \frac{{h_2}^3}{6}\,\nabla^2\boldsymbol{u}_2 - \frac{{h_2}^2}{2}(\nabla\cdot\boldsymbol{u}_2)\,\nabla h_2\\ - \frac{{h_2}^2}{2}\nabla(\boldsymbol{u}_2\cdot\nabla h_2)\Bigg] = 0,\end{aligned}$$

$$\begin{aligned}\boldsymbol{u}_{2,t} + \boldsymbol{u}_2\cdot\nabla\boldsymbol{u}_2 + g\,\nabla\zeta &= -\nabla P/\rho_2 + \nabla\left[\nabla\cdot\left(\frac{{h_2}^2}{2}\boldsymbol{u}_{2,t}\right)\right]\\ &\quad - (\boldsymbol{u}_{2,t}\cdot\nabla h_2)\nabla h_2.\end{aligned}$$

On a final note, we point out that, instead of assuming zero physical vorticity, a regularized model for uneven bottom accounting for weak horizontal vorticity effects could also be easily obtained with replacing $\nabla^2\boldsymbol{u}_2$ in the second term on the right-hand side of (4.5) by $\nabla(\nabla\cdot\boldsymbol{u}_2)$.

5. Discussion

A regularized model for two-dimensional long internal waves of finite amplitude is proposed in a two-layer system and some of its asymptotic limits are examined. Weakly two-dimensional and weakly nonlinear approximate models are derived and it is shown, in particular, that our model extends the classical KP equation. The new regularized strongly nonlinear model is asymptotically equivalent to that proposed by Choi and Camassa [5], and reduces in the one-dimensional case to the model recently obtained by Choi et al. [30].

We would like to stress that the idea of writing the model in terms of the horizontal velocity evaluated at different vertical levels (thus being able to improve the dispersive behavior of the model) is not new in the theory of surface waves (see e.g. [43–46]), but clearly finds a new usage in the context of internal waves. For surface waves, this flexibility has been used to investigate the linear well-posedness (around the rest state) of Boussinesq-type equations and force their dispersion relations to meet the one given by the original Euler equations. However, choosing preferred

[1] We remark that the definition of $\overline{F}_2$ in Choi and Camassa [5] is incorrect. The correct form is given by (4.3) in this paper.

levels based on the comparison of the dispersion relation with the Euler equations is irrelevant for internal waves since the Euler equations already suffer from shear instability. The levels that we are interested in are those that guarantee stability in the presence of a background shear between the layers. For the rigid-lid case, it has been shown [30] that the best possible scenario is attained when the vertical levels at the top and bottom boundaries are chosen although a similar result is not known yet for the case of the top free surface [37]. Here it is shown that this finding for the one-dimensional case is still valid for the two-dimensional case and the new system of equations is stable to local transverse perturbations near the crest, or trough, depending on the wave polarity.

The model derivation puts in evidence that the physical assumptions behind this new regularized model differ slightly from the original strongly nonlinear model. The regularized model is more restrictive as we rely on (2.10) for a relationship between the two velocities, and is unable to describe of the original three-dimensional flow with vertical vorticity of $O(1)$. It should also be pointed out that the regularized model could be less suited for analytical studies and, in fact, the lack of conservation laws prevents one from finding exact solitary-wave solutions in the one-dimensional case [30]. Nevertheless, to study numerically the time evolution of large amplitude internal solitary waves, the new regularized model could be of great value. For example, it could be used to study the oblique interaction of internal solitary waves beyond the weakly nonlinear or weakly two-dimensional assumption, and investigate the finite amplitude effects on the complex patterns predicted by the weakly nonlinear KP equation.

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Appendix. On the magnitude of vertical vorticity

When the incompressible Euler equations (2.1)–(2.3) are non-dimensionalized with the characteristic length and time scales, they read

$$\nabla_3 \cdot (\boldsymbol{u}_i, w_i) = 0,$$
$$\rho_i \mathcal{D}_i \boldsymbol{U}_i + \nabla_3 p_i = -\nabla_3 \left(\rho_i F^{-2} z\right),$$

where $\nabla_3 = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$, $\mathcal{D}_i$ are the material derivatives defined by $\mathcal{D}_i = \partial/\partial t + (\boldsymbol{u}_i, w_i) \cdot \nabla_3$, F stands for the Froude number, and $\boldsymbol{U}_i$ are the vector fields defined by $\boldsymbol{U}_i = (\boldsymbol{u}_i, \epsilon^2 w_i)$ with $\boldsymbol{u}_i = (u_i, v_i)$. Define the vorticity vector field $\boldsymbol{\Omega}_i = (\epsilon^2 w_{i,y} - v_{i,z}, u_{i,z} - \epsilon^2 w_{i,x}, v_{i,x} - u_{i,y})$ as the curl of the vector field $\boldsymbol{U}_i$. Then, the dynamics of $\boldsymbol{\Omega}_i$ is governed by

$$\mathcal{D}_i \boldsymbol{\Omega}_i = (\boldsymbol{\Omega}_i \cdot \nabla_3)(\boldsymbol{u}_i, w_i). \tag{A.1}$$

Consider for the velocity field $(\boldsymbol{u}_i, w_i)$ the associated particle-trajectory mapping $\boldsymbol{\varphi}_i(\boldsymbol{X}_i, t)$ with initial condition $\boldsymbol{\varphi}_i(\boldsymbol{X}_i, t_0) = \boldsymbol{X}_i \equiv (X_i, Y_i, Z_i)$. Then, $\boldsymbol{\Omega}_i$ satisfy equation (A.1) if and only if (see e.g. Lemma 1.4 in [47])

$$\boldsymbol{\Omega}_i(\boldsymbol{\varphi}_i(\boldsymbol{X}_i, t), t) = \frac{\partial}{\partial \boldsymbol{X}_i} \boldsymbol{\varphi}_i(\boldsymbol{X}_i, t)\ \boldsymbol{\Omega}_i(\boldsymbol{X}_i, t_0), \tag{A.2}$$

where

$$\boldsymbol{\Omega}_i = \begin{pmatrix} \epsilon^2 w_{i,y} - v_{i,z} \\ u_{i,z} - \epsilon^2 w_{i,x} \\ v_{i,x} - u_{i,y} \end{pmatrix}, \qquad \frac{\partial \boldsymbol{\varphi}_i}{\partial \boldsymbol{X}_i} = \begin{pmatrix} \frac{\partial x}{\partial X_i} & \frac{\partial x}{\partial Y_i} & \frac{\partial x}{\partial Z_i} \\ \frac{\partial y}{\partial X_i} & \frac{\partial y}{\partial Y_i} & \frac{\partial y}{\partial Z_i} \\ \frac{\partial z}{\partial X_i} & \frac{\partial z}{\partial Y_i} & \frac{\partial z}{\partial Z_i} \end{pmatrix}. \tag{A.3}$$

Suppose the vertical vorticity is of $O(1)$ while the horizontal vorticity is weak to be of $O(\epsilon^4)$. Then, from (A.2)–(A.3), we would have

$$\frac{\partial x}{\partial Z_i} = O(\epsilon^4), \qquad \frac{\partial y}{\partial Z_i} = O(\epsilon^4), \qquad \frac{\partial z}{\partial Z_i} = O(1), \quad \forall t. \tag{A.4}$$

By taking the time derivative of (A.4), we have

$$\frac{\partial}{\partial t} \frac{\partial x}{\partial Z_i} = O(\epsilon^4), \quad \forall t,$$

which, by definition, yields

$$\frac{\partial}{\partial Z_i} u_i(\boldsymbol{\varphi}_i(\boldsymbol{X}_i, t), t) = O(\epsilon^4), \quad \forall t. \tag{A.5}$$

Using the chain rule, (A.5) can be written as

$$u_{i,x} \frac{\partial x}{\partial Z_i} + u_{i,y} \frac{\partial y}{\partial Z_i} + u_{i,z} \frac{\partial z}{\partial Z_i} = O(\epsilon^4), \quad \forall t.$$

In order for this to hold, we conclude from (A.4) that $u_{i,z} = O(\epsilon^4)$ for all time. Similarly, we can prove that $v_{i,z} = O(\epsilon^4)$. This, however, is not compatible with the imposed horizontal vorticity condition:

$$u_{i,z} = \epsilon^2 w_{i,x} + O(\epsilon^4), \qquad v_{i,z} = \epsilon^2 w_{i,y} + O(\epsilon^4), \quad \forall t.$$

The contradiction results from the assumption that the vertical vorticity is of $O(1)$. Therefore, we conclude that the new regularized model given by (2.13)–(2.14) has a weaker vorticity condition than its asymptotically equivalent model (2.5)–(2.6), and is valid when the vertical vorticity is of $O(\epsilon^2)$.

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