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On Rayleigh expansion for nonlinear long water waves *

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Abstract: We consider strongly nonlinear long waves on the surface of a homogeneous fluid layer. By modifying the formulation for the high-order spectral (HOS) method for waves in water of finite depth, we present a higher-order nonlinear system for the surface elevation and the velocity potential on the free surface to describe the two-dimensional evolution of large amplitude long waves. It is shown that the resulting system preserves the Hamiltonian structure of the Euler equations and can be transformed to the strongly nonlinear long-wave model for the depth-averaged velocity. Due to truncation of the linear dispersion relation for water waves, both the system for the surface velocity potential and that for the depth-averaged velocity are ill-posed when the order of approximation is odd and even, respectively. To avoid this ill-posedness, fully dispersive models are also proposed. Under the same order approximation, the long-wave model is found more effective for numeral studies of large amplitude long waves than the finite-depth model.

Key words: Long surface gravity waves, strongly nonlinear waves, Hamiltonian system, regularized model

Introduction

In recent years, broadband nonlinear surface waves in water of finite depth have been studied using the High-Order Spectral (HOS) method developed by West et al.[1], Dommermuth and Yue[2], Craig and Sulem[3], and others. In particular, the formulation of West et al.[1] has been shown effective for numerical computations and its numerical solutions have been found to compare well with laboratory experiments, for example, in Refs. [4-5] for non-breaking broadband waves and in Refs. [6-7] for breaking waves with an eddy viscosity model to represent energy dissipation due to wave breaking.

In the formulation of West et al.[1], the system of nonlinear evolution equations describes the dynamics of the surface elevation and the velocity potential evaluated at the free surface, which are the canonical variables of the Hamiltonian formulation for the original surface wave problem, as shown by Zakharov[8]. Although no small parameter is explicitly introduced, the right-hand sides of the time evolution equations are written in infinite series, which can be understood as asymptotic series under the assumption of small wave steepness: $\varepsilon \ll 1$, where $\varepsilon = a/\lambda$ with a and λ being the characteristic wave amplitude and wavelength, respectively. For numerical computations, one should truncate the series to a desired order of ε.

While the HOS formulation truncated at a relatively low order successfully describes the evolution of weakly nonlinear waves in water of finite depth, the order of approximation needs to be increased as the water depth decreases, which is computationally challenging. Therefore, it would be useful to find a model that could converge faster to the solutions of the Euler equations in the shallow water regime. Such efforts have been made extensively under the long-wave assumption.

There are two important parameters relevant to long surface waves: $\alpha = a/h$ and $\beta = h/\lambda$, where h is the water depth. When $\alpha = O(\beta^2) \ll 1$, one can obtain the well-known weakly nonlinear models, including the Kortwweg-de Vries (KdV) equation and the Boussinesq system. See, e.g., an earlier review by Miles[9].

In shallow water, the wave amplitude can be as large as the water depth so that α is no longer small. While the long wave approximation is valid $(\beta \ll 1)$, the parameter for nonlinearity α can be $O(1)$. This strongly nonlinear scenario has been investigated by Su and Gardner (SG)[10] and Green and Naghdi

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(GN)[11] for one and two-dimensional waves, respectively. Their strongly nonlinear models for the depth-averaged velocity are the shallow water equations combined with the leading-order (linear and nonlinear) dispersive terms that appear at $O(\beta^2)$.

It is interesting to notice that Rayleigh[12] obtained in 1876 the steady solitary-wave solution under the same assumption of $\alpha = O(1)$ and $\beta \ll 1$. In fact, the ordinary differential equation for solitary waves that Rayleigh[12] used is exactly the steady form of the SG or GN model. Almost 80 years later, the same steady formulation was also proposed independently by Serre[13]. Similarly to Stokes expansion for periodic waves, the expansion for strongly nonlinear long waves introduced by Rayleigh[12] will be referred to as Rayleigh expansion in this paper.

Validity of the Rayleigh expansion for large amplitude waves of $\alpha = a/h = O(1)$ needs to be tested in comparison with the Euler equations. Asymptotically, as long as the wave profile remains smooth in the sense that no short-wavelength disturbances are excited, the SG or GN model is expected to well describe the evolution of long waves of large amplitudes. For solitary wave collisions, this has been confirmed in Li et al.[14] by comparing the numerical solutions of the SG model with those of the Euler equations. As expected, a good agreement was observed for small to moderate wave amplitudes, but a discrepancy is found as the amplitude approaches its maximum value $(\alpha_{\max}/h_1 \approx 0.83)$.

Instead of the depth-averaged velocity, following Nwogu[15], nonlinear long-wave models are often written for the horizontal velocity at a vertical level for a better dispersive behavior. Wei et al.[16] developed a strongly nonlinear long-wave model and, for solitary waves propagating over bottom topography, showed that its numerical solutions are in good agreement with the Euler solutions before the slope becomes too high.

As the aforementioned strongly nonlinear long-wave models typically neglects any terms of $O(\beta^4)$ or higher, they need to be improved to describe evolving steep surface waves for which the long wave approximation could be inaccurate. A systematic higher-order approximation has been made by Agnon et al.[17], Madsen et al.[18], Wu[19-20], and many others. In particular, Wu[20] presented a few different systems for different velocities, including the depth-averaged velocity, the bottom velocity, the free surface velocity, and the velocity at an intermediate depth.

Here the formulation of West et al.[1] is modified for shallow water to obtain a higher-order nonlinear system for long waves. To preserve the Hamiltonian structure of the original water wave problem, two canonical surface variables (the surface elevation and the velocity potential evaluated at the free surface) are adopted for dependent variables. The resulting system is then studied in comparison with the finite-depth model for the HOS formulation and the depth-averaged velocity model.

1. Basic formulation

For inviscid water waves, the free surface boundary conditions can be written in terms of the surface variables[8] as

$$\zeta_t + \nabla\Phi\cdot\nabla\zeta = (1+|\nabla\zeta|^2)W \tag{1}$$

$$\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + g\zeta = \frac{1}{2}(1+|\nabla\zeta|^2)W^2 \tag{2}$$

where $\zeta(\boldsymbol{x},t)$ is the surface elevation, $\Phi(\boldsymbol{x},t)\equiv\phi(\boldsymbol{x}, z=\zeta,t)$ is the velocity potential evaluated at the free surface, W is the vertical velocity evaluated at the free surface defined by $W\equiv\partial\phi/\partial z|_{z=\zeta}$ and g is the gravitational acceleration. Here the subscript t represents partial differentiation with respect to time t and ∇ is the two-dimensional horizontal gradient given by $\nabla=(\partial/\partial x,\partial/\partial y)$. These equations can be regarded as a system of nonlinear evolution equations for ζ and Φ once W is expressed in terms of these two variables. Depending upon how to close this system, one can obtain various theoretical models, which in general need be solved numerically unless an additional approximation is made.

To close the system given by (1)-(2), the velocity potential is first expanded in Taylor series about a fixed vertical level at $z=\tilde{z}$:

$$\phi(\boldsymbol{x},z,t) = \sum_{j=0}^{\infty}\frac{(-1)^j(z-\tilde{z})^{2j}}{(2j)!}\nabla^{2j}\phi\Big|_{z=\tilde{z}} + \sum_{j=0}^{\infty}\frac{(-1)^j(z-\tilde{z})^{2j+1}}{(2j+1)!}\nabla^{2j+1}\cdot\nabla^{-1}\frac{\partial\phi}{\partial z}\Big|_{z=\tilde{z}} \tag{3}$$

By introducing

$$\tilde{\phi}\equiv\phi\big|_{z=\tilde{z}},\quad \tilde{w}\equiv\frac{\partial\phi}{\partial z}\Big|_{z=\tilde{z}} \tag{4}$$

$\phi(\boldsymbol{x},z,t)$ and $w(\boldsymbol{x},z,t)=\partial\phi/\partial z$ can be written[18] as:

$$\phi = \cos[(z-\tilde{z})\nabla]\tilde{\phi} + \sin[(z-\tilde{z})\nabla]\cdot\nabla^{-1}\tilde{w} \tag{5}$$

$$w = -\sin[(z-\tilde{z})\nabla]\cdot\nabla\tilde{\phi} + \cos[(z-\tilde{z})\nabla]\tilde{w} \tag{6}$$

For the HOS formulation, by evaluating ϕ and w at $z=\zeta$ in (5)-(6), the expressions of the free surface variables $\Phi(\boldsymbol{x},t)$ and $W(\boldsymbol{x},t)$ can be found, in terms of $\tilde{\phi}$ and $\tilde{w}$, as:

$$\Phi = \cos[(\zeta-\tilde{z})\nabla]\tilde{\phi} + \sin[(\zeta-\tilde{z})\nabla]\cdot\nabla^{-1}\tilde{w} \tag{7}$$

$$W = -\sin[(\zeta-\tilde{z})\nabla]\cdot\nabla\tilde{\phi} + \cos[(\zeta-\tilde{z})\nabla]\tilde{w} \tag{8}$$

1.1 *Finite-depth model for HOS*

For surface gravity waves on water of finite depth, following West et al.[1], one can choose the mean surface level for $\tilde{z}$, i.e., $\tilde{z}=0$. Then, the free surface variables Φ and W are given, from (7)-(8), by

$$\Phi = \cos(\zeta\nabla)\phi_0 + \sin(\zeta\nabla)\cdot\nabla^{-1}w_0 \tag{9}$$

$$W = \cos(\zeta\nabla)w_0 - \sin(\zeta\nabla)\cdot\nabla\phi_0 \tag{10}$$

where $\phi_0 = \phi|_{z=0}$ and $w_0 = w|_{z=0}$. If one can write w_0 in terms of ϕ_0, Φ can be expressed in terms of ϕ_0 and ζ from (9). With substituting $\tilde{z}=0$ and $z=-h$ and imposing the bottom boundary condition ($w=0$ at $z=-h$), equation (6) yields such a relationship between w_0 and ϕ_0:

$$w_0 = -\tan(h\nabla)\cdot\nabla\phi_0 \equiv -\mathcal{L}[\phi_0] \tag{11}$$

where $\sec(h\nabla)\sin(h\nabla) = \tan(h\nabla)$ has been used for constant h. Here the linear operator $\mathcal{L}$ is the Dirichlet-to-Neumann (DtN) operator for the lower half-plane given by

$$\mathcal{L} = \tan(h\nabla)\cdot\nabla \tag{12}$$

Notice that $\mathcal{L}$ is defined as an infinite series in physical space that requires truncation for its evaluation, but can be evaluated exactly in spectral space as

$$\mathcal{F}(\mathcal{L}[f]) = -k\tanh kh\,\mathcal{F}[f] \tag{13}$$

where $\mathcal{F}$ denotes the Fourier transform, and $k = |\boldsymbol{k}|$ with $\boldsymbol{k}$ being the two-dimensional wavenumber vector. In (13), we have used $\tan(\mathrm{i}kh) = \mathrm{i}\tanh(kh)$.

When (11) is substituted, (9) becomes

$$\Phi = [\cos(\zeta\nabla) - \sin(\zeta\nabla)\cdot\nabla^{-1}\mathcal{L}]\phi_0 \tag{14}$$

This equation needs to be inverted to express ϕ_0 in terms of ζ and Φ. While the inversion cannot be made explicitly, the expression of ϕ_0 can be obtained as an infinite series. Although it is not required to formally introduce a small parameter, it can be expected that the resulting expression is simply an asymptotic series under the assumption of small wave steepness, or, equivalently, $|\zeta\nabla\phi_0| = O(\epsilon\phi_0)$. Then, by substituting the resulting series into (10), one can obtain an expression for W in terms of ζ and Φ. Finally, by substituting the expression of W into (1)-(2), one can obtain a system of nonlinear evolution equations for ζ and Φ. After truncating the series to a desired order, the system can be solved numerically using a pseudo-spectral method. For example, when truncated at $O(\epsilon^3)$, the system is given[21] by:

$$\zeta_t = -\mathcal{L}[\Phi] - \nabla\cdot(\zeta\nabla\Phi) - \mathcal{L}(\zeta\mathcal{L}[\Phi]) - \mathcal{L}\left[\zeta\mathcal{L}(\zeta\mathcal{L}[\Phi]) + \frac{1}{2}\zeta^2\nabla^2\Phi\right] - \nabla^2\left(\frac{1}{2}\zeta^2\mathcal{L}[\Phi]\right) \tag{15}$$

$$\Phi_t = -g\zeta - \frac{1}{2}\nabla\Phi\cdot\nabla\Phi + \frac{1}{2}(\mathcal{L}[\Phi])^2 - \mathcal{L}\left[\zeta\mathcal{L}(\zeta\mathcal{L}[\Phi]) + \frac{1}{2}\zeta^2\nabla^2\Phi\right] \tag{16}$$

The explicit higher-order systems and their Hamiltonian structures can be found in Choi[22] in both physical and spectral spaces.

This approach has been referred to as the HOS method, which was first developed independently by West et al.[1] and Dommermuth and Yue[2]. A similar formulation has been presented by Craig and Sulem[3] by expanding the nonlinear DtN operator. While the HOS method can be described in various ways, the formulation consistent with the description here can be found, for example, in Choi[21] and Choi et al.[4]. A comparison of numerical solutions with different orders of approximation with laboratory experiments is presented in Goullet and Choi[5].

1.2 *Generalization of HOS to shallow water*

For long waves in shallow water, instead of $\tilde{z}=0$, it is more convenient to choose the bottom level for $\tilde{z}$, i.e., $\tilde{z}=-h$ to have, from (5)-(6),

$$\phi(\boldsymbol{x},z,t)=\cos[(z+h)\nabla]\phi_b \tag{17}$$

$$w(\boldsymbol{x},z,t)=-\sin[(z+h)\nabla]\cdot\nabla\phi_b \tag{18}$$

where the bottom boundary condition $(\tilde{w}=0)$ has been imposed. Likewise, from (7)-(8) with $\tilde{z}=-h$, Φ and W can be simplified to

$$\Phi=\cos[(\zeta+h)\nabla]\phi_b \tag{19}$$

$$W=-\sin[(\zeta+h)\nabla]\cdot\nabla\phi_b \tag{20}$$

where $\phi_b\equiv\phi|_{z=-h}$. Similarly to the finite-depth case, (19) needs to be inverted to express ϕ_b in terms of ζ and Φ. Contrary to (14), the resulting expression of ϕ_b is expected to be an asymptotic series for long waves $(\beta\ll 1)$, but no assumption on α needs to imposed so that $\alpha=O(1)$ for finite amplitude waves. Therefore, this expansion should be appropriate for finite amplitude long waves although $\epsilon=\alpha\beta$ is still small. Notice that this is exactly the same assumption that Rayleigh[12] adopted to find his solitary wave solution.

In this paper, using (17)-(20), we will obtain a higher-order nonlinear long-wave model for ζ and Φ, which can be considered a generalization of the HOS model to shallow water. Before we consider the nonlinear case in Section 3, we first examine the linear dispersion relation of various models mentioned previously.

2. Well-posedness of linear systems

When linearized for small amplitude waves, (1)-(2) can be reduced to

$$\zeta_t=W,\ \Phi_t=-g\zeta \tag{21}$$

Similarily, (19)-(20) can be linearized, with $a/h\ll 1$, to

$$\Phi=\cos(h\nabla)\phi_b,\ W=-\sin(h\nabla)\cdot\nabla\phi_b \tag{22}$$

which can be combined to find the linear relationship between W and Φ as

$$W=-\mathcal{L}[\Phi] \tag{23}$$

where $\mathcal{L}=\tan(h\nabla)\cdot\nabla$, as defined in (11). Then, the linear system given by (21) can be written as

$$\zeta_t=-\mathcal{L}[\Phi],\ \Phi_t=-g\zeta \tag{24}$$

which can be combined into $\zeta_{tt}=g\mathcal{L}[\zeta]$. The same equation also holds for Φ. For $(\zeta,\Phi)\sim\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)]$, using (13), equation (24) gives the full linear dispersion relation:

$$\omega^2=gk\tanh kh. \tag{25}$$

Therefore, without any further approximation, the finite-depth model (24) yields the exact linear dispersion relation for water waves.

For long waves, as $h\nabla=O(\beta)\ll 1$, the operator $\mathcal{L}$ in (24) can be expanded as

$$\mathcal{L}=h\sum_{n=0}^{\infty}\frac{4^{n+1}(4^{n+1}-1)B_{2(n+1)}(-h^2\nabla^2)^n}{[2(n+1)]!}\nabla^2 \tag{26}$$

where B_{2n} is the Bernoulli numbers given by $B_0=1$, $B_2=1/6$, $B_4=-1/30$, $B_6=1/42$, $\cdots$. When the $\mathcal{L}$-operator is truncated at $O(\beta^{2M})$, the linear system given by (24) becomes

$$\zeta_t=-\mathcal{L}_M[\Phi],\ \Phi_t=-g\zeta \tag{27}$$

where $\mathcal{L}_M$ is given by (26) with changing the upper limit of the summation to M. Then the linear dispersion relation for the truncated system (27), or $\zeta_{tt}=g\mathcal{L}_M[\zeta]$, is given, with $kh=O(\beta)$, by

$$\omega^2\simeq ghk^2\left[1-\frac{1}{3}(kh)^2+\frac{2}{15}(kh)^4+\cdots+\frac{4^{M+1}(4^{M+1}-1)B_{2(M+1)}}{(2M+2)!}(kh)^{2M}\right] \tag{28}$$

Now one can see that, when the operator $\mathcal{L}$ is truncated with an odd integer for M, the system given by (27) becomes ill-posed in the sense that short waves of large k are unstable with unbounded growth rates as the right-hand side of (28) becomes negative for large k with $B_{2(M+1)}<0$ for odd M. On the other hand, it is well-posed for even M. Therefore, the long-wave model for the surface variables (ζ and Φ) given by (27) should be truncated at $O(\beta^{2M})$ with even M for numerical studies.

While the long-wave model for ζ and Φ truncated at $O(\beta^2)$ should be avoided, most well-known nonlinear long-wave models including the Boussinesq equations contain dispersive terms of $O(\beta^2)$ as they are written in terms of different variables: the surface elevation and the depth-averaged velocity.

From its definition given by

$$\bar{\boldsymbol{u}} = \frac{1}{(h+\zeta)}\int_{-h}^{\zeta} \nabla\phi \mathrm{d}z \tag{29}$$

the depth-averaged velocity $\bar{\boldsymbol{u}}$ can be written, using (17), as

$$\bar{\boldsymbol{u}} = \frac{1}{(h+\zeta)}\sin[(h+\zeta)\nabla]\phi_b \tag{30}$$

Once again, for small amplitude waves, $\bar{\boldsymbol{u}}$ can be approximated, after replacing $h+\zeta$ in (30) by h, to

$$\bar{\boldsymbol{u}} \simeq \frac{1}{h}\sin(h\nabla)\phi_b = \frac{1}{h}\tan(h\nabla)\Phi \tag{31}$$

from which $\nabla\Phi$ can be approximated, in terms of $\bar{\boldsymbol{u}}$, by

$$\nabla\Phi = \mathcal{J}[\bar{\boldsymbol{u}}] \tag{32}$$

where the linear operator $\mathcal{J}$ is defined as

$$\mathcal{J} = \sum_{n=0}^{\infty}\frac{4^n B_{2n}(-h^2\nabla^2)^n}{(2n)!} \tag{33}$$

Then, the linear system for ζ and $\bar{\boldsymbol{u}}$ can be written as

$$\zeta_t = -h\nabla\cdot\bar{\boldsymbol{u}}\,,\quad \mathcal{J}[\bar{\boldsymbol{u}}_i] = -g\nabla\zeta \tag{34}$$

which can be combined into $\mathcal{J}[\zeta_{tt}] = gh\nabla^2\zeta$. As the operator J becomes in Fourier space a Fourier multiplier given by

$$\mathcal{F}(J[f]) = kh\coth(kh)\mathcal{F}[f] \tag{35}$$

the linear system given by (34) yields the full linear dispersion relation: $\omega^2 = gk\tanh kh$. For long wave, when truncated at $O(\beta^{2M})$, the linear system (34) gives the following dispersion relation

$$\left[1+\frac{1}{3}(kh)^2 - \frac{1}{45}(kh)^4 + \cdots + \frac{4^M B_{2M}}{(2M)!}(kh)^{2M}\right]\omega^2 = ghk^2 \tag{36}$$

Contrary to the system for ζ and Φ, the depth-averaged system is well-posed for odd M, but ill-posed for even M. This implies that the depth-averaged system given by (34) should be truncated with odd M unless the dispersive terms are modeled differently. Therefore, a higher-order depth-averaged model should be used with care.

3. Rayleigh expansion for surface variables

3.1 *Closure*

From (19)-(20), Φ and W can be written, in terms of the velocity potential at the bottom, ϕ_b, as

$$\Phi = \cos(\eta\nabla)\phi_b\,,\quad W = -\sin(\eta\nabla)\cdot\nabla\phi_b \tag{37}$$

where $\eta = h+\zeta$. To find an expression for ϕ_b in terms of ζ and Φ, the first equation in (37) is inverted to

$$\phi_b = \sum_{m=1}^{\infty}\Phi_{2m-1}\,,\quad \Phi_1 = \Phi \tag{38}$$

$$\Phi_{2m-1} = -\sum_{j=1}^{m-1}A_{2j}[\Phi_{2(m-j)-1}] \quad \text{for} \quad m\geq 2 \tag{39}$$

where we have used (A4), (A4b) with

$$A_{2m} = (-1)^m\frac{\eta^{2m}}{(2m)!}\nabla^{2^m}\,,\quad A_{2m+1} = 0 \tag{40}$$

Then, by substituting this into the second equation in (37), the expression of W can be found as

$$W = \sum_{m=1}^{\infty}W_{2m} \tag{41}$$

$$W_{2m} = \sum_{j=0}^{m-1}C_{2j+1}[\Phi_{2(m-j)-1}] \quad \text{for} \quad m\geq 1 \tag{42}$$

where we have used (A6)-(A7) with

$$\mathcal{C}_{2m} = 0\,,\ \mathcal{C}_{2m+1} = (-1)^{m+1}\frac{\eta^{2m+1}}{(2m+1)!}\nabla^{2(m+1)} \tag{43}$$

With assuming that $\eta\nabla = O(\beta)$ and $\zeta / h = O(1)$, one can notice that $\mathcal{A}_{2m}[\Phi]/\Phi = O(\beta^{2m})$ and $\mathcal{C}_{2m+1}[\Phi]/|\nabla\Phi| = O(\beta^{2m+1})$ Therefore, $\Phi_{2m-1}/\Phi = O[\beta^{2(m-1)}]$ and $W_{2m}/|\nabla\Phi| = O(\beta^{2m-1})$ for $m \geq 1$. Although no small parameter is explicitly introduced, this formal expansion produces an asymptotic series in β^2 as Rayleigh[12] suggested.

3.2 *Nonlinear evolution equations for surface variables*

By substituting (41) into (1)-(2), the nonlinear evolution equations for ζ (or $\eta = h + \zeta$) and Φ can be found as

$$\eta_t = \sum_{m=1}^{\infty} Q_{2m}(\eta,\Phi)\,,\ \Phi_t = \sum_{m=1}^{\infty} R_{2m}(\eta,\Phi) \tag{44}$$

where Q_n and R_n are given by

$$Q_2 = W_2 - \nabla\Phi\cdot\nabla\eta \tag{45a}$$

$$Q_{2m} = W_{2m} + |\nabla\eta|^2 W_{2(m-1)} \quad \text{for } m \geq 2 \tag{45b}$$

$$R_2 = -g\eta - \frac{1}{2}|\nabla\Phi|^2\,,\ R_4 = \frac{1}{2}W_2^{\,2} \tag{46a}$$

$$R_{2m} = \frac{1}{2}\sum_{j=1}^{m-1} W_{2(m-j)}W_{2j} + \frac{1}{2}|\nabla\eta|^2\sum_{j=1}^{m-2} W_{2(m-j-1)}W_{2j}$$

$$\text{for } m \geq 3 \tag{46b}$$

Relative to the left-hand sides, one can see that Q_{2m} and $O(\mathcal{R}_{2m})$ are $O[\beta^{2(m-1)}]$ Therefore the leading-order terms in (44) (with $m = 1$) are $O(1)$.

3.3 *Hamiltonian structure*

Zakharov[8] showed that the total energy defined by

$$E = \frac{1}{2}\int\left(g\zeta^2 + \Phi\frac{\partial\zeta}{\partial t}\right)\mathrm{d}\boldsymbol{x} \tag{47}$$

is the Hamiltonian so that the evolution equations for ζ and Φ can be written as Hamilton's equations:

$$\frac{\partial\zeta}{\partial t} = \frac{\delta E}{\delta\Phi}\,,\ \frac{\partial\Phi}{\partial t} = -\frac{\delta E}{\delta\zeta} \tag{48}$$

where $\delta E/\delta\zeta$ and $\delta E/\delta\Phi$ represent the functional derivatives of E with respect to the two canonical (or conjugate) variables ζ and Φ, respectively. Therefore, the system (44) conserves the total energy E.

After substituting (44) for $\partial\zeta/\partial t$ into (47), the total energy given by

$$E = \frac{1}{2}\int\left[g\zeta^2 + \Phi\sum_{m=1}^{\infty} Q_{2m}(\zeta,\Phi)\right]\mathrm{d}\boldsymbol{x} \tag{49}$$

can be expanded as

$$E = \sum_{m=0}^{\infty} E_{2m} \tag{50}$$

where $E_{2m} = O(\beta^{2m})$ for $m \geq 0$ are given by

$$E_0 = \int(g\zeta^2 + \Phi\mathcal{Q}_0[\Phi])\mathrm{d}\boldsymbol{x} \tag{51}$$

$$E_{2m} = \int\Phi\mathcal{Q}_{2m}[\Phi]\mathrm{d}\boldsymbol{x}\,,\ m \geq 1 \tag{52}$$

3.4 *Truncated systems*

From

$$\mathcal{A}_2 = -\frac{\eta^2}{2!}\nabla^2\,,\ \mathcal{A}_4 = \frac{\eta^4}{4!}\nabla^4 \tag{53}$$

$$\mathcal{C}_1 = -\eta\nabla^2\,,\ \mathcal{C}_3 = \frac{\eta^3}{3!}\nabla^4\,,\ \mathcal{C}_5 = -\frac{\eta^5}{5!}\nabla^6 \tag{54}$$

with $\nabla^{2n} = (\nabla^2)^n$, one can write explicitly Φ_j and W_j, correct to $O(\beta^4)$ and $O(\beta^5)$, respectively, as

$$\Phi_1 = \Phi\,,\ \Phi_3 = \frac{\eta^2}{2!}\nabla^2\Phi \tag{55}$$

$$\Phi_5 = \frac{\eta^2}{2!}\nabla^2\left(\frac{\eta^2}{2!}\nabla^2\Phi\right) - \frac{\eta^4}{4!}\nabla^4\Phi \tag{56}$$

$$W_2 = -\eta\nabla^2\Phi \tag{57}$$

$$W_4 = -\eta\nabla^2\left(\frac{\eta^2}{2!}\nabla^2\Phi\right) + \frac{\eta^3}{3!}\nabla^4\Phi \tag{58}$$

$$W_6 = -\eta\nabla^2\left[\frac{\eta^2}{2!}\nabla^2\left(\frac{\eta^2}{2!}\nabla^2\Phi\right) - \frac{\eta^4}{4!}\nabla^4\Phi\right] +$$

$$\frac{\eta^3}{3!}\nabla^4\left(\frac{\eta^2}{2!}\nabla^2\Phi\right) - \frac{\eta^5}{5!}\nabla^6\Phi \tag{59}$$

Then, by setting the upper limit to $M+1$ in (44) with $M=2$, the second-order model correct to $O(\beta^4)$ is given by

$$\eta_t + \nabla\cdot(\eta\nabla\Phi) + \nabla^2\left(\frac{1}{3}\eta^3\nabla^2\Phi\right) +$$

$$\nabla^2\left[\nabla\cdot\left(\frac{2}{15}\eta^5\nabla^2\nabla\Phi\right) +\right.$$

$$\left.\frac{1}{3}\eta^3\nabla\cdot(\eta\nabla\eta)(\nabla^2\Phi)\right] = 0 \tag{60}$$

$$\Phi_t + g\zeta + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi - \frac{1}{2}\eta^2(\nabla^2\Phi)^2 +$$

$$\eta^2\left[\nabla^2\left(\frac{1}{2}\eta^2\nabla^2\Phi\right) + \frac{1}{2}|\nabla\eta|^2\nabla^2\Phi -\right.$$

$$\left.\frac{1}{6}\eta^2\nabla^4\Phi\right]\nabla^2\Phi = 0 \tag{61}$$

which have errors of $O(\beta^6)$. Here the system correct to $O(\beta^{2M})$ is referred to as the M-th order system. When linearized, the second-order system (60)-(61) yields the linear dispersion relation given by (28) with $M=2$:

$$\omega^2 = ghk^2\left[1 - \frac{1}{3}(kh)^2 + \frac{2}{15}(kh)^4\right] > 0 \tag{62}$$

Therefore, the second-order system is well-posed and is suitable for numerical computations.

With the Hamiltonians are given, from (51)-(52), by

$$E_2 = \int \frac{1}{2}[g\zeta^2 + \eta\nabla\Phi\cdot\nabla\Phi]\mathrm{d}\boldsymbol{x} \tag{63}$$

$$E_4 = -\int \frac{1}{6}\eta^3(\nabla^2\Phi)^2\mathrm{d}\boldsymbol{x} \tag{64}$$

$$E_6 = \int\frac{1}{15}\eta^5\nabla^2(\nabla\Phi)\cdot\nabla^2(\nabla\Phi) -$$

$$\left.\frac{1}{6}\eta^3\nabla\cdot(\eta\nabla\eta)(\nabla^2\Phi)^2\right]\mathrm{d}\boldsymbol{x} \tag{65}$$

(60)-(61) can be also obtained from Hamilton's equations given by (48).

When any terms of $O(\beta^2)$ or higher are neglected from (60)-(61), the leading-order (or zeroth-order) system can be found as

$$\eta_t + \nabla\cdot(\eta\nabla\Phi) = 0 \tag{66}$$

$$\Phi_t + g\zeta + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi = 0 \tag{67}$$

which is equivalent to the shallow water equations.

On the other hand, the first-order system correct to $O(\beta^2)$ are given by

$$\eta_t + \nabla\cdot(\eta\nabla\Phi) + \nabla^2\left(\frac{1}{3}\eta^3\nabla^2\Phi\right) = 0 \tag{68}$$

$$\Phi_t + g\zeta + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi - \frac{1}{2}\eta^2(\nabla^2\Phi)^2 = 0 \tag{69}$$

This system is asymptotically equivalent to the GN system written in the depth-averaged velocity. Unfortunately, as shown previously, the first-order system given by (68)-(69) is ill-posed and, therefore, should not be used for numerical computations as it is.

For weakly nonlinear waves of $\alpha = O(\beta^2)$, the well-posed second-order model can be further simplified to

$$\zeta_t + \nabla\cdot[(h+\zeta)\nabla\Phi] + \frac{1}{3}h^3\nabla^4\Phi +$$

$$h^2\nabla^2(\zeta\nabla^2\Phi) + \frac{2}{15}h^5\nabla^6\Phi = 0 \tag{70}$$

$$\Phi_t + g\zeta + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi - \frac{1}{2}h^2(\nabla^2\Phi)^2 = 0 \tag{71}$$

where the errors relative to the leading-order terms are $O(\alpha^2\beta^2, \alpha\beta^4)$. It is worthwhile to mention that the weakly nonlinear system (70)-(71) can be also derived from the finite-depth model (15)-(16) under the second-order approximation valid to $O(\epsilon^2)$. See the

detailed derivation in Choi[21].

3.5 *Regularization*

As discussed in Section 2, the ill-posedness of a truncated system arises from an approximation to the original linear dispersion relation. Therefore, one can make the truncated system well-posed by changing its dispersion relation back to the full linear dispersion relation given by (25). As can be seen from (27), one possibility to achieve this is to modify the evolution equation for ζ in (44), under the M-th order approximation, to

$$\eta_t = \sum_{m=1}^{M+1} Q_{2m}(\eta, \Phi) + \mathcal{L}_M[\Phi] - \mathcal{L}[\Phi] \tag{72}$$

where the integral operator $\mathcal{L}$ is defined in (13) and the local operator $\mathcal{L}_M$ is given, from (26), by

$$\mathcal{L}_M = h\sum_{n=0}^{M} \frac{4^{n+1}(4^{n+1}-1)B_{2(n+1)}(-h^2\nabla^2)^n}{[2(n+1)]!}\nabla^2 \tag{73}$$

Compared with the original system, the evolution equation for η has two additional terms. The first additional term $\mathcal{L}_M[\Phi]$ eliminates the truncated linear dispersion relation that is contained in the summation of Q_{2m} while $\mathcal{L}[\Phi]$ provides the full linear dispersion relation. Then the system given by (72) along with the evolution equation for Φ becomes well-posed even for odd M. For example, from (72), the regularized first-order $(M=1)$ model can be obtained as

$$\eta_t + \nabla\cdot(\eta\nabla\Phi) + \nabla^2\left(\frac{1}{3}\eta^3\nabla^2\Phi\right) =$$

$$h\nabla^2\Phi + \frac{1}{3}h^3\nabla^2\nabla^2\Phi - \mathcal{L}[\Phi] \tag{74}$$

along with (69) for the evolution of Φ. Furthermore, the regularized model has the full dispersion relation for any M. A similar idea was proposed by Whitham[23] who improved the dispersion relation of the KdV equation by replacing the third-order spatial derivative term in the KdV equation by an integral operator that produces the full linear dispersion relation for uni-directional waves.

3.6 *Comparison with the finite-depth model*

As $\epsilon = \alpha\beta$, the assumption of $\beta \ll 1$ still implies $\epsilon \ll 1$. Therefore, similarly to the finite-depth model, the wave steepness defined by $\epsilon = a/\lambda$ has to remain small for the long-wave model to be valid. Nevertheless, one can think that the finite-depth and long-wave models are applicable to weakly and strongly nonlinear waves, respectively, in the following sense. First one should notice that the characteristic horizontal velocities for the two models are different: $U_1 = (g\lambda)^{1/2}$ and $U_2 = (gh)^{1/2}$ for the finite-depth and long-wave models, respectively. From the free surface boundary conditions, or, equivalently, from (1)-(2), the horizontal velocity $|\nabla\Phi| = O(\Phi/\lambda)$ can be scaled as $|\nabla\Phi|/U_1 = O(\epsilon) \ll 1$ for the finite-depth model, $|\nabla\Phi|/U_2 = O(\alpha) = O(1)$ for the long-wave model. Therefore, the long-wave model with $\alpha = O(1)$ can be considered a strongly nonlinear model.

In general, the long-wave model can be derived from the finite-depth model by taking the long-wave limit. It should be pointed out that, as $\epsilon = \alpha\beta = O(\beta)$ when $\alpha = O(1)$, one should use the $2(M+1)$-th order finite-depth model to obtain the M-th order long-wave model. More specifically, the second order long-wave model (68)-(69) can be obtained from the sixth-order finite-depth model valid to $O(\epsilon^6)$. Solving a higher-order finite-depth model that has a large number of terms is computationally expensive. Thus the lower-order long-wave model is more advantageous for numerical simulations of large amplitude long waves.

Numerically, the finite-depth model (15)-(16) needs to be solved using a pseudo-spectral method, where an integral operator $\mathcal{L}$ can be easily evaluated in Fourier space as a Fourier multiplier $(k\tanh kh)$, as shown in (13). When truncated at $O(\beta^{2M})$ with an odd integer for M, the system (44) can be solved with either a finite difference or a pseudo-spectral method. On the other hand, the regularized model (72) should be solved using a pseudo-spectral method for the evaluation of $\mathcal{L}$. As the order of approximation increases, the long-wave model introduces terms of higher-order spatial derivates and therefore a small time step might be required if an explicit time integration scheme is adopted.

4. System for depth-averaged variables

4.1 *Transformation*

The expression of $\bar{\boldsymbol{u}}$ in (30) can be inverted to

$$\nabla\phi_b = \sum_{m=1}^{\infty} \bar{\boldsymbol{u}}_{2m-1} \tag{75}$$

where $\bar{\boldsymbol{u}}_1 = \bar{\boldsymbol{u}}$ and

$$\bar{\boldsymbol{u}}_{2m-1} = -\sum_{j=1}^{m-1} D_{2j}[\bar{\boldsymbol{u}}_{2(m-j)-1}] \quad \text{for} \quad m \geq 2 \tag{76}$$

with $\mathcal{D}_j$ given by

$$D_{2j} = \frac{(-1)^j \eta^{2j}}{(2j+1)!}\nabla^{2j}, \quad D_{2j+1} = 0 \tag{77}$$

Also, from (37) and (75), W can be expanded, in terms of η and $\bar{\boldsymbol{u}}$, as

$$W = \sum_{m=1}^{\infty} \bar{W}_{2m} \tag{78}$$

$$\bar{W}_{2m} = \sum_{j=0}^{m-1} \boldsymbol{K}_{2j+1} \cdot [\bar{\boldsymbol{u}}_{2(m-j)-1}] \tag{79}$$

where the vector operators $\boldsymbol{K}_j$ are given by

$$\boldsymbol{K}_{2j} = \boldsymbol{0}, \quad \boldsymbol{K}_{2j+1} = (-1)^{j+1} \frac{\eta^{2j+1}}{(2j+1)!}\nabla^{2j}\nabla \tag{80}$$

In addition, from

$$\nabla \Phi = \nabla \phi|_{z=\zeta} + W \nabla \eta \tag{81}$$

along with $\nabla \phi|_{z=\zeta} = \cos(\eta \nabla)\nabla \phi_b$ from (17), $\boldsymbol{v} = \nabla \Phi$ can be written, in terms of $\bar{\boldsymbol{u}}$ and ζ, as

$$\boldsymbol{v} = \sum_{m=1}^{\infty} \boldsymbol{v}_{2m-1} \tag{82}$$

where $\boldsymbol{v}_1 = \bar{\boldsymbol{u}}$ and

$$\boldsymbol{v}_{2m-1} = \bar{W}_{2(m-1)}\nabla \eta + \sum_{j=0}^{m-1} A_{2j}[\bar{\boldsymbol{u}}_{2(m-j)-1}] \quad \text{for} \quad m \geq 2 \tag{83}$$

where $\bar{\boldsymbol{u}}_j$ and $\bar{W}_j$ are given by (76) and (79), respectively, and $\mathcal{A}_{2j}$ are given by (40). When the expressions for W and $\boldsymbol{v}$ given by (78) and (82) are substituted into (1)-(2), the system of nonlinear evolution equations for η and $\bar{\boldsymbol{u}}$ can be obtained.

Interestingly, at any order of approximation, the evolution equation for η is always given by

$$\eta_t + \nabla \cdot (\eta \bar{\boldsymbol{u}}) = 0 \tag{84}$$

This is not so surprising as (84) implies conservation of mass. It is in fact well-known that equation (84) is the exact evolution equation for η and can be obtained by integrating the continuity equation over a fluid column.

On the other hand, after taking the gradient of (2), the evolution equation for $\boldsymbol{v} = \nabla \Phi$ can be found as

$$\boldsymbol{v}_t = \nabla\left[\sum_{m=1}^{\infty} \boldsymbol{r}_{2m}(\eta, \bar{\boldsymbol{u}})\right] \tag{85}$$

where $\boldsymbol{r}_{2m}$ are given by

$$\boldsymbol{r}_2 = -(g\eta + \boldsymbol{v}_1 \cdot \boldsymbol{v}_1), \quad \boldsymbol{r}_4 = \frac{1}{2}\overline{W_2}^2 + \boldsymbol{v}_1 \cdot \boldsymbol{v}_3 \tag{86a}$$

$$\boldsymbol{r}_{2m} = \frac{1}{2}\sum_{j=1}^{m-1} \bar{W}_{2(m-j)}\bar{W}_{2j} + \frac{1}{2}|\nabla \eta|^2 \sum_{j=1}^{m-2} \bar{W}_{2(m-j-1)}\bar{W}_{2j} - \frac{1}{2}\sum_{j=1}^{m} \boldsymbol{v}_{2j-1} \cdot \boldsymbol{v}_{2(m-j)+1} \quad \text{for} \quad m \geq 3 \tag{86b}$$

Notice that (85) is written in conservative form and, once it is solved numerically for $\boldsymbol{v}$, one needs to find $\bar{\boldsymbol{u}}$ from (82) by inverting an semi-linear operator that depends on η.

Although the original three-dimensional velocity field is irrotational, the depth-averaged velocity field $\bar{\boldsymbol{u}}$ is no longer irrotational. From (83), the vorticity of the two-dimensional velocity field $\bar{\boldsymbol{u}}$ is given by

$$\nabla \times \bar{\boldsymbol{u}} = -\sum_{m=2}^{\infty} \nabla \times \boldsymbol{v}_{2m-1} \tag{87}$$

where $\nabla \times \boldsymbol{v} = 0$ has been used as $\boldsymbol{v} = \nabla \Phi$.

It was shown by Matsuno[24] that the system for $\bar{\boldsymbol{u}}$ also preserves the Hamiltonian structure of the original Euler equations for surface waves, and conserves the total energy defined by

$$E = \frac{1}{2}\int (g\zeta^2 + \Phi \zeta_t)\,\mathrm{d}\boldsymbol{x} = \frac{1}{2}\int (g\zeta^2 + \eta \boldsymbol{v} \cdot \bar{\boldsymbol{u}})\,\mathrm{d}\boldsymbol{x} \tag{88}$$

where $\boldsymbol{v}$ can be expressed in terms of η and $\bar{\boldsymbol{u}}$ using (82) with (76) and (79).

4.2 *Truncated systems*

When the right-hand side of (85) is evaluated with $m = M+1$ for the upper limit of the summation, the system is valid to $O(\beta^{2M})$. For example, to find the evolution equations valid to $O(\beta^2)$, one should

first find the expression for $\boldsymbol{v} = \nabla \Phi$ correct to $O(\beta^2)$ given, from (82), by

$$\boldsymbol{v} = \sum_{m=1}^{2} \boldsymbol{v}_{2m-1} + O(\beta^4) \tag{89}$$

where $\boldsymbol{v}_j$ are given by

$$\boldsymbol{v}_1 = \bar{\boldsymbol{u}}, \quad \boldsymbol{v}_3 = -\frac{1}{\eta}\nabla\left(\frac{\eta^3}{3}\nabla\cdot\bar{\boldsymbol{u}}\right) \tag{90}$$

Then the evolution equation for $\bar{\boldsymbol{u}}$ can be found, from (85) with $\bar{\boldsymbol{r}}_2 + \bar{\boldsymbol{r}}_4$ on the right-hand side, as

$$\left[\bar{\boldsymbol{u}} - \frac{1}{\eta}\nabla\left(\frac{\eta^3}{3}\nabla\cdot\bar{\boldsymbol{u}}\right)\right]_t = -\nabla(g\eta + \bar{\boldsymbol{u}}\cdot\bar{\boldsymbol{u}}) + \nabla\left[\frac{\eta^2}{2}(\nabla\cdot\bar{\boldsymbol{u}})^2 - \frac{\bar{\boldsymbol{u}}}{\eta}\cdot\nabla\left(\frac{\eta^3}{3}\nabla\cdot\bar{\boldsymbol{u}}\right)\right] \tag{91}$$

Using the following expression obtained from (87)

$$\nabla\times\bar{\boldsymbol{u}} = -\frac{1}{3}\eta\nabla\eta\times\nabla^2\bar{\boldsymbol{u}} + O(\beta^4) \tag{92}$$

Equation (91) can be rewritten as

$$\bar{\boldsymbol{u}}_t + \bar{\boldsymbol{u}}\cdot\nabla\bar{\boldsymbol{u}} + g\nabla\zeta = \frac{1}{\eta}\nabla\left\{\frac{\eta^3}{3}[\nabla\cdot\bar{\boldsymbol{u}}_t + \bar{\boldsymbol{u}}\cdot\nabla(\nabla\cdot\bar{\boldsymbol{u}}) - (\nabla\cdot\bar{\boldsymbol{u}})^2]\right\} \tag{93}$$

where we have used $\nabla(\nabla\cdot\bar{\boldsymbol{u}}) = \nabla^2\bar{\boldsymbol{u}} + O(\beta^2)$.

The first-order system given by (84) and (93) was previously derived by Su and Gardner[10] for one-dimensional waves. Later, using a different approach, Green and Naghdi[11] obtained the system for two-dimensional waves, which is often referred to as the GN system. It was shown[14] that the numerical solutions of the GN system agrees much better with those of the Euler equations than the weakly nonlinear Boussinesq equations.

It is interesting to notice that Rayleigh[12] first derived the steady form of the GN system for traveling waves and found the solitary wave solution propagating in the x-direction

$$\eta(X) = h + a\,\mathrm{sech}^2(kX), \quad \bar{u}(X) = c\left(1 - \frac{h_0}{\eta}\right) \tag{94}$$

where $X = x - ct$ and k and the wave speed c is are given by

$$k^2h^2 = \frac{3a}{4(h+a)}, \quad \frac{c^2}{gh} = 1 + \frac{a}{h} \tag{95}$$

respectively.

As discussed previously, the next-order approximation to (85) for $M = 2$ will result in an ill-posed system. Similarly to the system for (ζ, Φ), one can also regularize the system by subtracting the truncated linear dispersive terms and adding a full linear dispersive term to the evolution equation for $\bar{\boldsymbol{u}}$. Also, if one wants to include the full linear dispersion even for the well-posed case, the system can be modified in the same way. For example, the GN system with the full linear dispersion relation is given by (84) and

$$(\boldsymbol{v} - \mathcal{J}_M[\bar{\boldsymbol{u}}] + \mathcal{J}[\bar{\boldsymbol{u}}])_t = \nabla\left[\sum_{m=1}^{M+1}\boldsymbol{r}_{2m}(\eta, \bar{\boldsymbol{u}})\right] \tag{96}$$

where $\mathcal{J}_M$ is the local operator given by (26) with replacing by M the upper limit of the summation and $\mathcal{J}$ is the linear integral operator given by (35). For example, the fully-dispersive GN system is given by

$$\left(\mathcal{J}[\bar{\boldsymbol{u}}]\frac{h^2}{3}\nabla^2\bar{\boldsymbol{u}}\right)_t + \bar{\boldsymbol{u}}\cdot\nabla\bar{\boldsymbol{u}} + g\nabla\zeta = \frac{1}{\eta}\nabla\left\{\frac{\eta^3}{3}[\nabla\cdot\bar{\boldsymbol{u}}_t + \bar{\boldsymbol{u}}\cdot\nabla(\nabla\cdot\bar{\boldsymbol{u}}) - (\nabla\cdot\bar{\boldsymbol{u}})^2]\right\} \tag{97}$$

5. Conclusion

To describe the evolution of large amplitude long waves, we have presented a higher-order nonlinear system for the surface elevation and the free surface velocity potential by assuming $\beta = h/\lambda \ll 1$, but $\alpha = a/h = O(1)$. The system is a generalization of the finite depth water model for the HOS method to shallow water with preserving the original Hamiltonian structure. It has been shown that the system for the surface variables is ill-posed when it is truncated at $O(\beta^{2M})$ with odd M while the widely-used system for the depth-averaged velocity suffers from ill-posedness with even M. Nevertheless, both systems can be regularized by replacing linear dispersive terms by a term with full linear dispersion. In comparison with the finite-depth model, the long-wave model can be truncated at a lower order that has a less number of terms and is more beneficial

for numerical studies.

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References

[1] West B. J., Brueckner K. A., Janda R. S. et al. A new numerical method for surface hydrodynamics [J]. *Journal of Geophysical Research: Oceans*, 1987, 92(C11): 11803-11824.
[2] Dommermuth D. G., Yue D. K. P. A high-order spectral method for the study of nonlinear gravity waves [J]. *Journal of Fluid Mechanics*, 1987, 184: 267-288.
[3] Craig W., Sulem C. Numerical simulation of gravity waves [J]. *Journal of Computational Physics*, 1993, 108(1): 73-83.
[4] Choi W., Kent C. P., Schillinger C. J. Numerical modeling of nonlinear surface waves and its validation (Chwang A. T., Teng M. H., Valentine D. T. Advances in Engineering Mechanics–Reflections and Outlooks: In Honor of Theodore Y.-T. Wu) [M]. Singapore: World Scientific, 2005, 94-110.
[5] Goullet A., Choi W. A numerical and experimental study on the nonlinear evolution of long-crested irregular waves [J]. *Physics of Fluids*, 2011, 23(1): 016601.
[6] Tian Z., Perlin M., Choi W. Energy dissipation in two-dimensional unsteady plunging breakers and an eddy viscosity model [J]. *Journal of Fluid Mechanics*, 2010, 655: 217-257.
[7] Perlin M., Choi W., Tian Z. Breaking waves in deep and intermediate waters [J]. *Annual Review of Fluid Mechanics*, 2013, 45: 115-145.
[8] Zakharov V. E. Stability of periodic waves of finite amplitude on the surface of a deep fluid [J]. *Journal of Applied Mechanics and Technical Physics*, 1968, 9(2): 190-194.
[9] Miles J. W. Solitary waves [J]. *Annual Review of Fluid Mechanics*, 1980, 12(1): 11-43.
[10] Su C. H., Gardner C. S. Korteweg-de Vries equation and generalizations. III. Derivation of the Korteweg-de Vries equation and Burgers equation [J]. *Journal of Mathematical Physics*, 1969, 10(3): 536-539.
[11] Green A. E., Naghdi P. M. A derivation of equations for wave propagation in water of variable depth [J]. *Journal of Fluid Mechanics*, 1976, 78: 237-246.
[12] Rayleigh (Lord) J. W. S. On waves [J]. P*hilosophical Magazine, Series 5*, 1876, 1: 257-279.
[13] Serre F. Contribution a letude des ecoulements permanents et variables dans les canaux [J]. *LaHouille Blanche*, 1953, 6: 830-872.
[14] Li Y. A., Hyman J. M., Choi W. A numerical study of the exact evolution equations for surface waves in water of finite depth [J]. *Studies in Applied Mathematics*, 2004, 113(3): 303-324.
[15] Nwogu O. Alternative form of Boussinesq equations for nearshore wave propagation [J]. *Journal of Waterway, Port, Coastal, and Ocean Engineering*, 1993, 119(6): 618-638.
[16] Wei G., Kirby J. T., Grilli S. T. et al. A fully nonlinear Boussinesq model for surface waves. Part 1. Highly nonlinear unsteady waves [J]. *Journal of Fluid Mechanics*, 1995, 294: 71-92.
[17] Agnon Y., Madsen P. A., Schäffer H. A. A new approach to high-order Boussinesq models [J]. *Journal of Fluid Mechanics*, 1999, 399: 319-333.
[18] Madsen P. A., Bingham H. B., Liu H. A new Boussinesq method for fully nonlinear waves from shallow to deep water [J]. *Journal of Fluid Mechanics*, 2002, 462: 1-30.
[19] Wu T. Y. Modeling nonlinear dispersive water waves [J]. *Journal of Engineering Mechanics*, 1999, 125(7): 747-755.
[20] Wu T. Y. A unified theory for modeling water waves [J]. *Advances in Applied Mechanics*, 2001, 37: 1-88.
[21] Choi W. Nonlinear evolution equations for two-dimensional surface waves in a fluid of finite depth [J]. *Journal of Fluid Mechanics*, 1995, 295: 381-394.
[22] Choi W. Fifth-order nonlinear spectral model for surface gravity waves: From pseudo-spectral to spectral formulations [C]. *Proceedings of the RIMS workshop on nonlinear water waves in honor of Mitsuhiro Tanaka*, Kyoto, Japan: Kyoto University, 2019, 47-60.
[23] Whitham G. B. Linear and nonlinear waves [M]. New York, USA: Wiley, 1974.
[24] Matsuno Y. Hamiltonian formulation of the extended Green–Naghdi equations [J]. *Physica D: Nonlinear Phenomena*, 2015, 301-302: 1-7.

Appendix A: Inversion and composition of infinite series

Consider an infinite series

$$f(x)=\sum_{n=0}^{\infty}\mathcal{F}_n[g(x)] \tag{A1}$$

where operators $\mathcal{F}_n$ are given. Using the Cauchy product, its inversion can be found, in infinite series, as

$$g(x)=\sum_{n=0}^{\infty}\mathcal{G}_n[f(x)]=\sum_{n=1}^{\infty}g_n(x) \tag{A2}$$

where $\mathcal{F}_0\mathcal{G}_0=\mathcal{I}$ with $\mathcal{I}$ being the identity operator and

$$\mathcal{F}_0\mathcal{G}_n=-\sum_{j=0}^{n-1}\mathcal{F}_{n-j}\mathcal{G}_j \quad \text{for} \quad n\geq 1 \tag{A3}$$

and $g_n(x)=\mathcal{G}_{n-1}[f(x)]$ are given by

$$g_1(x)=\mathcal{F}_0^{-1}[f(x)] \tag{A4a}$$

$$g_n(x) = -\mathcal{F}_0^{-1} \sum_{j=1}^{n-1} \mathcal{F}_j [g_{n-j}(x)] \quad \text{for} \quad n \geq 2 \tag{A4b}$$

When $q(x)$ is defined as

$$q(x) = \sum_{n=0}^{\infty} \mathcal{H}_n [p(x)] \quad \text{with} \quad p(x) = \sum_{n=1}^{\infty} p_n(x) \tag{A5}$$

it can be expressed as

$$q(x) = \sum_{n=1}^{\infty} q_n(x) \tag{A6}$$

where $q_n(x)$ are given by

$$q_n(x) = \sum_{j=0}^{n-1} \mathcal{H}_j [p_{n-j}] \quad \text{for} \quad n \geq 1 \tag{A7}$$