Fifth-order nonlinear spectral model for surface gravity waves: From pseudo-spectral to spectral formulations

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Abstract

We present a fifth-order nonlinear spectral model describing the spectral evolution of nonlinear surface gravity waves in water of finite depth. Using the equivalence between pseudo-spectral and spectral formulations, it is shown that the spectral model can be easily obtained using a truncated Hamiltonian from the pseudo-spectral formulation. The fifth-order model is written explicitly in terms of two canonical variables (the Fourier transforms of the surface elevation and the free surface velocity potential) and preserves the Hamiltonian structure of the original water wave problem. Under discrete approximation, the time-periodic solutions of the spectral model for progressive and standing waves are shown to be consistent with the classical solutions of Stokes and Rayleigh, respectively, when truncated at the third order.

1 Introduction

For three-dimensional water waves, the free surface boundary conditions can be written (Zakharov 1967), in terms of the surface elevation $\zeta(\boldsymbol{x},t)$ and the free surface velocity potential $\Phi(\boldsymbol{x},t)$, as

$$\frac{\partial\zeta}{\partial t}+\nabla\Phi\cdot\nabla\zeta=\left(1+|\nabla\zeta|^2\right)W,\qquad \frac{\partial\Phi}{\partial t}+\frac{1}{2}|\nabla\Phi|^2+g\zeta=\frac{1}{2}\left(1+|\nabla\zeta|^2\right)W^2, \tag{1.1}$$

where $\boldsymbol{x}$ is the the horizontal coordinate, t is time, ∇ is the horizontal gradient, and g is the gravitational acceleration. In (1.1), $\Phi(\boldsymbol{x},t)\equiv\phi(\boldsymbol{x},z=\zeta,t)$ and $W\equiv\partial\phi/\partial z(\boldsymbol{x},z=\zeta,t)$ are the velocity potential and the vertical velocity evaluated at the free surface, respectively, where ϕ and z are the three-dimensional velocity potential and the vertical coordinate, respectively. The two equations in (1.1) can be regarded as a system of nonlinear evolution equations for ζ and Φ once W is expressed in terms of ζ and Φ. Depending upon how to close this system, various theoretical models have been developed.

A theoretical model particularly useful for numerical computations of the evolution of broadband nonlinear surface waves was proposed by West *et al.* (1987), who wrote W in an

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infinite series that depend on ζ and Φ. By substituting the infinite series into (1.1), a closed system of nonlinear evolution equations for ζ and Φ was obtained. After assuming the wave steepness is small, the series can be truncated at a desired order of nonlinearity and the resulting system has been studied numerically using a pseudo-spectral method by numerous researchers, including, for example, Tanaka (2001a, b), Bateman *et al.* (2001), Choi *et al* (2005), and Goullet & Choi (2011). Similar approaches have been proposed by Dommermuth & Yue (1987), Criag & Sulem (1993), and Clamond & Grue (2001).

An alternative approach to describe the evolution of broadband nonlinear waves was proposed by Zakharov (1968), who obtained a nonlinear integro-differential equation in spectral space for a single complex amplitude, which is a linear combination of the Fourier transforms of ζ and Φ. As a number of multiple integrals are required to be evaluated, the evolution equation of Zakharov is less efficient for numerical computations than the pseudo-spectral model of West *et al.* (1987). Nevertheless, his evolution equation is so useful for further analysis to describe the time evolution of wave spectra. For example, in his seminal work, Zakharov (1968) reduced the third-order equation to a relatively simpler form for resonant four-wave interactions. This equation is also often referred to as the (reduced) Zakharov equation, which has been studied numerically (Annenkov & Shrira 2001). The spectral models of Zakharov (1968) have been further extended to the fourth order by Stiassnie & Shemer (1984) to describe the five-wave interactions of gravity waves. Later the spectral models were reformulated by Krasitskii (1994) directly from a Hamiltonian approach along with canonical transformations to simplify the Hamiltonian. For the earlier development of of the spectral formulation, see, for example, Yuen & Lake (1982) and Mei *et al.* (2005).

As one can imagine, the formulation of Zakharov (1968) should be equivalent to that of West *et al.* (1987). Therefore, it is expected to be straightforward to recover one formulation from the other. This is particularly useful if one is interested in a spectral model valid at a high order as the pseudo-spectral model of West *et al.* (1987) can be found conveniently at any order of nonlinearity through recursion formulas. Here it is shown that a fifth-order spectral model can be indeed obtained in a straightforward manner from the pseudo-spectral model of West *et al.* (1987) by taking advantage of its Hamiltonian structure.

2 Pseudo-spectral formulation

2.1 Expansion

By expansing W in Taylor series about $z = 0$, it was shown by West *et al.* (1987) that the expression for W can be written in infinite series as

$$W = \sum_{n=1}^{\infty} W_n, \qquad W_n = \sum_{j=0}^{n-1} \mathcal{C}_j[\Phi_{n-j}] \quad \text{for } n \geq 1\,, \tag{2.1}$$

where Φ_n are given by

$$\Phi_1 = \Phi, \quad \Phi_n = \sum_{j=1}^{n-1} \mathcal{A}_j\left[\Phi_{n-j}\right] \quad \text{for } n \geq 2\,, \tag{2.2}$$

and operators $\mathcal{A}_n$ and $\mathcal{C}_n$ are defined, with $\triangle = \nabla^2$, by

$$\mathcal{A}_{2m} = (-1)^{m+1}\frac{\zeta^{2m}}{(2m)!}\triangle^m, \quad \mathcal{A}_{2m+1} = (-1)^m\frac{\zeta^{2m+1}}{(2m+1)!}\triangle^m\mathcal{L}, \tag{2.3}$$

$$\mathcal{C}_{2m} = (-1)^{m+1}\frac{\zeta^{2m}}{(2m)!}\triangle^m\mathcal{L}, \quad \mathcal{C}_{2m+1} = (-1)^{m+1}\frac{\zeta^{2m+1}}{(2m+1)!}\triangle^{m+1}. \tag{2.4}$$

The linear operator $\mathcal{L}[f]$ is given by $\mathcal{L}[f] = \mathcal{F}^{-1}\Big[-k\tanh(kh)\,\mathcal{F}[f]\Big]$, where h is the water depth, and $\mathcal{F}$ and $\mathcal{F}^{-1}$ represent the Fourier transform and its inverse, respectively. Alternatively, the linear operator $\mathcal{L}$ can be written as $\mathcal{L}[f] = \int K(\boldsymbol{x}-\boldsymbol{\xi})\,f(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}$, where the kernel $K(\boldsymbol{x})$ is defined in Fourier space as $\mathcal{F}[K(\boldsymbol{x})] = -k\tanh(kh)$.

Although the expansion for W given by (2.1) requires no formal introduction of a small parameter, except for the existence of Taylor series, the series given by (2.1)–(2.2) can be considered as an expansion in terms of (small) wave steepness, in particular, when the infinite series need to be truncated for numerical simulations or further approximations. From $\zeta\nabla = O(\epsilon)$ and $\zeta\mathcal{L} = O(\epsilon)$, where $\epsilon = a/\lambda$ with a and λ being the characteristic wave amplitude and wavelength, respectively, one can see that $\Phi_n = O(\epsilon^n)$ and $W_n = O(\epsilon^n)$. Therefore, the rate of convergence is expected to improve as ϵ decreases.

2.2 System of West *et al.* (1987)

By substituting into (1.1) the expansion for W given by (2.1), the evolution equations for ζ and Φ are given by

$$\frac{\partial\zeta}{\partial t} = \sum_{n=1}^{\infty}\mathrm{Q}_n(\zeta,\Phi)\,, \qquad \frac{\partial\Phi}{\partial t} = \sum_{n=1}^{\infty}\mathrm{R}_n(\zeta,\Phi)\,, \tag{2.5}$$

where Q_n and R_n are given by

$$\mathrm{Q}_1 = W_1\,, \quad \mathrm{Q}_2 = W_2 - \boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\zeta\,, \quad \mathrm{Q}_n = W_n + |\boldsymbol{\nabla}\zeta|^2 W_{n-2} \quad \text{for } n \geq 3\,, \tag{2.6}$$

$$\begin{aligned} &\mathrm{R}_1 = -g\,\zeta\,, \quad \mathrm{R}_2 = -\tfrac{1}{2}|\boldsymbol{\nabla}\Phi|^2 + \tfrac{1}{2}{W_1}^2\,, \quad \mathrm{R}_3 = W_1\,W_2\,, \\ &\mathrm{R}_n = \tfrac{1}{2}\sum_{j=0}^{n-2} W_{n-j-1}\,W_{j+1} + \tfrac{1}{2}|\boldsymbol{\nabla}\zeta|^2\sum_{j=0}^{n-4} W_{n-j-3}\,W_{j+1} \quad \text{for } n \geq 4\,. \end{aligned} \tag{2.7}$$

Here the expressions of W_n are given by (2.1). Notice that $Q_n = O(\epsilon^n)$ are linear in Φ while $R_n = O(\epsilon^n)$ are quadratic in Φ.

For small amplitude waves, the system (2.5) can be linearized, with $W_1 = -\mathcal{L}[\Phi]$, to

$$\frac{\partial\zeta}{\partial t} = -\mathcal{L}[\Phi], \qquad \frac{\partial\Phi}{\partial t} = -g\zeta\,, \tag{2.8}$$

which can be combined into $\partial^2\zeta/\partial t^2 = g\mathcal{L}[\zeta]$. The same equation also holds for Φ. Substituting $(\zeta,\Phi) \sim \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)]$ into (2.8) yields the linear dispersion relation given by

$$\omega^2 = g\,k\tanh kh\,, \tag{2.9}$$

where we have used $\mathcal{L}\left[\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\right] = -k\tanh kh\,\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}$. While the leading-order terms (Q_1 and R_1) represent linear dispersive effects, Q_n and R_n for $n \geq 2$ describe nonlinear dispersive effects and nonlinear wave interactions.

Following West *et al.* (1987), the system given by (2.5) has been studied extensively in recent years using a pseudo-spectral method based on Fast Fourier Transform (FFT), for example, by Tananka (2001a, b) and many others. For numerical computations, after assuming ζ and Φ are doubly periodic in space so that they can be written in Fourier series, the linear operators $\triangle$ and $\mathcal{L}$ in (2.3)–(2.4) are evaluated in Fourier space:

$$\triangle = -k_{\mathbf{j}}^2\,, \qquad \mathcal{L} = -k_{\mathbf{j}}\,T_{\mathbf{j}}\,, \tag{2.10}$$

where $\mathbf{j} = (j, l)$, $\boldsymbol{k}_{\mathbf{j}} = (jK_x, lK_y)$, $k_{\mathbf{j}} = |\boldsymbol{k}_{\mathbf{j}}|$, $T_{\mathbf{j}} = \tanh(k_{\mathbf{j}}h)$, and with K_x and K_y being the fundamental wavenumbers in the x and y directions, respectively. Then the two nonlinear operators $\mathcal{A}_n$ and $\mathcal{C}_n$ defined by (2.3) and (2.4) are computed as

$$\mathcal{A}_{2m} = -\frac{\zeta^{2m}}{(2m)!}\,k_{\mathbf{j}}^{2m}\,, \qquad \mathcal{A}_{2m+1} = -\frac{\zeta^{2m+1}}{(2m+1)!}\,k_{\mathbf{j}}^{2m+1}\,T_{\mathbf{j}}\,, \tag{2.11}$$

$$\mathcal{C}_{2m} = \frac{\zeta^{2m}}{(2m)!}\,k_{\mathbf{j}}^{2m+1}\,T_{\mathbf{j}}\,, \qquad \mathcal{C}_{2m+1} = \frac{\zeta^{2m+1}}{(2m+1)!}\,k_{\mathbf{j}}^{2m+2}\,. \tag{2.12}$$

In (2.11), to compute $\mathcal{A}_{2m}[f]$, the Fourier transform of f is multiplied by $-k_{\mathbf{j}}^{2m}/(2m)!$ in Fourier space and, then, its inverse Fourier transform is multiplied by ζ^{2m} in physical space. Finally, after evaluating its right-hand sides up to a desired order of nonlinearity, the system given by (2.5) is integrated in time.

2.3 Hamiltonians

Zakharov (1968) showed that the totoal energy defined by

$$E = \frac{1}{2}\int\left(g\,\zeta^2 + \Phi\,\frac{\partial\zeta}{\partial t}\right)\mathrm{d}\boldsymbol{x}\,, \tag{2.13}$$

is the Hamiltonian for the water wave problem so that the evolution equations for ζ and Φ can be written as

$$\frac{\partial\zeta}{\partial t} = \frac{\delta E}{\delta\Phi}\,, \qquad \frac{\partial\Phi}{\partial t} = -\frac{\delta E}{\delta\zeta}\,, \tag{2.14}$$

where $\delta E/\delta\zeta$ and $\delta E/\delta\Phi$ represent the functional derivatives of E with respect to the two conjugate variables ζ and Φ, respectively. Therefore, the total energy E is conserved. From (2.5), the total energy E defined in (2.13) can be expanded, in infinite series, as

$$E = \frac{1}{2}\int\left(g\,\zeta^2 + \Phi\sum_{n=1}^{\infty}Q_n\right)\mathrm{d}\boldsymbol{x} = \sum_{n=2}^{\infty}E_n\,, \tag{2.15}$$

where Q_n are given by (2.7) and the n-th order energy E_n is given by

$$E_2 = \frac{1}{2}\int\left(g\zeta^2 + \Phi\,Q_1\right)\mathrm{d}\boldsymbol{x}, \qquad E_n = \frac{1}{2}\int\Phi\,Q_{n-1}\,\mathrm{d}\boldsymbol{x} \quad \text{for } n \geq 3\,. \tag{2.16}$$

2.4 Fifth-order model

When truncated at $O\left(\epsilon^5\right)$, the fifth-order nonlinear evolution equations for ζ and Φ can be obtained as

$$\frac{\partial \zeta}{\partial t} = \sum_{n=1}^{5} \mathrm{Q}_n(\zeta, \Phi)\,, \qquad \frac{\partial \Phi}{\partial t} = \sum_{n=1}^{5} \mathrm{R}_n(\zeta, \Phi)\,, \tag{2.17}$$

where Q_n and R_n are given, explicitly, by

$$Q_1 = -\mathcal{L}[\Phi]\,, \tag{2.18}$$

$$Q_2 = -\boldsymbol{\nabla}\cdot(\zeta\boldsymbol{\nabla}\Phi) - \mathcal{L}\Big[\zeta\mathcal{L}[\Phi]\Big]\,, \tag{2.19}$$

$$Q_3 = -\mathcal{L}\left[\zeta\mathcal{L}\Big[\zeta\mathcal{L}[\Phi]\Big] + \tfrac{1}{2}\zeta^2\nabla^2\Phi\right] - \nabla^2\left(\tfrac{1}{2}\zeta^2\mathcal{L}[\Phi]\right)\,, \tag{2.20}$$

$$\begin{aligned} Q_4 = -\mathcal{L}\Big[&\zeta\mathcal{L}[\zeta\mathcal{L}[\zeta\mathcal{L}[\Phi]] + \tfrac{1}{2}\zeta^2\nabla^2\Phi] + \tfrac{1}{2}\zeta^2\nabla^2\Big(\zeta\mathcal{L}[\Phi]\Big) - \tfrac{1}{6}\zeta^3\nabla^2\mathcal{L}[\Phi]\Big] \\ &-\nabla^2\left(\tfrac{1}{2}\zeta^2\mathcal{L}\Big[\zeta\mathcal{L}[\Phi]\Big] + \tfrac{1}{3}\zeta^3\nabla^2\Phi\right)\,, \end{aligned} \tag{2.21}$$

$$\begin{aligned} Q_5 = -\mathcal{L}\Big[&\zeta\mathcal{L}\left[\zeta\mathcal{L}\left[\zeta\mathcal{L}\Big[\zeta\mathcal{L}[\Phi]\Big] + \tfrac{1}{2}\zeta^2\nabla^2\Phi\right] + \tfrac{1}{2}\zeta^2\nabla^2\Big(\zeta\mathcal{L}[\Phi]\Big) - \tfrac{1}{6}\zeta^3\nabla^2\mathcal{L}[\Phi]\right]\,, \\ &+\tfrac{1}{2}\zeta^2\nabla^2\left(\zeta\mathcal{L}\Big[\zeta\mathcal{L}[\Phi]\Big] + \tfrac{1}{2}\zeta^2\nabla^2\Phi\right) - \tfrac{1}{6}\zeta^3\nabla^2\mathcal{L}\Big[\zeta\mathcal{L}[\Phi]\Big] - \tfrac{1}{24}\zeta^4\nabla^2\nabla^2\Phi\Big] \\ &-\nabla^2\left(\tfrac{1}{2}\zeta^2\mathcal{L}\left[\zeta\mathcal{L}\Big[\zeta\mathcal{L}[\Phi]\Big] + \tfrac{1}{2}\zeta^2\nabla^2\Phi\right] + \tfrac{1}{3}\zeta^3\nabla^2\left(\zeta\mathcal{L}[\Phi]\right) - \tfrac{1}{8}\zeta^4\nabla^2\mathcal{L}[\Phi]\right)\,, \end{aligned} \tag{2.22}$$

$$R_1 = -g\,\zeta\,, \tag{2.23}$$

$$R_2 = -\tfrac{1}{2}\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi + \tfrac{1}{2}(\mathcal{L}[\Phi])^2\,, \tag{2.24}$$

$$R_3 = \mathcal{L}[\Phi]\left(\mathcal{L}\Big[\zeta\mathcal{L}[\Phi]\Big] + \zeta\nabla^2\Phi\right)\,, \tag{2.25}$$

$$\begin{aligned} R_4 = \mathcal{L}[\Phi]\,\mathcal{L}\left[\zeta\mathcal{L}[\zeta\mathcal{L}[\Phi]] + \tfrac{1}{2}\zeta^2\nabla^2\Phi\right] + \nabla^2\left[\tfrac{1}{4}\zeta^2(\mathcal{L}[\Phi])^2\right] + \tfrac{1}{2}\Big(\mathcal{L}[\zeta\mathcal{L}[\Phi]] + \zeta\nabla^2\Phi\Big)^2 \\ +\tfrac{1}{2}\zeta(\nabla^2\zeta)(\mathcal{L}[\Phi])^2 - \tfrac{1}{2}\zeta^2(\nabla\mathcal{L}[\Phi])^2\,, \end{aligned} \tag{2.26}$$

$$\begin{aligned} R_5 = \mathcal{L}[\Phi]\mathcal{L}\Big[&\zeta\mathcal{L}\left[\zeta\mathcal{L}\Big[\zeta\mathcal{L}[\Phi]\Big] + \tfrac{1}{2}\zeta^2\nabla^2\Phi\right] + \tfrac{1}{2}\zeta^2\nabla^2\Big(\zeta\mathcal{L}[\Phi]\Big) - \tfrac{1}{6}\zeta^3\nabla^2\mathcal{L}[\Phi]\Big] \\ &+\nabla^2\left(\tfrac{1}{2}\zeta^2\mathcal{L}[\Phi]\mathcal{L}\Big[\zeta\mathcal{L}[\Phi]\Big] + \tfrac{1}{3}\zeta^3\mathcal{L}[\Phi]\nabla^2\Phi\right) - \zeta^2\left(\nabla\mathcal{L}[\Phi]\right)\cdot\left(\nabla\mathcal{L}\Big[\zeta\mathcal{L}[\Phi]\Big] + \tfrac{2}{3}\zeta\nabla(\nabla^2\Phi)\right) \\ &+\tfrac{1}{6}\zeta^3(\nabla^2\mathcal{L}[\Phi])(\nabla^2\Phi) + \Big(\mathcal{L}\left[\zeta\mathcal{L}[\Phi]\right] + \zeta\nabla^2\Phi\Big)\left(\mathcal{L}\left[\zeta\mathcal{L}\Big[\zeta\mathcal{L}[\Phi]\Big] + \tfrac{1}{2}\zeta^2\nabla^2\Phi\right] + \tfrac{1}{2}\zeta(\nabla^2\zeta)\mathcal{L}[\Phi]\right)\,. \end{aligned} \tag{2.27}$$

The expressions of the corresponding Hamiltonians E_n $(n = 2, \cdots, 6)$ are explicitly given by

$$E_2 = \frac{1}{2}\int\left(g\,\zeta^2 - \Phi\mathcal{L}[\Phi]\right)\mathrm{d}\boldsymbol{x}, \tag{2.28}$$

$$E_3 = \frac{1}{2}\int\left\{\zeta\nabla\Phi\cdot\nabla\Phi - \zeta\left(\mathcal{L}[\Phi]\right)^2\right\}\mathrm{d}\boldsymbol{x}, \tag{2.29}$$

$$E_4 = -\frac{1}{2}\int\zeta\mathcal{L}[\Phi]\left(\mathcal{L}\left[\zeta\mathcal{L}[\Phi]\right] + \zeta\nabla^2\Phi\right)\mathrm{d}\boldsymbol{x}, \tag{2.30}$$

$$E_5 = -\frac{1}{2}\int \Big\{\zeta\Big(\mathcal{L}\left[\zeta\mathcal{L}[\Phi]\right]\Big)^2 + \tfrac{1}{3}\zeta^3\left(\nabla^2\Phi\right)^2 + \zeta\mathcal{L}[\Phi]\left(\mathcal{L}\left[\zeta^2\nabla^2\Phi\right] + \tfrac{1}{2}\zeta\nabla^2(\zeta\mathcal{L}[\Phi]) - \tfrac{1}{6}\zeta^2\nabla^2\mathcal{L}[\Phi]\right)\Big\}\,\mathrm{d}\boldsymbol{x}\,, \tag{2.31}$$

$$E_6 = -\frac{1}{2}\int \Big\{\zeta\mathcal{L}\left[\zeta\mathcal{L}\Big[\zeta\mathcal{L}[\Phi]\Big]\right]\left(\mathcal{L}\left[\zeta\mathcal{L}[\Phi]\right] + \zeta\nabla^2\Phi\right) + \mathcal{L}\left[\zeta\mathcal{L}[\Phi]\right]\left(\zeta^2\nabla^2(\zeta\mathcal{L}[\Phi]) - \tfrac{1}{3}\zeta^3\nabla^2\mathcal{L}[\Phi]\right) + \left(\tfrac{1}{2}\zeta^2\nabla^2\Phi\right)\mathcal{L}\left[\tfrac{1}{2}\zeta^2\nabla^2\Phi\right] + \zeta^2\mathcal{L}[\Phi]\left(\tfrac{1}{2}\nabla^2\left(\zeta^2\nabla^2\Phi\right) - \tfrac{1}{12}\zeta^2\nabla^2\left(\nabla^2\Phi\right)\right)\Big\}\,\mathrm{d}\boldsymbol{x}\,. \tag{2.32}$$

When truncated at $O(\epsilon^3)$, the system given by (2.17) becomes the third-order system obtained by Choi (1995), who showed that the truncated system also preserves the Hamiltonian structure. Likewise, it can be shown that the fifth-order model given by (2.17) is a Hamiltonian system.

3 Spectral Formulation

3.1 System for continuous spectrum

To obtain a nonlinear system in spectral space, ζ and Φ are expressed as

$$\zeta(\boldsymbol{x},t) = \int a(\boldsymbol{k},t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{k}, \quad \Phi(\boldsymbol{x},t) = \int b(\boldsymbol{k},t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{k}\,, \tag{3.1}$$

where $a(\boldsymbol{k},t)$ and $b(\boldsymbol{k},t)$ representing the Fourier transforms of ζ and Φ, respectively. Notice that $a(-\boldsymbol{k},t) = a^*(\boldsymbol{k},t)$ and $b(-\boldsymbol{k},t) = b^*(\boldsymbol{k},t)$, with the asterisks representing the complex conjugates, as ζ and Φ are real-valued functions.

One way to find such a system is to take the Fourier transform of (2.5), which would yield the nonlinear evolution equations for $a(\boldsymbol{k},t)$ and $b(\boldsymbol{k},t)$ as

$$\frac{\partial a}{\partial t} - kT\,b = \sum_{n=2}^{\infty} q_n\,, \qquad \frac{\partial b}{\partial t} + g\,a = \sum_{n=2}^{\infty} r_n\,, \tag{3.2}$$

where q_n and r_n representing the Fourier transforms of Q_n and R_n given by (2.6)–(2.7) can be written as

$$q_n = \iint\cdot\cdot\int \alpha^{(n)}_{0,1,\cdots,n}\,b_1\,a_2\,a_3\cdots a_n\,\delta_{0-1-\cdots-n}\,\mathrm{d}\boldsymbol{k}_1\mathrm{d}\boldsymbol{k}_2\cdots\mathrm{d}\boldsymbol{k}_n\,, \tag{3.3}$$

$$r_n = \iint\cdot\cdot\int \beta^{(n)}_{0,1,\cdots,n}\,b_1\,b_2\,a_3\cdots a_n\,\delta_{0-1-\cdots-n}\,\mathrm{d}\boldsymbol{k}_1\mathrm{d}\boldsymbol{k}_2\cdots\mathrm{d}\boldsymbol{k}_n\,. \tag{3.4}$$

In (3.3)–(3.4), we have used the following short-hand notations

$$a_j = a(\boldsymbol{k}_j,t), \quad b_j = b(\boldsymbol{k}_j,t), \quad \delta_{j-l} = \delta(\boldsymbol{k}_j - \boldsymbol{k}_l), \quad \boldsymbol{k}_0 = \boldsymbol{k}\,, \tag{3.5}$$

where δ is the Dirac delta function. Under the third-order approximation, the system given by (3.2) was derived first by Zakharov (1968) for infinitely deep water and by Stiassnie & Shemer (1984) for finite depth water. The system has been also extended to $O(\epsilon^4)$ by Stiassnie & Shemer (1984).

Although it is straightforward, finding the explicit expressions of $\alpha^{(n)}_{0,1,\cdots,n}$ and $\beta^{(n)}_{0,1,\cdots,n}$ by taking the Fourier transform of (2.5) is lengthy and cumbersome, in particular, as the order

of nonlinearity increases. An alternative and more convenient way is to use the Hamiltonian, as shown by Krasitskii (1994), whose approach will be adopted here to obtain the fifth-order system. From (2.16), the n-th order Hamiltonian $H_n = E_n/(2\pi)^2$ can be written in spectral space as

$$H_2 = \frac{1}{2}\iint \Big(g\, a_1\, a_2 + k_1 T_1\, b_1\, b_2\Big)\, \delta_{1+2}\, \mathrm{d}\boldsymbol{k}_1 \mathrm{d}\boldsymbol{k}_2\,, \tag{3.6}$$

$$H_n = \frac{1}{2}\iint\cdot\cdot\int h^{(n)}_{1,2,3,\cdots,n}\; b_1\, b_2\, a_3\, \cdots\, a_n\, \delta_{1+\cdots+n}\, \mathrm{d}\boldsymbol{k}_1 \mathrm{d}\boldsymbol{k}_2\, \mathrm{d}\boldsymbol{k}_3 \cdots \mathrm{d}\boldsymbol{k}_n \quad \text{for } n \geq 3\,. \tag{3.7}$$

For example, under the fifth-order approximation, $h^{(n)}_{1,2,3,\cdots,n}$ for $n = 3, 4, 5, 6$ can be written explicitly as

$$h^{(3)}_{1,2,3} = -\left(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 + \theta_1\theta_2\right)\,, \tag{3.8}$$

$$h^{(4)}_{1,2,3,4} = -\Big(k_2^2\, \theta_1 - \theta_1\, \theta_2\, \theta_{2+3}\Big)\,, \tag{3.9}$$

$$h^{(5)}_{1,2,3,4,5} = -\Big[\Big(\tfrac{1}{6}k_2^2 - \tfrac{1}{2}k_{2+3}^2 + \theta_{1+3}\, \theta_{2+4}\Big)\, \theta_1\theta_2 \; - k_2^2\, \theta_1\theta_{2+3+4} + \tfrac{1}{3}k_1^2 k_2^2\Big]\,, \tag{3.10}$$

$$h^{(6)}_{1,2,3,4,5,6} = \Big[\Big(\theta_1\theta_{1+3} - k_1^2\Big)\, \theta_2\theta_{2+4}\theta_{2+4+5} + \Big(\tfrac{1}{3}k_2^2 - k_{2+4}^2\Big)\, \theta_1\theta_2\theta_{1+3} + \tfrac{1}{4}k_1^2 k_2^2 \theta_{2+5+6} + \tfrac{1}{2}k_1^2 k_{1+3+4}^2 \theta_2 - \tfrac{1}{12}k_1^4\theta_2\Big]\,, \tag{3.11}$$

where

$$k_j = |\boldsymbol{k}_j|\,, \quad \theta_j = k_j T_j\,, \quad T_j = \tanh(k_j h)\,, \quad k_{m+n} = |\boldsymbol{k}_m + \boldsymbol{k}_n|\,, \quad T_{m+n} = \tanh(k_{m+n} h)\,. \tag{3.12}$$

The evolution equations for $a(\boldsymbol{k}, t)$ and $b(\boldsymbol{k}, t)$ can be then obtained from Hamilton's equations:

$$\frac{\partial a}{\partial t} = \frac{\delta H}{\delta b^*}, \qquad \frac{\partial b}{\partial t} = -\frac{\delta H}{\delta a^*}\,. \tag{3.13}$$

From (3.6)–(3.7) and (3.13), the expressions of $\alpha^{(n)}_{0,1,\cdots,n}$ and $\beta^{(n)}_{0,1\cdots,n}$ in (3.3)–(3.4) are found, in terms of $h^{(n)}_{1,2,3,\cdots,n}$, as

$$\alpha^{(n)}_{0,1,\cdots,n} = \tfrac{1}{2}\left(h^{(n+1)}_{-0,1,2,\cdots,n} + h^{(n+1)}_{1,-0,2,\cdots,n}\right)\,, \tag{3.14}$$

$$\beta^{(n)}_{0,1\cdots,n} = -\tfrac{1}{2}\left(h^{(n+1)}_{1,2,-0,3,\cdots,n} + \cdots + h^{(n+1)}_{1,2,3,\cdots,n,-0}\right)\,. \tag{3.15}$$

Notice that the interaction coefficients $h^{(n)}_{1,2,\cdots,n}$ in (3.8)–(3.11) are not symmetric. In other words, except for $h^{(3)}_{1,2,3}$, they change when indices 1 and 2 are interchanged although their Hamiltonians given by (3.7) remain unchanged. This is also true for indices $3, \cdots, n$. Nevertheless, if necessary, they can be easily made symmetric, as shown by Krasitskii (1994).

3.2 System for discrete spectrum

When a nonlinear wave field can be represented by a superposition of discrete modes, $a(\boldsymbol{k}, t)$ and $b(\boldsymbol{k}, t)$ can be written as

$$a(\boldsymbol{k}, t) = \sum_{\mathbf{j}} \delta(\boldsymbol{k} - \boldsymbol{k}_{\mathbf{j}})\, a_{\mathbf{j}}(t)\,, \qquad b(\boldsymbol{k}, t) = \sum_{\mathbf{j}} \delta(\boldsymbol{k} - \boldsymbol{k}_{\mathbf{j}})\, b_{\mathbf{j}}(t)\,, \tag{3.16}$$

where $\boldsymbol{k}_{-\mathbf{j}} = -\boldsymbol{k}_{\mathbf{j}}$. In (3.16), the summations should be in general taken over all discrete modes involved in nonlinear wave interactions unless an additional approximation is made.

When truncated at $O(\epsilon^M)$, the amplitude equations for $a_{\mathbf{j}}$ and $b_{\mathbf{j}}$ under the M-th order approximation are given, from (3.2), by

$$\frac{\mathrm{d}a_{\mathbf{j}}}{\mathrm{d}t} - k_j T_j\, b_{\mathbf{j}} = \sum_{n=2}^{M} \left[\sum_{\mathbf{j_1},\mathbf{j_2},\cdots,\mathbf{j_n}} \alpha^{(n)}_{\mathbf{j},\mathbf{j_1},\cdots,\mathbf{j_n}}\; b_{\mathbf{j_1}}\, a_{\mathbf{j_2}}\, a_{\mathbf{j_3}} \cdots a_{\mathbf{j_n}}\; \delta_{0-1-\cdots-n} \right], \tag{3.17}$$

$$\frac{\mathrm{d}b_{\mathbf{j}}}{\mathrm{d}t} + g\, a_{\mathbf{j}} = \sum_{n=2}^{M} \left[\sum_{\mathbf{j_1},\mathbf{j_2},\cdots,\mathbf{j_n}} \beta^{(n)}_{\mathbf{j},\mathbf{j_1},\cdots,\mathbf{j_n}}\; b_{\mathbf{j_1}}\, b_{\mathbf{j_2}}\, a_{\mathbf{j_3}} \cdots a_{\mathbf{j_n}}\; \delta_{0-1-\cdots-n} \right], \tag{3.18}$$

where $\delta_{0-1-\cdots-n} = \delta_{\mathbf{j}-\mathbf{j_1}-\cdots-\mathbf{j_n}}$. The corresponding Hamiltonians are given by

$$H_2 = \frac{1}{2} \sum_{\mathbf{j_1}} \sum_{\mathbf{j_2}} \left(g\, a_{\mathbf{j_1}}\, a_{\mathbf{j_2}} + k_{\mathbf{j_1}} T_{\mathbf{j_1}}\, b_{\mathbf{j_1}}\, b_{\mathbf{j_2}} \right) \delta_{1+2}\,, \tag{3.19}$$

$$H_n = \frac{1}{2} \sum_{\mathbf{j_1},\mathbf{j_2},\cdots,\mathbf{j_n}} h^{(n)}_{\mathbf{j_1},\mathbf{j_2},\cdots,\mathbf{j_n}}\; b_{\mathbf{j_1}}\, b_{\mathbf{j_2}}\, a_{\mathbf{j_3}}\; \cdots\; a_{\mathbf{j_n}}\, \delta_{1+2+\cdots+n}\,, \tag{3.20}$$

from which the amplitude equations given by (3.17)–(3.18) can be obtained from the Hamilton's equations:

$$\frac{\partial a_{\mathbf{j}}}{\partial t} = \frac{\partial H}{\partial b^*_{\mathbf{j}}}, \qquad \frac{\partial b_{\mathbf{j}}}{\partial t} = -\frac{\partial H}{\partial a^*_{\mathbf{j}}}\,, \tag{3.21}$$

where $H = \sum_n H_n$. For $M = 5$, the expressions of $h^{(n)}_{\mathbf{j_1},\mathbf{j_2},\cdots,\mathbf{j_n}}$ are defined by (3.8)–(3.11) and $\alpha^{(n)}_{\mathbf{j},\mathbf{j_1},\cdots,\mathbf{j_n}}$ and $\beta^{(n)}_{\mathbf{j},\mathbf{j_1},\cdots,\mathbf{j_n}}$ are given by (3.14)–(3.15).

When $\boldsymbol{k}_{\mathbf{j}} = (jK_x, lK_y)$, (3.16) represent the Fourier series of ζ and Φ and equations (3.17)–(3.18) determine their evolution of the Fourier coefficients, $a_{\mathbf{j}}$ and $b_{\mathbf{j}}$. If a finite number of Fourier modes are used for numerical computations, solving the ordinary differential equations given by (3.17)–(3.18) is equivalent to solving (2.5) using the pseudo-spectral method described in §2.2. Unfortunately, the evaluation of the right-hand sides is computationally expensive and, therefore, solving a dynamical system is in general less effective than the pseudo-spectral method based on FFT.

3.3 Time-periodic solutions of the third-order spectral model

Under the third-order approximation ($M = 3$), the amplitude equations for $a_{\mathbf{j}}$ and $b_{\mathbf{j}}$ can be written, from (3.17)–(3.18), as

$$\begin{aligned}\frac{\mathrm{d}a_{\mathbf{j}}}{\mathrm{d}t} &= k_{\mathbf{j}}\, T_{\mathbf{j}}\, b_{\mathbf{j}} + \sum_{\mathbf{j_1},\mathbf{j_2}} \Big(\boldsymbol{k}_{\mathbf{j}} \cdot \boldsymbol{k}_{\mathbf{j_1}} - k_{\mathbf{j}}\, T_{\mathbf{j}}\, k_{\mathbf{j_1}}\, T_{\mathbf{j_1}} \Big)\, b_{\mathbf{j_1}} a_{\mathbf{j_2}}\, \delta_{0-1-2} \\ &\quad + \sum_{\mathbf{j_1},\mathbf{j_2},\mathbf{j_3}} \Big[k_{\mathbf{j}}\, T_{\mathbf{j}} \Big(k_{\mathbf{j_1}}\, T_{\mathbf{j_1}}\, k_{\mathbf{j_1}+\mathbf{j_2}}\, T_{\mathbf{j_1}+\mathbf{j_2}} - \tfrac{1}{2} k^2_{\mathbf{j_1}} \Big) - \tfrac{1}{2} k^2_{\mathbf{j}}\, k_{\mathbf{j_1}}\, T_{\mathbf{j_1}} \Big]\, b_{\mathbf{j_1}} a_{\mathbf{j_2}} a_{\mathbf{j_3}}\, \delta_{0-1-2-3}\,, \\ \frac{\mathrm{d}b_{\mathbf{j}}}{\mathrm{d}t} &= -g\, a_{\mathbf{j}} + \sum_{\mathbf{j_1},\mathbf{j_2}} \tfrac{1}{2} \Big(\boldsymbol{k}_{j_1} \cdot \boldsymbol{k}_{\mathbf{j_2}} + k_{\mathbf{j_1}}\, T_{\mathbf{j_1}}\, k_{\mathbf{j_2}} T_{\mathbf{j_2}} \Big) b_{\mathbf{j_1}} b_{\mathbf{j_2}}\, \delta_{0-1-2}\end{aligned} \tag{3.22}$$

$$+\sum_{\mathbf{j_1},\mathbf{j_2},\mathbf{j_3}}\left[k_{\mathbf{j_1}}T_{\mathbf{j_1}}\,\left(-k_{\mathbf{j_2}}T_{\mathbf{j_2}}\,k_{\mathbf{j-j_1}}T_{\mathbf{j-j_1}}+k_{\mathbf{j_2}}^2\right)\right]b_{\mathbf{j_1}}b_{\mathbf{j_2}}a_{\mathbf{j_3}}\,\delta_{0-1-2-3}\,. \tag{3.23}$$

When we assume that the waves are propagating in the x-direction so that $\boldsymbol{k}_{\mathbf{j}}=(k_j,0)$ with $k_j=jk$ and $(a_{\mathbf{j}},b_{\mathbf{j}})=(a_j,b_j)$, equations (3.22)–(3.23) describe the evolution of the Fourier coefficients of ζ and Φ. Furthermore, we assume that the first harmonics are initially dominant and all other higher-harmonics are excited through nonlinearity so that

$$a_j=O(\epsilon)\,,\quad a_0=O(b_0)=O(b_{2j})=O(a_{2j})=O(\epsilon^2),\quad a_{3j}=O(b_{3j})=O(\epsilon^3)\,. \tag{3.24}$$

Then, the third-order system (3.22)–(3.23) can be approximated by four ordinary differential equations: for the j-th mode,

$$\begin{aligned}\frac{\mathrm{d}a_j}{\mathrm{d}t}-k_j\,T_j\,b_j=2k_j^2\,(1-T_jT_{2j})\;a_j^*b_{2j}-k_j^2\left(1+T_j^2\right)\,a_{2j}b_j^*\\ -2k_j^3T_j\,(1-T_jT_{2j})\,|a_j|^2\,b_j-k_j^3T_ja_j^2\,b_j^*\,,\end{aligned} \tag{3.25}$$

$$\frac{\mathrm{d}b_j}{\mathrm{d}t}+g\,a_j=-2k_j^2\,(1-T_jT_{2j})\;b_j^*b_{2j}+2k_j^3T_j\,(1-T_jT_{2j})\,a_j\,|b_j|^2+k_j^3T_j\,a_j^*\,b_j^2\,, \tag{3.26}$$

and, for the $2j$-th mode,

$$\frac{\mathrm{d}a_{2j}}{\mathrm{d}t}-k_{2j}\,T_{2j}\,b_{2j}=2k_j^2\,(1-T_j\,T_{2j})\;a_jb_j\,, \tag{3.27}$$

$$\frac{\mathrm{d}b_{2j}}{\mathrm{d}t}+g\,a_{2j}=\tfrac{1}{2}k_j^2\left(1+T_j^2\right)\,b_j^2\,. \tag{3.28}$$

The Hamiltonian for the system is given, by imposing (3.24) to (3.19)–(3.20), by

$$\begin{aligned}H&=\left(g\,|a_j|^2\,+k_jT_j\,|b_j|^2\right)+\left(g\,|a_{2j}|^2\,+k_{2j}T_{2j}\,|b_{2j}|^2\right)\\&\quad+\tfrac{1}{2}\left[h^{(3)}_{j,j,-2j}\,b_j^2\,a_{2j}^*+h^{(3)}_{-j,-j,2j}\,b_j^{*2}\,a_{2j}+2h^{(3)}_{2j,-j,-j}\,b_{2j}\,b_j^*\,a_j^*+2h^{(3)}_{-2j,j,j}\,b_{2j}^*\,b_j\,a_j\right]\\&\quad+\tfrac{1}{2}\left[h^{(4)}_{j,j,-j,-j}\,b_j^2\,a_j^{*2}+h^{(4)}_{-j,-j,j,j}\,b_j^{*2}\,a_j^2\right.\\&\qquad\left.+\left(h^{(4)}_{j,-j,j,-j}+h^{(4)}_{j,-j,-j,j}+h^{(4)}_{-j,j,j,-j}+h^{(4)}_{-j,j,-j,j}\right)\,|b_j|^2\,|a_j|^2\right]\\&=\left(g\,|a_j|^2\,+k_jT_j\,|b_j|^2\right)+\left(g\,|a_{2j}|^2\,+k_{2j}T_{2j}\,|b_{2j}|^2\right)\\&\quad-\tfrac{1}{2}k_j^2\,\left(1+T_j^2\right)\left(b_j^2\,a_{2j}^*+b_j^{*2}\,a_{2j}\right)+2k_j^2\,(1-T_jT_{2j})\left(b_{2j}\,b_j^*\,a_j^*+b_{2j}^*\,b_j\,a_j\right)\\&\quad-\tfrac{1}{2}k_j^3T_j\left(b_j^2\,a_j^{*2}+b_j^{*2}\,a_j^2\right)-2k_j^3T_j\,(1-T_jT_{2j})\;|b_j|^2\,|a_j|^2\,,\end{aligned} \tag{3.29}$$

where we have used $h^{(3)}_{2j,-j,-j}=h^{(3)}_{-j,2j,-j}$ and $h^{(3)}_{-2j,j,j}=h^{(3)}_{j,-2j,j}$. Here the amplitude equations of the third-harmonics (a_{3j} and b_{3j}) as not written as they have no effect on the dynamics of the first harmonics of interest unless the higher-order nonlinearity is included. Therefore, we consider only the first two harmonics here.

3.3.1 Progressive waves

When linearized, (3.25)–(3.26) can be reduced to

$$\frac{\mathrm{d}a_j}{\mathrm{d}t} = k_j\, T_j\, b_j\,, \qquad \frac{\mathrm{d}b_j}{\mathrm{d}t} = -g\, a_j\,, \tag{3.30}$$

whose solution can be written as

$$a_j = \overline{a}_j\, \mathrm{e}^{\mathrm{i}\omega_j t}\,, \qquad b_j = \mathrm{i}\ (g/\omega_j)\ \overline{a}_j\, \mathrm{e}^{\mathrm{i}\omega_j t}\,, \tag{3.31}$$

so that

$$b_j = \mathrm{i}\ (g/\omega_j)\ a_j\,. \tag{3.32}$$

Here $\omega_j > 0$ satisfies the linear dispersion relation (2.9):

$$\omega_j^2 = g\, k_j T_j\,. \tag{3.33}$$

At the second order, the particular solutions of (3.27)–(3.28) for the second harmonics can be obtained, using $\mathrm{d}a_{2j}/\mathrm{d}t = 2\omega_j a_{2j}$ and $\mathrm{d}b_{2j}/\mathrm{d}t = 2\omega_j b_{2j}$, as

$$a_{2j} = \frac{1}{\omega_{2j}^2 - 4\omega_j^2}\left[4\mathrm{i}\,\omega_j k_j^2 (1 - T_j T_{2j})\, a_j b_j + k_j^3 T_{2j}(1 + T_j^2)\, b_j^2\right]\,, \tag{3.34}$$

$$b_{2j} = \frac{1}{\omega_{2j}^2 - 4\omega_j^2}\left[-2g k_j^2 (1 - T_j T_{2j})\, a_j b_j + \mathrm{i}\,\omega_j k_j^2 (1 + T_j^2)\, b_j^2\right]\,, \tag{3.35}$$

where $\omega_{2j}^2 = g k_{2j} T_{2j}$ is the natural frequency of the second harmonics of wavenumber $k_{2j} = 2k_j$ and we have assumed that $\omega_{2j} \neq 2\omega_j$. When substituting the linear solution (3.31) into (3.34)–(3.35), the second harmonic solutions can be found as

$$a_{2j} = \alpha_{2j}\, \overline{a}_j^2\, \mathrm{e}^{2\mathrm{i}\omega_j t}\,, \qquad \alpha_{2j} = g k_j^2 \left(\frac{4 - 3T_j T_{2j} + T_{2j}/T_j}{4\omega_j^2 - \omega_{2j}^2}\right) = k_j\, \left(\frac{3 - T_j^2}{2\, T_j^3}\right)\,, \tag{3.36}$$

$$b_{2j} = \mathrm{i}\,\beta_{2j}\, \overline{a}_j^2\, \mathrm{e}^{2\mathrm{i}\omega_j t}\,, \qquad \beta_{2j} = \frac{g^2 k_j^2}{\omega_j}\left(\frac{3 - 2T_j T_{2j} + T_j^2}{4\omega_j^2 - \omega_{2j}^2}\right) = \omega_j\, \left(\frac{3 + T_j^4}{4\, T_j^4}\right)\,, \tag{3.37}$$

where we have used, for the last expressions of α_{2j} and β_{2j},

$$T_{2j} = 2T_j/(1 + T_j^2)\,. \tag{3.38}$$

From (3.31), a_{2j} and b_{2j} given by (3.36)–(3.37) can be expressed, in terms of a_j and b_j, as

$$a_{2j} = \alpha_{2j}\, a_j^2 + O(\epsilon^3)\,, \qquad b_{2j} = \mathrm{i}\,\beta_{2j}\, a_j^2 + O(\epsilon^3)\,. \tag{3.39}$$

To study the nonlinear behavior of the first harmonics (a_j and b_j), although not necessary, it is convenient to use a single amplitude equation, for example, for a_j. After substituting (3.32) and (3.39) into the right-hand sides of (3.25)–(3.26), the time evolution equation for a_j correct to $O(\epsilon^3)$ can be found as

$$\frac{\mathrm{d}^2 a_j}{\mathrm{d}t^2} + \omega_j^2\left(1 + \alpha_j\, |a_j|^2\right)\, a_j = 0\,, \tag{3.40}$$

where α_j is given by

$$\alpha_j = k_j^2 \left[\frac{16\,T_j + (1 - 18\,T_j^2 + 9\,T_j^4)T_{2j}}{2\,T_j(2\,T_j - T_{2j})} \right] = k_j^2 \left(\frac{9\,T_j^4 - 10\,T_j^2 + 9}{2\,T_j^3} \right) > 0\,. \tag{3.41}$$

Similarly, the amplitude equation for b_j can be found as

$$\frac{\mathrm{d}^2 b_j}{\mathrm{d}t^2} + \omega_j^2 \left(1 + \beta_j\,|b_j|^2\right)\, b_j = 0\,, \qquad \beta_j = \left(\omega_j^2/g^2\right)\alpha_j\,. \tag{3.42}$$

For a time-periodic solution of (3.40), $a_j(t)$ is written as

$$a_j(t) = A_j\,\mathrm{e}^{\mathrm{i}\Omega_j t}\,, \qquad \Omega_j = \omega_j\left[1 + \delta_j + O(\epsilon^4)\right]\,, \tag{3.43}$$

where $\delta = O(\epsilon^2)$ is the nonlinear correction to the wave frequency. By substituting (3.43) into (3.40), one can find, at the order of $O\left(\epsilon^3\right)$, that

$$\delta_j = \frac{1}{2}\alpha_j A_j^2 = \left(\frac{9\,T_j^4 - 10\,T_j^2 + 9}{4\,T_j^3} \right) k_j^2 A_j^2\,, \tag{3.44}$$

which is the nonlinear frequency correction of Stokes waves in water of finite depth, as shown, for example, in Whitham (1976, §13.13). As a special case, for infinitely deep water ($T_j \to 1$ and $T_{2j} \to 1$), the expression of α_j, δ_j, and a_{2j} are given by

$$\alpha_j = 4k_j^2\,, \qquad \delta_j = 2\,k_j^2 A_j^2\,, \qquad a_{2j} = k_j A_j^2\,\mathrm{e}^{2\mathrm{i}\omega_j t}\,. \tag{3.45}$$

This solution corresponds to that of Stokes (1847), where the wave amplitude $\overline{a}_j$ is defined as $\overline{a}_j = 2A_j$ so that $\delta_j = k_j^2\overline{a}_j^2/2$.

3.3.2 Standing waves

For standing wave solutions, we must have

$$a_j = a_{-j} = a_j^*\,, \qquad b_j = b_{-j} = b_j^*\,, \qquad a_0 = 0\,, \tag{3.46}$$

so that ζ and Φ can be written as

$$\zeta(x,t) = 2\sum_{j\geq 0} a_j(t)\,\cos(k_j x)\,, \qquad \Phi(x,t) = 2\sum_{j\geq 0} b_j(t)\,\cos(k_j x)\,, \tag{3.47}$$

which satisfy the side-wall boundary conditions at $x = 0$ and L with

$$k_j = j\pi/L\,, \tag{3.48}$$

where L is the tank length. Then, as a_j and b_j are real, their evolution equations are given, from (3.22)–(3.23), or directly from (3.25)–(3.26), by

$$\frac{\mathrm{d}a_j}{\mathrm{d}t} - k_j\,T_j\,b_j = 2k_j^2\,(1 - T_jT_{2j})\;a_jb_{2j} - k_j^2\left(1 + T_j^2\right)\,a_{2j}b_j - k_j^3T_j\,(3 - 2T_jT_{2j})\,a_j^2b_j\,, \tag{3.49}$$

$$\frac{\mathrm{d}b_j}{\mathrm{d}t} + g\,a_j = -2k_{\mathbf{j}}^2\,(1 - T_jT_{2j})\;b_jb_{2j} + k_j^3T_j\,(3 - 2T_jT_{2j})\;a_jb_j^2\,, \tag{3.50}$$

while the time evolution of the second harmonics a_{2j} and b_{2j} are governed by (3.27)–(3.28).

When we linearize (3.49)–(3.50), the leading-order solutions can be found as

$$a_j = A_j\,\mathrm{e}^{\mathrm{i}\,\omega_j t} + C.C.\,, \qquad b_j = \mathrm{i}\,(g/\omega_j)\,A_j\,\mathrm{e}^{\mathrm{i}\,\omega_j t} + C.C.\,, \tag{3.51}$$

where the complex conjugates ($C.C.$) are needed as a_j and b_j are real functions. At the second-order, the particular solutions of (3.27)–(3.28) for the second harmonics can be found as

$$a_{2j} = \left(\alpha_{2j}\,A_j^2\,\mathrm{e}^{2\mathrm{i}\,\omega_j t} + C.C.\right) + \gamma_{2j}|A_j|^2\,, \qquad b_{2j} = \left(\mathrm{i}\,\beta_{2j}\,A_j^2\,\mathrm{e}^{2\mathrm{i}\,\omega_j t} + C.C.\right), \tag{3.52}$$

with α_{2j}, β_{2j}, and γ_{2j} given by

$$\alpha_{2j} = k_j\left(\frac{3-T_j^2}{2\,T_j^3}\right), \quad \beta_{2j} = \frac{gk_j}{\omega_j}\left(\frac{3+T_j^4}{4\,T_j^3}\right), \quad \gamma_{2j} = k_j\left(\frac{1+T_j^2}{T_j}\right). \tag{3.53}$$

For standing waves, since it is not possible to write a_{2j} and b_{2j} in terms of a_j or b_j, the system cannot be reduced to a single equation for a_j or b_j. Therefore, in general, it is necessary to solve the system given by (3.49)–(3.50) along with (3.27)–(3.28), except for the infinitely deep water case, for which $a_{2j} = k_j a_j^2$ as $\alpha_{2j} = k_j$ and $\gamma_{2j} = 2k_j$.

For a time-periodic solution, we write a_j and b_j as

$$a_j = A_j\,\mathrm{e}^{\mathrm{i}\,\Omega_j t} + A_{3j}\,\mathrm{e}^{3\mathrm{i}\,\Omega_j t} + C.C. + O(\epsilon^5)\,, \quad \Omega_j = \omega_j\left[1+\delta_j+O(\epsilon^4)\right] + O(\epsilon^5)\,, \tag{3.54}$$

$$b_j = B_j\,\mathrm{e}^{\mathrm{i}\,\Omega_j t} + B_{3j}\,\mathrm{e}^{3\mathrm{i}\,\Omega_j t} + C.C. + O(\epsilon^5)\,, \tag{3.55}$$

where $A_j = O(\epsilon)$, $B_j = O(\epsilon)$, $A_{3j} = O(\epsilon^3)$, $B_{3j} = O(\epsilon^3)$, and $\delta_j = O(\epsilon^2)$ have been assumed real. By substituting (3.54)–(3.55) into (3.49)–(3.50) with (3.52), the nonlinear correction to the wave frequency can be determined at $O(\epsilon^3)$ as

$$\delta_j = \left(\frac{9-12\,T_j^2-3\,T_j^4-2\,T_j^6}{4\,T_j^4}\right)k_j^2 A_j^2\,, \tag{3.56}$$

which has been obtained by Tadjbakhsh & Keller (1960). As pointed out by Tadjbakhsh & Keller (1960), δ_j is negative for $k_j h > 1.058$, implying that the frequency decreases as the wave amplitude increases, which is observed for a soft spring. On the other hand, for $k_j h < 1.058$, δ_j is positive, which corresponds to the case of a hard spring.

For infinitely deep water ($h \to \infty$, $T_j \to 1$), (3.56) can be reduced to $\delta_j = -2k_j^2 A_j^2$, which is the result obtained by Rayleigh (1915), where the wave amplitude is defined as $\overline{a}_j = 4A_j$ so that $\delta_j = -k_j^2\overline{a}_j^2/8$.

4 Conclusion

Using the equivalence between the spectral formulation of Zakharov (1968) and the pseudo-spectral formulation of West *et al.* (1987), we obtain an explicit fifth-order spectral model that governs the evolution of the Fourier transforms of the surface elevation ζ and the free surface velocity potential Φ. Compared with a lower-order one, the fifth-order model would improve

the description of the spectral evolution of broadband nonlinear waves of finite amplitudes. When discretized, the model provides a dynamical system for any number of discrete modes, which would be useful to study nonlinear standing waves in a sloshing tank. Although only the third-order solutions for traveling and standing waves have been presented, the fifth-order time-periodic solutions can be easily obtained from the model presented here. It should be remarked that, as the higher-order Hamiltonians are also available from the pseudo-spectral formulation of *West et al.* (1987) via recursion formulas, it is straightforward to find a higher-order spectral model although it would be complicated.

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