Received: 26 July 2018 | Revised: 29 October 2018 | Accepted: 23 November 2018

DOI: 10.1111/sapm.12249

WILEY STUDIES IN APPLIED MATHEMATICS

SPECIAL ISSUE

On resonant interactions of gravity-capillary waves without energy exchange

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Funding information
NSF Division of Ocean Sciences, Grant/Award Number: OCE-1634939; NSF Division of Mathematical Sciences, Grant/Award Number: DMS-1517456; National Science of Foundation

Abstract

We consider resonant triad interactions of gravity-capillary waves and investigate in detail special resonant triads that exchange no energy during their interactions so that the wave amplitudes remain constant in time. After writing the resonance conditions in terms of two parameters (or two angles of wave propagation), we first identify a region in the two-dimensional parameter space, where resonant triads can be always found, and then describe the variations of resonant wavenumbers and wave frequencies over the resonance region. Using the amplitude equations recovered from a Hamiltonian formulation for water waves, it is shown that any resonant triad inside the resonance region can interact without energy exchange if the initial wave amplitudes and relative phase satisfy the two conditions for fixed point solutions of the amplitude equations. Furthermore, it is shown that the symmetric resonant triad exchanging no energy forms a transversely modulated traveling wave field, which can be considered a two-dimensional generalization of Wilton ripples.

KEYWORDS

gravity-capillary waves, resonant triad interactions, symmetric Wilton ripples

1 | INTRODUCTION

Resonant interactions of weakly nonlinear waves on the surface of water have been considered one of the main mechanisms for the long-term evolution of wave spectrum and have been studied extensively since the pioneering work of Phillips.[1] For surface gravity waves, resonant interactions occur

among four waves, whose time evolution can be studied with a system of four amplitude equations[2,3] as confirmed experimentally by Longuet-Higgins and Smith[4] and McGoldrick et al.[5] A more general theory of resonant four-wave interactions was developed by Zakharov[6] and has been further extended by numerous researchers. See, for example, an extensive review of Dias and Kharif[7] and a textbook of Mei et al.[8]

While it can be neglected for surface waves of relatively long wavelengths, the surface tension must be included for short waves of a few centimeters or less. For surface gravity-capillary waves, as shown in Ref. 9, resonant wave interactions occur among three waves that satisfy the following conditions for wavenumber vectors $\boldsymbol{k}_j$ and wave frequencies ω_j $(j = 1, 2, 3)$

$$\boldsymbol{k}_1 = \boldsymbol{k}_2 + \boldsymbol{k}_3, \qquad \omega_1 = \omega_2 + \omega_3, \tag{1}$$

where ω_j and $k_j = |\boldsymbol{k}_j|$ satisfy the linear dispersion relation given by

$$\omega_j^2 = g_j k_j T_j, \qquad g_j = g + \sigma k_j^2, \qquad k_j = |\boldsymbol{k}_j|, \qquad T_j = \tanh(k_j d), \tag{2}$$

with g and σ being the gravitational acceleration and the surface tension coefficient divided by the fluid density, respectively, and d is the water depth.

The nonlinear evolution of a resonant triad has been studied theoretically using a system of three equations for the complex wave amplitudes of the triad $\mathcal{A}_j$ $(j = 1, 2, 3)$. The system was first obtained via a method of multiple scale expansion[9] or a variational formulation.[10] McGoldrick[9] showed that the solutions of the amplitude equations can be written in terms of Jacobian elliptic functions and, in general, the resonant triad exchanges their energies (proportional to $|\mathcal{A}_j|^2$ with $\mathcal{A}_j = |\mathcal{A}_j| \exp(\mathrm{i}\varphi_j)$) periodically in time, while the total energy is conserved in the absence of viscosity. Laboratory experiments were also performed by McGoldrick,[11] Bannerjee and Korpel,[12] Henderson and Hammack,[13] and Perlin et al.[14] They observed the periodic exchange of energy through (often successive) resonant triad interactions. See a review of Hammack and Henderson.[15]

In this paper, we consider a special resonant triad interaction during which no energy exchange between the three waves occurs. In other words, even though they interact resonantly, their amplitudes or $|\mathcal{A}_j|$ remain unchanged in time. The only sign of their interaction can be observed in their phases φ_j. When this happens, each wave under this special resonance would propagate with its own constant speed whose nonlinear correction to the linear wave speed is proportional to wave steepness, as discussed in detail in Section 5. This is more significant than Stokes's correction for monochromatic waves that is proportional to the square of wave steepness. Simmons[10] discussed possible triad solutions of constant amplitudes $|\mathcal{A}_j|$ in addition to time-periodic and constant-phase (φ_j) solutions. Nevertheless, no explicit conditions for the constant amplitude solutions were given. Recently, similar constant-amplitude solutions were studied for the resonant four- and five-wave interactions of gravity waves by Xu et al.[16] and Shrira et al.,[17] respectively.

While the resonant waves of constant amplitudes propagate with constant wave speeds, each of them has a different wave speed and a different propagation direction. Therefore, the resulting wave field is in general unsteady. For a resonant triad of constant amplitudes to become stationary in a moving reference frame, or to become a traveling wave, more restrictive conditions are required beyond the constant amplitude condition. Such solution was studied for one-dimensional gravity-capillary waves by Wilton,[18] who showed that the traveling wave solution can be found when the self-interaction of the first harmonic of wavenumber k_2 $(= k_3)$ generates the wave frequency of the second harmonic $(2k_2 = k_1)$ at the second order such that $2\omega(k_2) = \omega(k_1)$. This condition is satisfied when $k_2 = (g/2\sigma)^{1/2}$

and the corresponding solution often referred to as Wilton ripples was described in terms of a special resonant triad of constant amplitudes by McGoldrick.[19]

Here, we explore explicit conditions for resonant triads of two-dimensional gravity-capillary waves that exchange no energy during their interactions. Furthermore, we attempt to find special resonant triads producing traveling wave fields in two horizontal dimensions, which can be considered a generalization of one-dimensional Wilton ripples.

Starting with a second-order nonlinear pseudo-spectral model and its Hamiltonian in Section 2, the amplitude equations for multiple resonant triad interactions are rederived and are compared with those obtained previously using different methods in Section 3. With re-examining the resonant conditions given by (1), we identify in Section 4 an explicit region of resonance in a two-dimensional parameter space and describe the variations of wavenumbers and wave frequencies in the region. After finding the conditions for resonant triads of no energy exchange, we find traveling wave solutions resulting from such resonant interactions in Section 5.

2 | SECOND-ORDER MODELS FOR GRAVITY-CAPILLARY WAVES

2.1 | Pseudo-spectral model

The weakly nonlinear evolution of surface gravity-capillary waves of small wave steepness can be described by a second-order asymptotic model (eg, Ref. 20) written, in terms of the surface displacement $\zeta(\boldsymbol{x}, t)$ and the surface velocity potential $\Phi(\boldsymbol{x}, t)$, as

$$\frac{\partial \zeta}{\partial t} = -\mathcal{L}[\Phi] - \boldsymbol{\nabla}\cdot(\zeta\boldsymbol{\nabla}\Phi) - \mathcal{L}[\zeta\mathcal{L}[\Phi]], \qquad \frac{\partial \Phi}{\partial t} = -g\,\zeta + \sigma\nabla^2\zeta - \tfrac{1}{2}\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi + \tfrac{1}{2}(\mathcal{L}[\Phi])^2, \tag{3}$$

where ∇ is the horizontal gradient and $\mathcal{L}$ is the linear operator defined by

$$\mathcal{L}[f] = \int_{-\infty}^{\infty} K(\boldsymbol{x}-\boldsymbol{\xi})\, f(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}, \qquad \mathcal{F}[K(\boldsymbol{x})] = -kT, \quad \mathcal{L}\left[\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\right] = -kT\,\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{4}$$

with $\mathcal{F}$ representing the Fourier transform, $k = |\boldsymbol{k}|$, and $T = \tanh kd$. Although the system given by (3) can be extended to arbitrary order of nonlinearity,[21–23] the second-order model valid to $O(\epsilon^2)$ with ϵ being the wave steepness (defined by the ratio of wave amplitude to characteristic wavelength) is adopted here to study resonant three-wave interactions that occur at the second order of nonlinearity.

The second-order system given by (3) can also be written as Hamilton's equations:[6]

$$\frac{\partial \zeta}{\partial t} = \frac{\delta E}{\delta \Phi}, \qquad \frac{\partial \Phi}{\partial t} = -\frac{\delta E}{\delta \zeta}, \tag{5}$$

where the total energy E can be written as $E = E_2 + E_3 + O(\epsilon^4)$ with $E_n = O(\epsilon^n)$ given by

$$E_2 = \frac{1}{2}\int\left(g\,\zeta^2 - \Phi\mathcal{L}[\Phi] + \sigma\nabla\zeta\cdot\nabla\zeta\right)\mathrm{d}\boldsymbol{x}, \qquad E_3 = \frac{1}{2}\int\left\{\zeta\nabla\Phi\cdot\nabla\Phi - \zeta(\mathcal{L}[\Phi])^2\right\}\mathrm{d}\boldsymbol{x}. \tag{6}$$

2.2 | Spectral model

The second-order model (3) can be written in spectral space, when ζ and Φ are expressed as

$$\zeta(\boldsymbol{x},t) = \int a(\boldsymbol{k},t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{k}, \qquad \Phi(\boldsymbol{x},t) = \int b(\boldsymbol{k},t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{k}, \tag{7}$$

where $a(\boldsymbol{k},t)$ and $b(\boldsymbol{k},t)$ represent the Fourier transforms of ζ and Φ, respectively. As ζ and Φ are real functions, one can see that $a(-\boldsymbol{k},t) = a^*(\boldsymbol{k},t)$ and $b(-\boldsymbol{k},t) = b^*(\boldsymbol{k},t)$, where the asterisks represent complex conjugates. Then, by taking the Fourier transform of (3), one can obtain the second-order evolution equations for $a(\boldsymbol{k},t)$ and $b(\boldsymbol{k},t)$ in spectral space as

$$\frac{\partial a}{\partial t} - kT\,b = \iint \alpha_{0,1,2}\,b_1\,a_2\,\delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2}, \quad \frac{\partial b}{\partial t} + (g+\sigma k^2)a = \iint \beta_{0,1,2}\,b_1\,b_2\,\delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2}, \tag{8}$$

where $\mathrm{d}\boldsymbol{k}_{1,2} = \mathrm{d}\boldsymbol{k}_1\mathrm{d}\boldsymbol{k}_2$ and the coefficients $\alpha_{0,1,2}$ and $\beta_{0,1,2}$ are given by

$$\alpha_{0,1,2} = \boldsymbol{k}_0\cdot\boldsymbol{k}_1 - k_0k_1T_0T_1, \qquad \beta_{0,1,2} = \frac{1}{2}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2 + k_1k_2T_1T_2), \tag{9}$$

with $\boldsymbol{k}_0 = \boldsymbol{k}$ and $T_0 = T = \tanh(kd)$. In (8) and (9), just for brevity, the following short-hand notations have been used

$$a_j = a(\boldsymbol{k}_j,t), \quad b_j = b(\boldsymbol{k}_j,t), \quad \delta_{0-1-2} = \delta(\boldsymbol{k}_0 - \boldsymbol{k}_1 - \boldsymbol{k}_2), \tag{10}$$

where $\delta(\boldsymbol{k})$ is the Dirac delta function, and k_j and T_j are defined in (2).

As expected from (5), Equation (8) can also be written as a Hamiltonian system:

$$\frac{\partial a}{\partial t} = \frac{\delta H}{\delta b^*}, \qquad \frac{\partial b}{\partial t} = -\frac{\delta H}{\delta a^*}, \tag{11}$$

where the Hamiltonian H is given by $H = H_2 + H_3 + O(\epsilon^3)$ with $H_n = E_n/(2\pi)^2$ given, from (6), by

$$H_2 = \frac{1}{2}\int [(g+\sigma k^2)a\,a^* + kT\,b\,b^*]\,\mathrm{d}\boldsymbol{k}, \quad H_3 = \frac{1}{2}\iiint h_{1,2,3}\,b_1\,b_2\,a_3\,\delta_{1+2+3}\,\mathrm{d}\boldsymbol{k}_{1,2,3}. \tag{12}$$

As the subscripts 1 and 2 can be interchanged without altering the integrals, $h_{1,2,3}$ given by

$$h_{1,2,3} = -(\boldsymbol{k}_1\cdot\boldsymbol{k}_2 + k_1k_2T_1T_2), \tag{13}$$

satisfies the symmetry condition of $h_{1,2,3} = h_{2,1,3}$. Then, from (11), $\alpha_{0,1,2}$ and $\beta_{0,1,2}$ defined in (9) can be expressed, in terms of $h_{1,2,3}$, as

$$\alpha_{0,1,2} = \frac{1}{2}(h_{-0,1,2} + h_{1,-0,2}), \qquad \beta_{0,1,2} = -\frac{1}{2}h_{1,2,-0}. \tag{14}$$

The evolution equations for a and b given by (8) were previously obtained by Krasitskii[24] directly from the approximate Hamiltonians given by (12). Here, it is shown that the same evolution equations can be obtained from the pseudo-spectral model and therefore the two formulations are equivalent. In general, when it is combined with fast Fourier transform, the pseudo-spectral model is useful for numerical computations, while the spectral model is advantageous for theoretical analysis.

3 | AMPLITUDE EQUATIONS FOR RESONANT TRIADS

Although the complex amplitude equations for a single resonant triad have been previously obtained by McGoldrick[9] and Simmons,[10] we present, from a Hamiltonian system (11) of Krasitskii,[24] a relatively simple derivation of the complex amplitude equations for resonant triads of continuous spectrum without introducing Krasitskii's canonical transformation. Some differences of the resulting amplitude equations from the previous models are briefly discussed.

3.1 | Reduced Hamiltonian for resonant three-wave interactions

As shown by Zakharov,[6] when introducing $z(\boldsymbol{k},t)$ defined by

$$a = \left(\frac{\omega_k}{2g_k}\right)^{1/2} [z(\boldsymbol{k},t) + z^*(-\boldsymbol{k},t)], \qquad b = \mathrm{i}\left(\frac{g_k}{2\omega_k}\right)^{1/2} [z(\boldsymbol{k},t) + z^*(-\boldsymbol{k},t)], \tag{15}$$

with $g_k = g + \sigma k^2$ and $\omega_k^2 = g_k kT$, Hamilton's equations (11) can be reduced to a single equation for $z(\boldsymbol{k},t)$:

$$\frac{\partial z}{\partial t} = \mathrm{i}\,\frac{\delta H}{\delta z^*}. \tag{16}$$

When (15) is substituted into (12), the Hamiltonian H can be expressed, in terms of $z(\boldsymbol{k},t)$, as[24]

$$\begin{aligned} H = &\int \omega_k\, zz^*\, \mathrm{d}\boldsymbol{k} + \iiint U^{(1)}_{1,2,3}\, \left(z_1^*\, z_2\, z_3 + z_1\, z_2^*\, z_3^*\right)\, \delta_{1-2-3}\, \mathrm{d}\boldsymbol{k}_{1,2,3} \\ &+ \frac{1}{3}\iiint U^{(2)}_{1,2,3}\, \left(z_1^*\, z_2^*\, z_3^* + z_1\, z_2\, z_3\right)\, \delta_{1+2+3}\, \mathrm{d}\boldsymbol{k}_{1,2,3}, \end{aligned} \tag{17}$$

where $z_j = z(\boldsymbol{k}_j,t)$ and $U^{(1)}_{1,2,3}$ and $U^{(2)}_{1,2,3}$ are given by

$$U^{(1)}_{1,2,3} = U_{2,3,-1} - U_{-1,2,3} - U_{3,-1,2}, \qquad U^{(2)}_{1,2,3} = U_{1,2,3} + U_{2,3,1} + U_{3,1,2}, \tag{18}$$

with

$$U_{1,2,3} = -\left(\frac{g_1 g_2\, \omega_3}{32\,\omega_1\omega_2\, g_3}\right)^{1/2} h_{1,2,3} = \left(\frac{g_1 g_2\, \omega_3}{32\,\omega_1\omega_2\, g_3}\right)^{1/2} \left(\boldsymbol{k}_1\cdot\boldsymbol{k}_2 + \frac{\omega_1^2\omega_2^2}{g_1 g_2}\right). \tag{19}$$

In (18), $U^{(j)}_{1,2,3}$ $(j = 1,2)$ are symmetric[24] in the sense that

$$U^{(1)}_{1,2,3} = U^{(1)}_{1,3,2}, \qquad U^{(2)}_{1,2,3} = U^{(2)}_{2,1,3} = U^{(2)}_{3,1,2}. \tag{20}$$

When $z(\boldsymbol{k},t)$ is introduced in (15), only a single amplitude equation given by (16) needs to be solved and, therefore, is more convenient for further analysis than the original system (8) for two variables a and b. Meanwhile, when solved numerically, both (8) and (16) require the same computational efforts because $z(\boldsymbol{k},t)$ has no relationship with its complex conjugate, while $a(-\boldsymbol{k},t) = a^*(\boldsymbol{k},t)$ and $b(-\boldsymbol{k},t) = b^*(\boldsymbol{k},t)$.

When $z(\boldsymbol{k},t)$ is written as

$$z(\boldsymbol{k},t) = \mathcal{Z}(\boldsymbol{k},t)\,\mathrm{e}^{\mathrm{i}\,\omega t}, \tag{21}$$

the Hamiltonian H can be expressed, from (17), in terms of $\mathcal{Z}$, as

$$H = \int \omega_k \mathcal{Z}\mathcal{Z}^* \mathrm{d}\boldsymbol{k} + \iiint U^{(1)}_{1,2,3} \left(\mathcal{Z}^*_1 \mathcal{Z}_2 \mathcal{Z}_3 \, \mathrm{e}^{-\mathrm{i}(\omega_1-\omega_2-\omega_3)t} + \mathcal{Z}_1 \mathcal{Z}^*_2 \mathcal{Z}^*_3 \, \mathrm{e}^{\mathrm{i}(\omega_1-\omega_2-\omega_3)t} \right) \delta_{1-2-3} \, \mathrm{d}\boldsymbol{k}_{1,2,3}$$
$$+ \frac{1}{3} \iiint U^{(2)}_{1,2,3} \left(\mathcal{Z}^*_1 \mathcal{Z}^*_2 \mathcal{Z}^*_3 \, \mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2+\omega_3)t} + \mathcal{Z}_1 \mathcal{Z}_2 \mathcal{Z}_3 \, \mathrm{e}^{\mathrm{i}(\omega_1+\omega_2+\omega_3)t} \right) \delta_{1+2+3} \, \mathrm{d}\boldsymbol{k}_{1,2,3}, \tag{22}$$

where $\mathcal{Z}_j = \mathcal{Z}(\boldsymbol{k}_j, t)$. Following Zakharov,[6] under the resonant conditions (1), the exponential term (oscillating in fast time) disappears. Therefore, the second integral denoted by $\mathcal{H}$:

$$\mathcal{H} = \iiint U^{(1)}_{1,2,3} \left(\mathcal{Z}^*_1 \mathcal{Z}_2 \mathcal{Z}_3 + \mathcal{Z}_1 \mathcal{Z}^*_2 \mathcal{Z}^*_3 \right) \delta_{1-2-3} \, \mathrm{d}\boldsymbol{k}_{1,2,3}, \tag{23}$$

can be considered the Hamiltonian responsible for the slowly varying time evolution of resonant triads. On the other hand, the last integral in (22) represents nonresonant wave interactions that vary in fast time. Notice that $\mathcal{H}$ denotes the reduced Hamiltonian given by (23), while H represents the original Hamiltonian given by (17). Then, from (16) and (21), the evolution equation for resonant triad interactions can be obtained as

$$\frac{\partial \mathcal{Z}}{\partial t} = \mathrm{i} \, \frac{\delta \mathcal{H}}{\delta \mathcal{Z}^*}, \tag{24}$$

which yields

$$\frac{\partial \mathcal{Z}}{\partial t} = \mathrm{i} \iint U^{(1)}_{0,1,2} \, \mathcal{Z}_1 \, \mathcal{Z}_2 \, \delta_{0-1-2} \, \mathrm{d}\boldsymbol{k}_{1,2} + 2\,\mathrm{i} \iint U^{(1)}_{2,1,0} \, \mathcal{Z}^*_1 \, \mathcal{Z}_2 \, \delta_{0+1-2} \, \mathrm{d}\boldsymbol{k}_{1,2}, \tag{25}$$

where $U^{(1)}_{2,0,1} = U^{(1)}_{2,1,0}$ has been used for the last integral and $\mathcal{Z}(\boldsymbol{k}, t)$ is assumed to vary slowly in time. This is the Zakharov equation for resonant gravity-capillary waves, which describes the time evolution of all possible resonant triads in spectral space. In addition to conservation of the reduced Hamiltonian $\mathcal{H}$:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\delta \mathcal{H}}{\delta \mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial t} + \frac{\delta \mathcal{H}}{\delta \mathcal{Z}^*} \frac{\partial \mathcal{Z}^*}{\partial t} = 0, \tag{26}$$

it can be shown that (25) conserves energy and momentum:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \omega \, |\mathcal{Z}|^2 \, \mathrm{d}\boldsymbol{k} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t} \int \boldsymbol{k} \, |\mathcal{Z}|^2 \, \mathrm{d}\boldsymbol{k} = 0, \tag{27}$$

where the resonance conditions (1) along with the symmetry condition for $U^{(1)}_{0,1,2}$ in (20) have been used.

With a canonical transformation to eliminate the last integral of (17) at the second order, Krasitskii[24] obtained the same Hamiltonian as (23) in terms of a new canonical variable, but his evolution equation for the new variable describes both resonant and nonresonant wave interactions (or both fast and slow time evolutions) when the original variable $z(\boldsymbol{k}, t)$ is recovered. As we are interested in the evolution of the slowly varying complex amplitudes of resonant waves, Equation (25) is written for $Z(\boldsymbol{k}, t)$, not the canonical variable of Krasitskii.[24]

Previously, in their amplitude equations for a single resonant triad, McGoldrick[9] and Simmons[10] used a different slowly varying complex amplitude $\mathcal{A}(\boldsymbol{k},t)$, but it can be shown that $\mathcal{A}$ and $\mathcal{Z}$ are related as

$$\mathcal{Z}(\boldsymbol{k},t) = \left(\frac{2g_k}{\omega}\right)^{1/2} \mathcal{A}(\boldsymbol{k},t). \tag{28}$$

In Appendix A, the amplitude equation of $\mathcal{A}(\boldsymbol{k},t)$ is discussed in comparison with that of $\mathcal{Z}(\boldsymbol{k},t)$.

To derive (25), $\mathcal{Z}(\boldsymbol{k},t)$ is assumed to vary slowly in time, but no assumption on its bandwidth in spectral space (or, equivalently, its spatial variation in physical space) has been imposed. Further reductions can be made when the bandwidths around discrete resonant wavenumbers are non-zero (although they are assumed small), or in particular when the amplitudes of resonant waves vary slowly in space. For example, Simmons[10] obtained a system of three partial differential equations for the amplitudes of a single resonant triad that vary slowly in both time and space. Another interesting resonant triad interaction can occur between long gravity waves and short gravity-capillary waves in water of finite depth. A system of partial differential equations describing a long wave and the slowly varying envelope of short waves was proposed by Benney[25,26] and Djordjevic and Redekopp.[27] It is worthwhile to remark that these reduced models can be readily obtained from the slowly varying amplitude equation (25) by assuming that the wave spectrum is narrow-banded.

In the following discussions, focusing on discrete wave modes, (25) will be reduced to a system of ordinary differential equations using discrete approximations to $\mathcal{Z}$.

3.2 | Discrete spectrum

When we assume that a nonlinear wave field can be represented by a superposition of N discrete modes so that $\mathcal{Z}(\boldsymbol{k},t)$ can be written as

$$\mathcal{Z}(\boldsymbol{k},t) = \sum_{j=-N}^{N} \mathcal{Z}_j(t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_j), \tag{29}$$

with $\boldsymbol{k}_{-j} = -\boldsymbol{k}_j$, the reduced Hamiltonian is given by

$$\begin{aligned}\mathcal{H} &= \sum_{l,m,n>0} U^{(1)}_{l,m,n}\left(\mathcal{Z}^*_l\,\mathcal{Z}_m\,\mathcal{Z}_n + \mathcal{Z}_l\,\mathcal{Z}^*_m\,\mathcal{Z}^*_n\right)\delta_{l-m-n} \\ &= 2\sum_{l,m,n>0} U^{(1)}_{l,m,n}\,|\mathcal{Z}_l||\mathcal{Z}_m||\mathcal{Z}_n|\cos\Delta_{l,m,n}\,\delta_{l-m-n},\end{aligned} \tag{30}$$

where $\mathcal{Z}_j$ is expressed as

$$\mathcal{Z}_j = |\mathcal{Z}_j|\,\mathrm{e}^{\mathrm{i}\varphi_j}, \tag{31}$$

and $\Delta_{l,m,n}$ is defined by $\Delta_{l,m,n} = \varphi_l - \varphi_m - \varphi_n$. Then, from

$$\mathrm{d}\mathcal{Z}_j/\mathrm{d}t = \mathrm{i}\,(\delta\mathcal{H}/\delta\mathcal{Z}^*_j), \tag{32}$$

the amplitude equations can be obtained as

$$\frac{\mathrm{d}\mathcal{Z}_j}{\mathrm{d}t} = \mathrm{i}\sum_{m,n>0}\left(U^{(1)}_{j,m,n}\,\mathcal{Z}_m\,\mathcal{Z}_n\,\delta_{j-m-n} + 2\,U^{(1)}_{n,m,j}\,\mathcal{Z}^*_m\,\mathcal{Z}_n\,\delta_{j+m-n}\right), \tag{33}$$

where $U^{(1)}_{n,j,m} = U^{(1)}_{n,m,j}$ has been used. From (33), the amplitude equations for any number of resonant triads can be obtained.

For a single resonant triad ($N = 3$), the reduced Hamiltonian $\mathcal{H}$ is given, from (30), by

$$\mathcal{H} = 2\,U^{(1)}_{1,2,3}\,\left(\mathcal{Z}_1^*\mathcal{Z}_2\mathcal{Z}_3 + \mathcal{Z}_1\mathcal{Z}_2^*\mathcal{Z}_3^*\right) = 4\,U^{(1)}_{1,2,3}\,|\mathcal{Z}_1||\mathcal{Z}_2||\mathcal{Z}_3|\cos\Delta, \tag{34}$$

where Δ is defined as

$$\Delta = \varphi_1 - \varphi_2 - \varphi_3. \tag{35}$$

From this Hamiltonian, the three amplitude equations can be obtained, from (33), as

$$\frac{\mathrm{d}\mathcal{Z}_1}{\mathrm{d}t} = \mathrm{i}\,\gamma_0\,\mathcal{Z}_2\mathcal{Z}_3, \quad \frac{\mathrm{d}\mathcal{Z}_2}{\mathrm{d}t} = \mathrm{i}\,\gamma_0\,\mathcal{Z}_1\mathcal{Z}_3^*, \quad \frac{\mathrm{d}\mathcal{Z}_3}{\mathrm{d}t} = \mathrm{i}\,\gamma_0\,\mathcal{Z}_1\mathcal{Z}_2^*, \tag{36}$$

where $\gamma_0 = 2\,U^{(1)}_{1,2,3}$ is given by

$$\gamma_0 = \left(\frac{g_1 g_2 \omega_3}{8\omega_1\omega_2 g_3}\right)^{1/2} h_{-1,2,3} + \left(\frac{g_3 g_1 \omega_2}{8\omega_3\omega_1 g_2}\right)^{1/2} h_{3,-1,2} - \left(\frac{g_2 g_3 \omega_1}{8\omega_2\omega_3 g_1}\right)^{1/2} h_{2,3,-1}, \tag{37}$$

with $h_{1,2,3}$ given by (13). From (1) with $\omega_j > 0$, it should be noticed that $\mathcal{Z}_1$ is the amplitude of the highest frequency mode in the triad. As the coefficients of the three amplitude equations (36) are the same, they can be made one by rescaling t if convenient. As shown in Appendix A, the coefficients are all different if $\mathcal{A}_j$ are used for the amplitudes.

As discussed previously for continuous spectrum, in addition to conservation of the reduced Hamiltonian $\mathcal{H}$ for a resonant triad, the amplitude equations given by (36) have two additional conservation laws, which can be written, from (27), as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{j=1}^{3}\omega_j\left|\mathcal{Z}_j\right|^2\right] = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{j=1}^{3}\boldsymbol{k}_j\left|\mathcal{Z}_j\right|^2\right] = 0. \tag{38}$$

It can be shown that these conservation laws along with the resonance conditions (1) are equivalent to the Manley-Rowe relations,[28] which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|\mathcal{Z}_1|^2 + |\mathcal{Z}_2|^2\right) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(|\mathcal{Z}_1|^2 + |\mathcal{Z}_3|^2\right) = 0. \tag{39}$$

Hereafter, with focusing on gravity-capillary waves whose wavelengths are generally small compared with water depth, we assume $k_j d \gg 1$, or, equivalently, $d \to \infty$, for which γ_0 is given by

$$\gamma_0 = \left(\frac{\omega_1\omega_2\omega_3}{8k_1k_2k_3}\right)^{1/2}\left[\frac{k_3}{\omega_3}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2) + \frac{k_2}{\omega_2}(\boldsymbol{k}_3\cdot\boldsymbol{k}_1) + \frac{k_1}{\omega_1}(\boldsymbol{k}_2\cdot\boldsymbol{k}_3) + k_1k_2k_3\left(\frac{1}{\omega_1} - \frac{1}{\omega_2} - \frac{1}{\omega_3}\right)\right], \tag{40}$$

where we have used the linear dispersion relation in deep water, $\omega_j^2 = g_j k_j$.

4 | RESONANCE CONDITIONS IN DEEP WATER

Simmons[10] discussed graphically how to find a resonant triad, but explicit expressions of $\boldsymbol{k}_j$ satisfying the resonance conditions are often convenient. When $\boldsymbol{k}_j$ are written as

$$\boldsymbol{k}_j = k_j\,(\cos\theta_j, \sin\theta_j), \tag{41}$$

there are six unknowns (k_j and θ_j for $j = 1, 2, 3$) among which three parameters can be chosen arbitrarily after the three scalar conditions of resonance given by (1) are applied. After assuming $\theta_1 = 0$ (for the highest frequency mode) without loss of generality, one should be able to express k_j ($j = 1, 2, 3$) in terms of θ_2 and θ_3. Previously, McGoldrick[9] described the resonance conditions in terms of, for example, k_2 and k_3. As can be seen in the followings, the description in terms of θ_2 and θ_3 is convenient to identify a region of resonance, where all resonant triads reside, and to obtain explicit expressions of $\boldsymbol{k}_j$ from a bi-quadratic equation.

4.1 | General case

After nondimensionalizing k_j and ω_j as

$$K_j = (\sigma/g)^{1/2}\,k_j, \quad \Omega_j = \left(\sigma/g^3\right)^{1/4}\omega_j, \tag{42}$$

the linear dispersion relation is given by

$$\Omega_j^2 = K_j\left(1 + K_j^2\right). \tag{43}$$

Then, the resonance conditions (1) can be rewritten as

$$K_2\cos\theta_2 + K_3\cos\theta_3 = K_1, \qquad K_2\sin\theta_2 + K_3\sin\theta_3 = 0, \tag{44}$$

$$K_1^{1/2}\left(1 + K_1^2\right)^{1/2} = K_2^{1/2}\left(1 + K_2^2\right)^{1/2} + K_3^{1/2}\left(1 + K_3^2\right)^{1/2}, \tag{45}$$

where $\Omega_j > 0$ have been assumed. Then, K_2 and K_3 can be found as

$$K_2 = \frac{\sin\theta_3}{\sin\theta_{32}}K_1, \qquad K_3 = -\frac{\sin\theta_2}{\sin\theta_{32}}K_1, \tag{46}$$

where $\theta_{32} = \theta_3 - \theta_2$ vanishes only for one-dimensional waves, which will be discussed separately.

When (46) is substituted into (45), an equation for K_1 can be obtained, in terms of θ_2 and θ_3, as a bi-quadratic equation given by

$$\left[\left(1 + \frac{\sin^3\theta_2}{\sin^3\theta_{32}} - \frac{\sin^3\theta_3}{\sin^3\theta_{32}}\right)^2 + 4\frac{\sin^3\theta_2\sin^3\theta_3}{\sin^6\theta_{32}}\right]K_1^4$$
$$+ 2\left[\left(1 + \frac{\sin\theta_2}{\sin\theta_{32}} - \frac{\sin\theta_3}{\sin\theta_{32}}\right)\left(1 + \frac{\sin^3\theta_2}{\sin^3\theta_{32}} - \frac{\sin^3\theta_3}{\sin^3\theta_{32}}\right)\right.$$

$$+ 2\left(\frac{\sin^2\theta_2}{\sin^2\theta_{32}} + \frac{\sin^2\theta_3}{\sin^2\theta_{32}}\right)\frac{\sin\theta_2\sin\theta_3}{\sin^2\theta_{32}}\Bigg] K_1^2$$

$$+ \left[\left(1 + \frac{\sin\theta_2}{\sin\theta_{32}} - \frac{\sin\theta_3}{\sin\theta_{32}}\right)^2 + 4\frac{\sin\theta_2\sin\theta_3}{\sin^2\theta_{32}}\right] = 0. \tag{47}$$

Before finding a region in the (θ_2, θ_3)-plane where Equation (47) has positive real roots for K_1^2, one should notice that, because K_2 and K_3 in (46) must be positive, the resonance region must be contained inside two triangular regions in the second and fourth quadrants of the (θ_2, θ_3)-plane, bounded by

$$\theta_3 = \theta_2 + \pi, \qquad -\pi \le \theta_2 \le 0, \qquad 0 \le \theta_3 \le \pi, \tag{48}$$

and

$$\theta_3 = \theta_2 - \pi, \qquad 0 \le \theta_2 \le \pi, \qquad -\pi \le \theta_3 \le 0. \tag{49}$$

As it can be confirmed numerically that the second and third coefficients of the quadratic equation for K_1^2 are negative inside these regions while the discriminant is always positive, (47) would have only one positive root for K_1^2 along with one negative (or nonphysical) root when the first coefficient is positive:

$$f(\theta_2, \theta_3) \equiv \left(\sin^3\theta_{32} + \sin^3\theta_2 - \sin^3\theta_3\right)^2 + 4\sin^3\theta_2\sin^3\theta_3 > 0. \tag{50}$$

Otherwise, no positive real solutions for K_1^2 can be found. One more care must be taken as (47) is equivalent to $(\omega_1^2 - \omega_2^2 - \omega_3^2)^2 = (\pm 2\omega_2\omega_3)^2$. In other words, one should exclude a region, where $\Omega_1 = \Omega_2 - \Omega_3$, instead of $\Omega_1 = \Omega_2 + \Omega_3$, as $\boldsymbol{K}_1 = \boldsymbol{K}_2 + \boldsymbol{K}_3$ has been assumed.

Figure 1 shows the resonance region (shaded) in the (θ_2, θ_3)-plane, in which a single positive solution for K_1 always exists. Therefore, all resonant triads must lie in the shaded region. The boundaries of the resonance region given by $f(\theta_2, \theta_3) = 0$ are tangent to the θ_2 and θ_3-axes at $\theta_2 = \pm\pi/2$ and $\theta_3 = \pm\pi/2$, respectively. Notice that the θ_2 and θ_3-axes should be excluded except for the origin (which corresponds to one-dimensional waves) because the resonance conditions given by (44) cannot be fulfilled if two waves are propagating in the x-direction, while the third wave is propagating obliquely from the x-axis.

Figures 2 and 3 show the variations of K_j and Ω_j inside the resonance region located in the fourth quadrant of the (θ_2, θ_3)-plane. The similar behaviors can be found in the second quadrant. In particular, Figure 3 showing contour lines of Ω_j are useful for experimental studies, where wave frequencies are controlled. To better understand Figure 3, we consider an experiment, where the frequency Ω_1 is fixed to be Ω_1^{exp}. Then, in Figure 3(A), along a contour line corresponding to the value of Ω_1^{exp} (chosen to be 3 and denoted by a dashed line), one can see the variation of θ_2 and θ_3 along the contour line, or the functional relationship between θ_2 and θ_3, for possible resonant triads. For a given value of Ω_1, notice that only a certain range of θ_2 (or θ_3) is allowed. By drawing the same contour line (dashed) for $\Omega_1 = \Omega_1^{\text{exp}}$ in Figure 3(B), the variation of Ω_2 can be observed from the intersections between the contour curve for $\Omega_1 = \Omega_1^{\text{exp}}$ and contour lines for Ω_2. Similarly, the variation of Ω_3 can be seen from Figure 3(C).

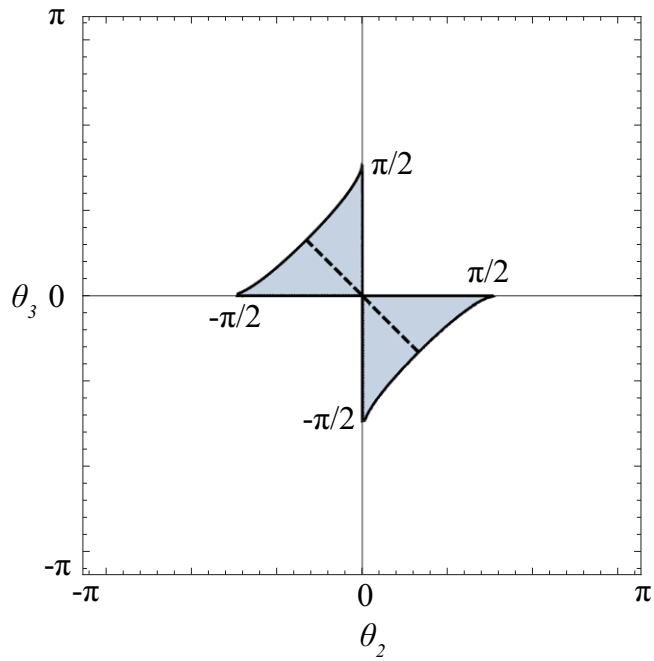

FIGURE 1 Region for resonant three-wave interactions (shaded area) in the (θ_2, θ_3)-plane defined by (48), (49), and $f(\theta_1, \theta_2) > 0$, where f is defined by (50). The dashed line represents the symmetric case of $\theta_3 = -\theta_2$

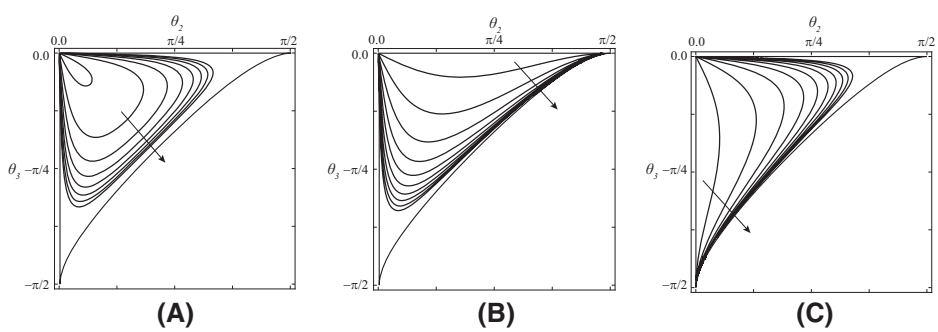

FIGURE 2 Contour plots of (dimensionless) resonant wavenumbers (K_j) in the fourth quadrant of the (θ_2, θ_3)-plane: (A) $3 \le K_1 \le 5$; (B) $0.5 \le K_2 \le 5$; (C) $0.5 \le K_3 \le 5$. The increment between the two neighboring contour levels is 0.5 and the arrows indicate the direction of increasing contour levels. Notice that the plot in (C) can be obtained from the plot in (B) by replacing θ_2 and θ_3 by $-\theta_3$ and $-\theta_2$, respectively, as there is no real distinction between θ_2 and θ_3

4.2 | One-dimensional waves

For one-dimensional waves, it is more convenient to find the relationship between K_2 and K_3 directly from the resonance condition (45):

$$9K_2K_3(K_2 + K_3)^2 - 4\left(1 + K_2^2\right)\left(1 + K_3^2\right) = 0, \tag{51}$$

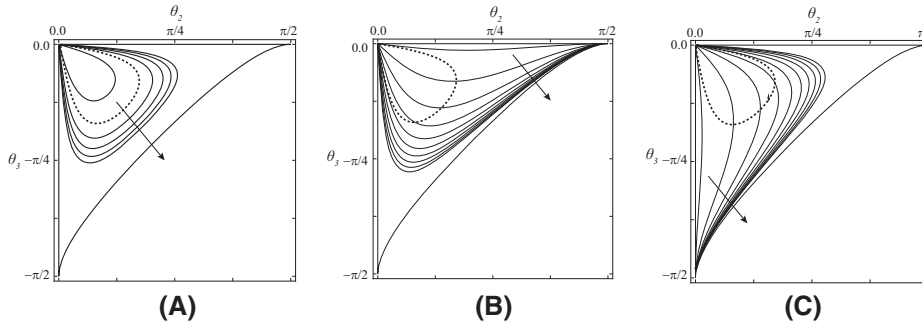

FIGURE 3 Contour plots of (dimensionless) resonant wave frequencies (Ω_j) in the fourth quadrant of the (θ_2, θ_3)-plane: (A) $2.5 \le \Omega_1 \le 5$; (B) $0.5 \le \Omega_2 \le 5$; (C) $0.5 \le \Omega_3 \le 5$. The increment between two neighboring contour levels is 0.5 and the arrows indicate the direction of increasing contour levels. To illustrate how to use these plots, as an example, the contour line of $\Omega_1 = 3$ is represented by a dashed curve in (A), which shows the relationship between θ_2 and θ_3 of all possible resonant triads with $\Omega_1 = 3$. Then, the values of Ω_2 and Ω_3 of the resonant triads with $\Omega_1 = 3$ can be determined by the levels of contour lines of Ω_2 and Ω_3 intersecting with the (dashed) contour line of $\Omega_1 = 3$, as shown in (B) and (C), respectively

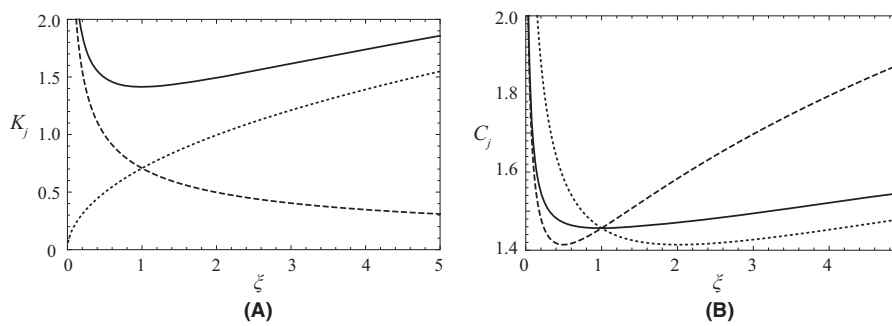

FIGURE 4 (A) Wavenumbers (K_j) and (B) linear wave speeds ($C_j = \Omega_j/K_j$) of one-dimensional resonant triads for varying ξ for $j = 1$ (solid), 2 (dashed), and 3 (dotted). Notice that the case of $\xi = 1$ corresponds to Wilton ripples and the variations of Ω_j are similar to those of K_j shown in (A)

with $K_1 = K_2 + K_3$ from (44). Notice that the real solution of (51) can be found only when K_j ($j = 1, 2, 3$) have the same sign, or the three waves are propagating in the same direction. More specifically, the solution of (51) can be parameterized in terms of ξ as

$$K_2^2 = \frac{2\left[(\xi^2+1) + (\xi^4 + 9\xi^3 + 16\xi^2 + 9\xi + 1)^{1/2}\right]}{\xi\,(9\xi^2 + 14\xi + 9)}, \quad K_3 = \xi K_2, \qquad K_1 = (1+\xi)K_2, \tag{52}$$

where ξ is any positive real constant in $0 < \xi < \infty$. This expression of K_2 can be found in Craik,[29] who suggested to write (52) as an algebraic equation for ξ whose coefficients depending on K_2. Alternatively, as (52) determines K_2 for any positive real value of ξ, the variations of K_j and C_j can be presented parametrically in terms of ξ, as shown in Figure 4. The variations of Ω_j for varying ξ are similar to those of K_j.

Of special interest is the case of $\xi = 1$, or, equivalently, $K_1 = 2K_2 = 2K_3 = 2^{1/2}$, for which the first ($K_2$) and second ($K_1 = 2K_2$) harmonics satisfy $\Omega_1 = 2\Omega_2$ from the resonance condition (1). This implies that the first and second harmonics for $\xi = 1$ have the same linear wave speed, or $\Omega_1/K_1 = \Omega_2/K_2$, The nonlinear behavior of this particular triad of $\xi = 1$ was studied by Wilton,[18] who showed

that the Stokes expansion for $K_2 = 2^{-1/2}$ becomes singular at the second order of nonlinearity. To avoid the singularity, Wilton[18] showed that the Stokes expansion needs to be modified to include the first (K_2) and second ($K_1 = 2K_2$) harmonics at the leading order, and found a traveling wave solution when the amplitude ratio of the second harmonic to the first is a half. These so-called Wilton ripples were described, in terms of resonant interaction of three copropagating waves, first by McGoldrick.[19] Finite amplitude solutions have been also obtained numerically in Schwartz and Vanden-Broeck.[30] More recent references can be found in Vanden-Broeck.[31]

4.3 | Symmetric two-dimensional waves

When the wavenumber vectors of two propagating waves ($\boldsymbol{k}_2$ and $\boldsymbol{k}_3$) are aligned symmetrically about $\boldsymbol{k}_1$, in other words, $\theta_2 = -\theta_3$, Equation (47) can be simplified to

$$\left(K_1^2 + 1\right)\left[\left(\cos^3\theta_3 - \tfrac{1}{2}\right)K_1^2 - (2 - \cos\theta_3)\cos^2\theta_3\right] = 0, \tag{53}$$

so that K_1 can be found as

$$K_1^2 = \frac{2(2 - \cos\theta_3)\cos^2\theta_3}{2\cos^3\theta_3 - 1} \qquad \text{for } -\theta_{\max} < \theta_3 < \theta_{\max}, \tag{54}$$

where the denominator must be positive so that the symmetric waves exist only when $|\theta_3| < \theta_{\max}$ with

$$\cos\theta_{\max} = 2^{-1/3}, \quad \text{or} \quad \theta_{\max} \simeq 37.467^o. \tag{55}$$

The same expression of the maximum angle was found by McGoldrick[9] from the resonance conditions for pure capillary waves. Because $K_1 \to \infty$ as $\theta \to \theta_{\max}$, the dispersion relation for gravity-capillary waves can be approximated by that for capillary waves as $\theta \to \theta_{\max}$. Therefore, the two expressions for $\theta_{\max}$ from the two dispersion relations coincide. Notice that the straight line in Figure 1 represents symmetric triads, and its intersection with the curve of $f(\theta_1, \theta_2) = 0$ defined by (50) represents the maximum angle $\theta_{\max}$.

For symmetric triads, from (43) and (46), the relationships between Ω_j's and K_j's are given by

$$\Omega_1 = 2\Omega_2 = 2\Omega_3, \quad K_1 = 2\cos\theta_3\, K_2, \quad K_2 = K_3. \tag{56}$$

Figure 5 shows the variations of K_j and Ω_j with the angle of symmetric waves, or θ_3, along with the linear wave speeds in the x-direction defined by $C_{jx} = \Omega_j / K_{jx}$ with $K_{jx} = K_j \cos\theta_j$. Both K_j and Ω_j increase with θ_3 to become infinity at $\theta_3 = \theta_{\max}$. For one-dimensional waves ($\theta_3 = 0$), the expression for K_1 given by (54) is equivalent to (52) with $\xi = 1$, for which the linear speeds of the three modes in the resonant triad coincide. For $\theta_3 \neq 0$, one can see, from (56), that the linear wave speeds in the x-direction defined by $C_{jx} = \Omega_j/(K_j \cos\theta_j)$ become identical, as shown in Figure 5(C). As seen for one-dimensional Wilton ripples, if the nonlinear wave speeds match, the symmetric resonant triad could form a wave field that travels in the x-direction with a constant speed although the wave field is transversely modulated. This will be discussed in detail in Section 5.2.

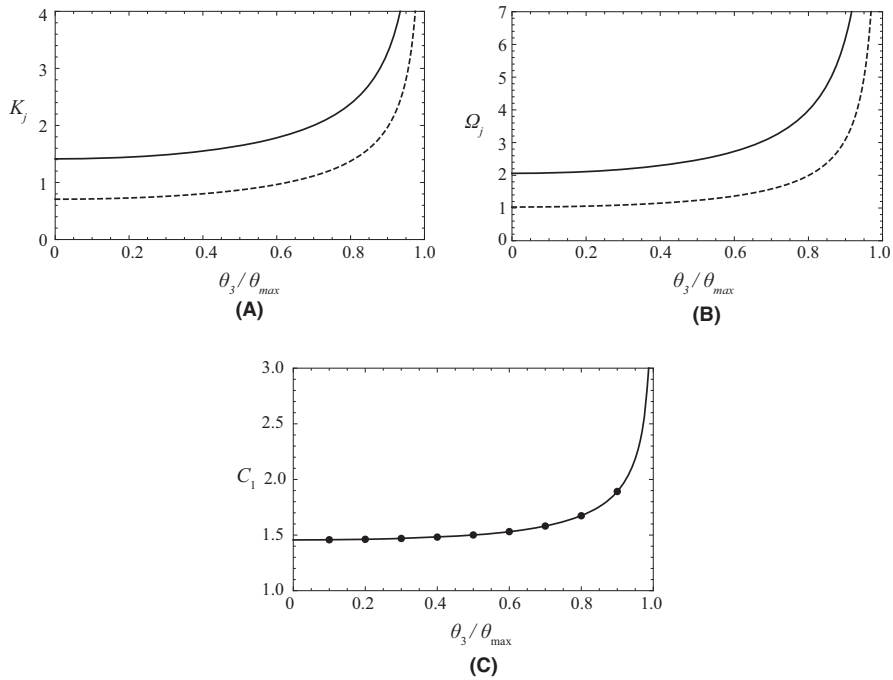

FIGURE 5 (A) Wavenumbers K_1 (solid line) and $K_2 = K_3$ (dashed) of symmetric resonant triads versus the angle of wave propagation ($\theta_3/\theta_{\max}$) with $\theta_{\max}$ defined by (55). (B) Wave frequencies Ω_1 (solid) and $\Omega_2 = \Omega_3$ (dashed) of symmetric resonant triads versus $\theta_3/\theta_{\max}$. (C) Linear wave speed C_1 versus $\theta_3/\theta_{\max}$. Notice that the wave speed of the K_1-mode given by $C_1 = \Omega_1/K_1$ (solid line) matches the x-components of C_2 and C_3, given by $C_{2x} = C_{3x} = \Omega_3/(K_3 \cos\theta_3)$ (symbols)

5 | RESONANT INTERACTIONS WITHOUT ENERGY EXCHANGE

Once the resonance conditions between three waves are satisfied, the amplitude equations are given, after nondimensionalizing (36), by

$$\frac{\mathrm{d}Z_1}{\mathrm{d}T} = \mathrm{i}\,\Gamma_0\, Z_2 Z_3, \quad \frac{\mathrm{d}Z_2}{\mathrm{d}T} = \mathrm{i}\,\Gamma_0\, Z_3^* Z_1, \quad \frac{\mathrm{d}Z_3}{\mathrm{d}T} = \mathrm{i}\,\Gamma_0\, Z_1 Z_2^*, \tag{57}$$

where $\mathcal{Z}_j$, t, and γ_0 are nondimensionalized as

$$Z_j = \mathcal{Z}_j/(\sigma^5/g^3)^{1/8}, \quad T = t/(\sigma/g^3)^{1/4}, \quad \Gamma_0 = \gamma_0/(g^9/\sigma^7)^{1/8}, \tag{58}$$

with Γ_0 given by (40) with replacing ω_j and k_j by Ω_j and K_j, respectively. If necessary, notice that Γ_0 can be scaled out from (57) by introducing a new dimensionless time, $T_0 = \Gamma_0 t$. It is well known that the amplitude equations given by (57) can be solved analytically in terms of elliptic functions (McGoldrick[9]) and, in general, the amplitudes oscillate periodically in time by exchanging energies proportional to $|Z_j|^2$.

An interesting question is if resonant interactions can occur without such energy exchange. If resonant interactions happen without energy exchange, $|Z_j|$ should be independent of time although its phase can vary in time so that $\mathrm{d}Z_j/\mathrm{d}t \neq 0$. Simmons[10] first noticed that it is possible to find time-independent solutions for $|Z_j|$, but provided little detailed description of resonant interactions with no

energy exchange. Later McGoldrick[19] described Wilton ripples as a special one-dimensional resonant triad for which the amplitudes remain constant, but his description was limited to a one-dimensional triad of $\xi = 1$.

Trivial resonant triads without energy exchange exist when any two modes have zero initial amplitudes, eg, $|Z_1| = |Z_2| = 0$, or $|Z_2| = |Z_3| = 0$. This implies that the third mode is a monochromatic wave train and should remain unchanged. If only one mode (eg, Z_1) has initially zero amplitude, the other two modes of nonzero wave amplitudes (Z_2 and Z_3) will interact resonantly and excite the mode that is initially absent, or Z_1, as can be seen from (57).

When all three resonant waves initially have nonzero amplitudes (no matter how small they are), the existence of resonant triad interactions without energy exchange would depend on their initial amplitudes and phases, which will be explored next.

5.1 | Fixed points

To find conditions under which no energy exchange occurs during resonant three wave interactions, it is useful to write the evolution equations for $|Z_j|$. Following Simmons,[10] the system for $|Z_j|$ can be written as

$$\frac{\mathrm{d}|Z_1|}{\mathrm{d}T} = \Gamma_0|Z_2||Z_3| \sin\Delta, \quad \frac{\mathrm{d}|Z_2|}{\mathrm{d}T} = -\Gamma_0|Z_1||Z_3| \sin\Delta, \quad \frac{\mathrm{d}|Z_3|}{\mathrm{d}T} = -\Gamma_0|Z_1||Z_2| \sin\Delta, \tag{59}$$

$$\frac{\mathrm{d}\Delta}{\mathrm{d}T} = \Gamma_0|Z_1||Z_2||Z_3| \left(\frac{1}{|Z_1|^2} - \frac{1}{|Z_2|^2} - \frac{1}{|Z_3|^2}\right) \cos\Delta, \tag{60}$$

where $\Delta = \varphi_1 - \varphi_2 - \varphi_3$, as defined in (35). Then, the evolution of φ_j ($j = 1, 2, 3$) is governed by

$$\frac{\mathrm{d}\varphi_j}{\mathrm{d}T} = \Gamma_0 \frac{|Z_p||Z_q|}{|Z_j|} \cos\Delta, \tag{61}$$

with p and q being two remaining indices different from j.

Looking for resonant triads that interact without exchange of energy is equivalent to finding fixed points of the coupled system given by (59) and (60). It can be easily seen that the fixed points exist, or $\mathrm{d}|Z_j|/\mathrm{d}t = 0$, when

$$\Delta = m\pi \quad \text{for } m = 0,\ 1, \tag{62}$$

$$\frac{1}{|Z_1|^2} - \frac{1}{|Z_2|^2} - \frac{1}{|Z_3|^2} = 0, \tag{63}$$

where $|Z_j| \neq 0$ have been assumed because the zero amplitude cases were previously discussed. When the two conditions for fixed points given by (62) and (63) are met, the right-hand side of (61) becomes time-independent and, then, φ_j can be obtained as

$$\varphi_j = \mu_j T + \varphi_{j,0}, \tag{64}$$

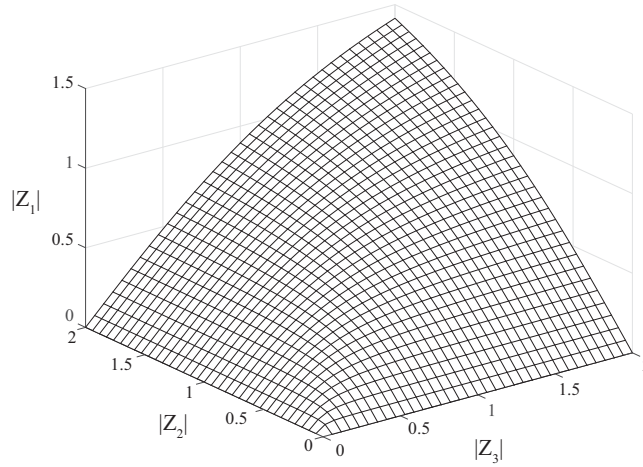

FIGURE 6 Surface of fixed points defined by (63). Any values of $|Z_j|$ on the surface are the fixed points of the system given by (59) and (60) if $\Delta_0 = m\pi$ ($m = 0, 1$)

where $\varphi_{j,0}$ are initial conditions for φ_j and μ_j represent (constant) nonlinear frequency corrections given by

$$\mu_j = (-1)^m \Gamma_0 \frac{|Z_p||Z_q|}{|Z_j|}. \tag{65}$$

When it is multiplied by $|Z_1||Z_2||Z_3|$, the condition (63) with (65) is equivalent to

$$\mu_1 - \mu_2 - \mu_3 = 0, \tag{66}$$

with which the condition for Δ given by (62) can be replaced by the condition for the initial phase difference $\Delta_0 = \varphi_{1,0} - \varphi_{2,0} - \varphi_{3,0}$:

$$\Delta_0 = m\pi \quad \text{for } m = 0, 1. \tag{67}$$

As shown in Figure 6, Equation (63) defines a surface in the three-dimensional $|Z_j|$-space on which fixed point solutions for $|Z_j|$ reside. Therefore, it can be concluded that any resonant triad can interact without exchange of energy if $|Z_j|$ are located on the surface shown in Figure 6 and their initial phase difference is 0 or π. Otherwise, the wave amplitudes will vary in time, or energy exchange occurs inside a resonant triad. It should be emphasized that the two conditions given by (63) and (67) must be met at the same time for no energy exchange.

Under the conditions for fixed points given by (63) and (67), each mode of constant amplitude has the wave speed given by $(\Omega_j + \mu_j)/K_j$. Therefore, the nonlinear correction (μ_j/K_j) resulting from the resonant interaction is $O(\epsilon)$ and is more significant than the Stokes correction that would appear at $O(\epsilon^2)$.

In general, even when $|Z_j|$ are independent of time, each mode in a resonant triad propagates with its own wave speed in its own direction. Therefore, the resulting wave field is not stationary in any moving reference frame. For one-dimensional waves, Wilton ripples, or a fixed point solution for $\xi = 1$ is the only resonant triad that propagates with a constant speed. All other one-dimensional resonant triads ($\xi \neq 1$) produce unsteady wave fields even though their initial wave amplitudes satisfy the conditions

for fixed points. Then, it is natural to ask if any two-dimensional resonant triad exchanging no energy can form a traveling wave field, which will be addressed in the next section.

5.2 | Traveling waves: symmetric Wilton ripples

As discussed in Section 4.3, the three modes of a symmetric resonant triad ($\theta_2 = -\theta_3$) have the same linear wave speed in the x-direction, necessary for a nonlinear traveling wave solution to exist. In addition, if the amplitudes of the three modes are independent of time and their nonlinear corrections to the linear wave speed match, the resonant triad becomes a traveling wave.

Surprisingly, when all symmetric resonant triads satisfy the conditions for fixed points given by (63) and (67), the x-components of the nonlinear wave speed corrections become identical as $K_1 = 2K_2 \cos\theta_2 = 2K_3 \cos\theta_3$ and $\mu_1 = 2\mu_2 = 2\mu_3$. This implies that any resonant symmetric triad without energy exchange is indeed a traveling wave and can be considered a two-dimensional generalization of Wilton ripples. These special symmetric waves will be referred to as *symmetric Wilton ripples* and exist for $|\theta_3| \leq \theta_{\max}$.

From (63) and (65), the amplitudes of the symmetric Wilton ripples satisfy

$$2^{1/2}|Z_1| = |Z_2| = |Z_3|, \tag{68}$$

while the nonlinear frequency corrections μ_j can be expressed as

$$\mu_1 = 2\,\mu_2 = 2\,\mu_3, \qquad \mu_3 = (-1)^m\,\Gamma_0\,|Z_1|\,(m = 0, 1). \tag{69}$$

Linear stability of these fixed points is examined in Appendix B and the fixed point solutions are found neutrally stable when only a single triad is considered.

To represent symmetric Wilton ripples in physical space, the surface displacement ζ nondimensionalized by $(\sigma/g)^{1/2}$ can be obtained from

$$\zeta = \sum_{j=1}^{3} \left[A_j(t)\,\mathrm{e}^{-\mathrm{i}\left(\boldsymbol{K}_j\cdot\boldsymbol{X} - \Omega_j T\right)} + C.C \right], \tag{70}$$

where A_j are the dimensionless amplitudes given by $A_j = \mathcal{A}_j/(\sigma/g)^{1/2}$ and $\boldsymbol{X} = \boldsymbol{x}/(\sigma/g)^{1/2}$. Using the following relationship between Z_j and A_j given by (28)

$$Z_j = \left(2\Omega_j/K_j\right)^{1/2} A_j, \tag{71}$$

the fixed point solutions given by (68) and (69) can be written, in terms of $|A_j|$, as

$$2|A_1|^2 = \cos\theta\,|A_3|^2 = \cos\theta\,|A_2|^2, \qquad \mu_1 = 2\mu_2 = 2\mu_3 = 2(-1)^m\,q(\theta)\Omega K\,|A_1|, \tag{72}$$

where $\theta = \theta_3$, $K = K_3$, $\Omega = \Omega_3$, and $q(\theta) = \cos^2\theta + 2\cos\theta - 2 > 0$. Then, by substituting $A_j = |A_j|\,\mathrm{e}^{\mathrm{i}(\mu_j T + \delta_{j,0})}$ with (67) into (70), the surface elevation ζ of symmetric Wilton ripples can be written, in physical space, as

$$\zeta(X, Y, T) = a_0\,\cos(K_y Y)\,\cos\psi \pm (\cos\theta/8)^{1/2}\,a_0 \cos 2\psi + O(\epsilon^2), \tag{73}$$

where

$$\psi = K_x X - \Omega\left[1 \pm (\cos\theta/32)^{1/2} q(\theta)\,K a_0\right] T, \tag{74}$$

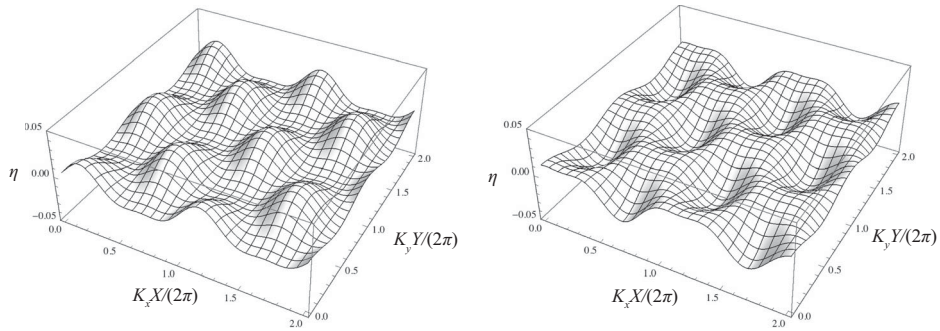

FIGURE 7 Symmetric Wilton ripples given by (73) with $K = 0.771$, $a_0 = 0.0154$, and $\theta = \theta_{\max}/3$=12.489^o: (A) $m = 0$; (B) $m = 1$. In each plot, the surface wave field over two wave wave periods is shown

with $K_x = K\cos\theta$, $K_y = K\sin\theta$, and $a_0 = 4|A_3|$. Here, $\delta_{2,0} = \delta_{3,0} = 0$ have been chosen and the plus and minus signs correspond to m=0 and 1, respectively. Figure 7 shows the surface profiles of symmetric Wilton ripples. When $\theta = 0$, the one-dimensional Wilton ripple solution is recovered from (73) and (74).

Using the classical Stokes expansion, it is confirmed in Appendix C that this traveling wave solution is indeed a solution of the original second-order model given by (3).

6 | CONCLUSION

We have re-examined three-wave resonant interactions of gravity-capillary waves using the Zakharov equation and its discrete approximation. After having identified the region of resonance, an alternative to the previous representations of resonant wavenumbers and wave frequencies of McGoldrick[9] and Simmons[10] has been proposed in terms of two propagation angles. This could provide a convenient way to understand possible resonant triads.

A special attention has been paid to resonant triad interactions, in which no energy exchange occurs so that the amplitudes of the triad remain constant during the interactions. The explicit conditions under which such interactions exist have been found in terms of initial wave amplitudes and phases. Any triad that violates the conditions must exchange energy and the amplitudes vary periodically in time. Among constant-amplitude resonant triads, it is shown that all symmetric triads (with one wavenumber vector bisecting the angle between the other two wavenumber vectors) can propagate with a constant wave speed to form a transversely modulated traveling wave when the angle is smaller than the maximum value given by Equation (55).

Previously, laboratory experiments of gravity-capillary waves have been performed to demonstrate resonant triads with periodic exchanges of energy. It would be interesting to test if the solutions of constant amplitudes, including symmetric Wilton ripples, described here can be observed. This question is related to stability of the symmetric Wilton ripples. The linear stability analysis of fixed points of a resonant triad presented in Appendix B is limited and more general perturbations beyond the resonant triad have to be considered. Recently, Trichtchenko et al.[32] investigated the stability of one-dimensional Wilton ripples and found extremely small growth rates when they are unstable. Then, such instability might be suppressed by viscosity.

Resonant triad interactions can also occur in density-stratified fluids[33–35] and the constant amplitude solutions could provide steady states or traveling wave solutions through resonant interactions between surface and internal wave modes or different internal wave modes.

ACKNOWLEDGMENTS

This work was supported by the US National Science Foundation through Grant Nos. DMS-1517456 and OCE-1634939. The authors are grateful to Diane Henderson for her valuable discussions.

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How to cite this article: Chabane M, Choi W. On resonant interactions of gravity-capillary waves without energy exchange. *Stud Appl Math*. 2019;142:528–550. https://doi.org/10.1111/sapm.12249

APPENDIX A: AMPLITUDE EQUATIONS FOR $\mathcal{A}$

Instead (15), one can introduce a slowly varying wave amplitude $\mathcal{A}$ as

$$a(\boldsymbol{k},t) = \mathcal{A}(\boldsymbol{k},t)\,\mathrm{e}^{\mathrm{i}\,\omega t} + \mathcal{A}^*(-\boldsymbol{k},t)\,\mathrm{e}^{-\mathrm{i}\,\omega t}, \qquad b(\boldsymbol{k},t) = \mathrm{i}\left(\frac{g_k}{\omega}\right)\left[\mathcal{A}(\boldsymbol{k},t)\,\mathrm{e}^{\mathrm{i}\,\omega t} - \mathcal{A}^*(-\boldsymbol{k},t)\,\mathrm{e}^{-\mathrm{i}\,\omega t}\right]. \tag{A.1}$$

Previously, McGoldrick[9] and Simmons[10] introduced this complex amplitude $\mathcal{A}$ to obtain their amplitude equations for a single resonant triad. By substituting into (23) the relationship between $\mathcal{A}$ and $\mathcal{Z}$ given by (28), the reduced Hamiltonian $\mathcal{H}_A$ in terms of $\mathcal{A}$ can be written as

$$\mathcal{H}_{\mathcal{A}} = \iiint V^{(1)}_{1,2,3}\left(\mathcal{A}^*_1\,\mathcal{A}_2\,\mathcal{A}_3 + \mathcal{A}_1\,\mathcal{A}^*_2\,\mathcal{A}^*_3\right)\delta_{1-2-3}\,\mathrm{d}\boldsymbol{k}_{1,2,3}, \tag{A.2}$$

where $V^{(1)}_{1,2,3}$ is given by

$$V^{(1)}_{1,2,3} = V_{2,3,-1} - V_{-1,2,3} - V_{3,-1,2}, \qquad V_{1,2,3} = \left(\frac{8\,g_1 g_2 g_3}{\omega_1\omega_2\omega_3}\right)^{1/2} U_{1,2,3}. \tag{A.3}$$

Then, the amplitude equation for $\mathcal{A}$ can be obtained as

$$\frac{\partial\mathcal{A}}{\partial t} = \mathrm{i}\left(\frac{\omega}{2g_k}\right)\frac{\delta\mathcal{H}_{\mathcal{A}}}{\delta\mathcal{A}^*} = \mathrm{i}\left(\frac{\omega}{2g_k}\right)\left(\iint V^{(1)}_{0,1,2}\mathcal{A}_1\,\mathcal{A}_2\delta_{0-1-2}\mathrm{d}\boldsymbol{k}_{1,2} + 2\iint V^{(1)}_{2,1,0}\mathcal{A}^*_1\mathcal{A}_2\delta_{0+1-2}\mathrm{d}\boldsymbol{k}_{1,2}\right), \tag{A.4}$$

where the symmetry condition of $V^{(1)}_{2,1,0} = V^{(1)}_{2,0,1}$ has been used. As shown previously for $\mathcal{Z}$, the evolution equation for $\mathcal{A}$ also conserves the reduced Hamiltonian $\mathcal{H}_{\mathcal{A}}$ along with the energy and linear

momenta given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int g_k\,|\mathcal{A}|^2\,\mathrm{d}\boldsymbol{k}=0,\qquad \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{g_k\boldsymbol{k}}{\omega}\,|\mathcal{A}|^2\,\mathrm{d}\boldsymbol{k}=0, \tag{A.5}$$

respectively.

For a resonant triad, the amplitude equations for $\mathcal{A}_j$ can be found, from (A.4), as

$$\frac{\mathrm{d}\mathcal{A}_1}{\mathrm{d}t}=\mathrm{i}\,\gamma_1\,\mathcal{A}_2\mathcal{A}_3,\qquad \frac{\mathrm{d}\mathcal{A}_2}{\mathrm{d}t}=\mathrm{i}\,\gamma_2\,\mathcal{A}_3^*\mathcal{A}_1,\qquad \frac{\mathrm{d}\mathcal{A}_3}{\mathrm{d}t}=\mathrm{i}\,\gamma_3\,\mathcal{A}_1\mathcal{A}_2^*, \tag{A.6}$$

where γ_j ($j=1,2,3$) are related to γ_0 as

$$\gamma_j=\left(\frac{\omega_j}{2g_j}\right)\left(\frac{8\,g_1g_2g_3}{\omega_1\omega_2\omega_3}\right)^{1/2}\gamma_0. \tag{A.7}$$

The explicit expressions of $\gamma_j>0$ are given by

$$\gamma_1=\frac{1}{2}\left[\left(\frac{\omega_2}{k_2T_2}+\frac{\omega_3}{k_3T_3}+\frac{\omega_2\omega_3\,k_1T_1}{\omega_1\,k_2k_3T_2T_3}\right)\boldsymbol{k}_2\cdot\boldsymbol{k}_3+\frac{\omega_2k_2}{T_2}+\frac{\omega_3k_3}{T_3}+\frac{k_1T_1}{\omega_1}\left(\omega_2\omega_3-\omega_1^2\right)\right], \tag{A.8}$$

$$\gamma_2=\frac{1}{2}\left[\left(\frac{\omega_3}{k_3T_3}-\frac{\omega_1}{k_1T_1}+\frac{\omega_1\omega_3\,k_2T_2}{\omega_2\,k_1k_3T_1T_3}\right)\boldsymbol{k}_3\cdot\boldsymbol{k}_1-\frac{\omega_3k_3}{T_3}+\frac{\omega_1k_1}{T_1}-\frac{k_2T_2}{\omega_2}\left(\omega_3\omega_1+\omega_2^2\right)\right], \tag{A.9}$$

$$\gamma_3=\frac{1}{2}\left[\left(-\frac{\omega_1}{k_1T_1}+\frac{\omega_2}{k_2T_2}+\frac{\omega_1\omega_2\,k_3T_3}{\omega_3k_1k_2T_1T_2}\right)\boldsymbol{k}_1\cdot\boldsymbol{k}_2+\frac{\omega_1k_1}{T_1}-\frac{\omega_2k_2}{T_2}-\frac{k_3T_3}{\omega_3}\left(\omega_1\omega_2+\omega_3^2\right)\right]. \tag{A.10}$$

In the limit of infinitely deep water ($d\to\infty$ and $T_j\to 1$), (A.6) can be reduced to the system of McGoldrick.[9] After replacing g_j by ω_j^2/k_j, the coefficients γ_j can be reduced to those in Craik.[29] From (A.5), one can see that the system given by (A.6) also has the following conservation laws

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{j=1}^{3} g_j\left|\mathcal{A}_j\right|^2\right]=0,\qquad \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{j=1}^{3}\frac{g_j\boldsymbol{k}_j}{\omega_j}\left|\mathcal{A}_j\right|^2\right]=0. \tag{A.11}$$

When necessary, $\mathcal{A}_j$, t, and γ_j can be nondimensionalized as

$$A_j=\mathcal{A}_j/(\sigma/g)^{1/2},\qquad T=t/\left(\sigma/g^3\right)^{1/4},\qquad \Gamma_j=\left(\sigma^3/g^5\right)^{1/4}\gamma_j, \tag{A.12}$$

and, then, the expressions of Γ_j can be obtained by replacing ω_j and k_j by Ω_j and K_j. Figure 8 shows the variations of Γ_j over the resonance region in the fourth quadrant of the (θ_2,θ_3)-plane.

APPENDIX B: LINEAR STABILITY OF FIXED POINTS

To study stability of fixed point solutions of the system given by (59) and (60), we assume that

$$|Z_j|=|Z_{j,f}|+|Z_j'|,\qquad \Delta=\Delta_f+\Delta', \tag{B.1}$$

where $|Z_{j,f}|$ for $j=1,2,3$ and Δ_f represent the fixed point solutions satisfying (62) and (63), and $|Z_j'|$ and Δ' denote small perturbations such that $|Z_j'|/|Z_{j,f}|\ll 1$ and $\Delta'/\Delta_f\ll 1$. Then, by substituting

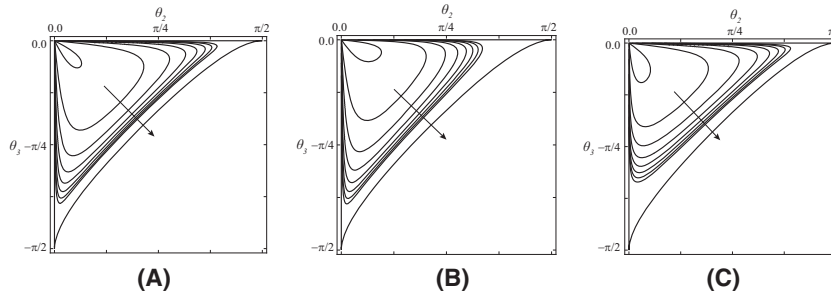

FIGURE 8 Contour plots of (dimensionless) coefficients of the amplitude equations $1.5 \le \Gamma_j \le 5$ in the fourth quadrant of the (θ_2, θ_3)-plane, where $\Gamma_j = (\sigma^3/g^5)^{1/4}\gamma_j$ with γ_j given by (A.8)–(A.10) for infinitely deep water ($T_j \to 1$). (A) Γ_1; (B) Γ_2; (C) Γ_3. The increment between the two neighboring contour levels is 0.5 and the arrows indicate the direction of increasing contour levels

(B.1) into (59) and (60) and linearizing the system about the fixed points, we obtain

$$\frac{\mathrm{d}|Z_1'|}{\mathrm{d}t} = \pm|Z_{2,f}||Z_{3,f}|\,\Delta', \quad \frac{\mathrm{d}|Z_2'|}{\mathrm{d}t} = \mp|Z_{1,f}||Z_{3,f}|\,\Delta', \quad \frac{\mathrm{d}|Z_3'|}{\mathrm{d}t} = \mp|Z_{1,f}||Z_{2,f}|\,\Delta', \tag{B.2}$$

$$\frac{\mathrm{d}\Delta'}{\mathrm{d}t} = \mp 2|Z_{1,f}||Z_{2,f}||Z_{3,f}|\left(\frac{|Z_1'|}{|Z_{1,f}|^3} - \frac{|Z_2'|}{|Z_{2,f}|^3} - \frac{|Z_3'|}{|Z_{3,f}|^3}\right), \tag{B.3}$$

where $+$ and $-$ signs need to be chosen for $\Delta_0 = 0$ and π, respectively, and we have used $\sin(\Delta_0 + \Delta') = \pm\sin\Delta' \simeq \pm\Delta'$ and $\cos(\Delta_0 + \Delta') = \pm\cos\Delta' \simeq \pm 1$. Then, from (B.2) to (B.3), one can obtain a single equation for Δ' as

$$\frac{\mathrm{d}^2\Delta'}{\mathrm{d}t^2} + \Omega'^2\Delta' = 0, \tag{B.4}$$

where Ω' is given by

$$\Omega'^2 = 2\left(\frac{|Z_{2,f}|^2|Z_{3,f}|^2}{|Z_{1,f}|^2} + \frac{|Z_{2,f}|^2|Z_{3,f}|^2}{|Z_{2,f}|^2} + \frac{|Z_{1,f}|^2|Z_{2,f}|^2}{|Z_{3,f}|^3}\right) > 0. \tag{B.5}$$

As Ω'^2 is always positive, Δ' is oscillatory in time and, therefore, the fixed points are neutrally stable.

APPENDIX C: STOKES EXPANSION FOR SYMMETRIC WILTON RIPPLES

Here, we assume that all physical variables are nondimensionalized with respect to g and σ or, equivalently, set $g = \sigma = 1$. To find a traveling wave solution, we assume the problem can be made steady in a reference frame moving in the X-direction with speed c. Then, the surface elevation ζ and the free surface velocity potential Φ can be written as

$$\zeta = \zeta(\xi, Y), \quad \Phi = \Phi(\xi, Y), \qquad \xi = X - cT. \tag{C.1}$$

With (C.1), the equations for ζ and Φ can be found, from (3), as

$$-c\frac{\partial\zeta}{\partial\xi} + \mathcal{L}[\Phi] = -\boldsymbol{\nabla}\cdot(\zeta\boldsymbol{\nabla}\Phi) - \mathcal{L}[\zeta\mathcal{L}[\Phi]], \quad -c\frac{\partial\Phi}{\partial\xi} + \zeta + \nabla^2\zeta = -\tfrac{1}{2}\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi + \tfrac{1}{2}(\mathcal{L}[\Phi])^2, \tag{C.2}$$

where $\boldsymbol{\nabla} = (\partial/\partial\xi, \partial/\partial Y)$. For weakly nonlinear waves, we further assume that ζ, Φ, and c can be expanded in small wave steepness ϵ as

$$\zeta = \zeta_1 + \zeta_2 + O(\epsilon^3), \quad \Phi = \Phi_1 + \Phi_2 + O(\epsilon^3), \quad c = c_0 + c_1 + O(\epsilon^2), \tag{C.3}$$

with $\zeta_n = O(\Phi_n) = O(c_n) = O(\epsilon^n)$.

By substituting (C.3) into (C.2), the first-order equations for ζ_1 and Φ_1 can be found, at $O(\epsilon)$, as

$$-c_0 \frac{\partial \zeta_1}{\partial \xi} + \mathcal{L}[\Phi_1] = 0, \qquad -c_0 \frac{\partial \Phi_1}{\partial \xi} + \zeta_1 + \nabla^2 \zeta_1 = 0. \tag{C.4}$$

For symmetric Wilton ripples given by (73), the first-order solutions are assumed to have both the first and second harmonics:

$$\zeta_1 = a_1 \cos(K_y Y)\, \mathrm{e}^{\mathrm{i}K_x\xi} + a_2\, \mathrm{e}^{2\mathrm{i}K_x\xi} + C.C., \qquad \Phi_1 = b_1 \cos(K_y Y)\, \mathrm{e}^{\mathrm{i}K_x\xi} + b_2\, \mathrm{e}^{2\mathrm{i}K_x\xi} + C.C., \tag{C.5}$$

where $K_x = K\cos\theta$ and $K_y = K\sin\theta$. By substituting (C.5) into (C.4), one can obtain, for the first harmonics,

$$b_1 = -\mathrm{i}c_0 (K_x/K)\, a_1, \qquad c_0^2 = (K + K^3)/K_x^2, \tag{C.6}$$

and, for the second harmonics,

$$b_2 = -\mathrm{i}c_0\, a_2, \qquad c_0^2 = (K_x/2 + 2K_x^3)/K_x^2. \tag{C.7}$$

For the consistency between the two different expressions for c_0^2 in (C.6)–(C.7), one can obtain a relationship between K and K_x, which is nothing but the condition for symmetric waves given by (54) with $K_1 = 2K_x$.

At the second order, ζ_2 and Φ_2 are governed by

$$-c_0 \frac{\partial \zeta_2}{\partial \xi} + \mathcal{L}[\Phi_2] = c_1 \frac{\partial \zeta_1}{\partial \xi} - \boldsymbol{\nabla}\cdot(\zeta_1 \boldsymbol{\nabla}\Phi_1) - \mathcal{L}[\zeta_1 \mathcal{L}[\Phi_1]], \tag{C.8}$$

$$-c_0 \frac{\partial \Phi_2}{\partial \xi} + \zeta_2 + \nabla^2 \zeta_2 = c_1 \frac{\partial \Phi_1}{\partial \xi} - \frac{1}{2}\boldsymbol{\nabla}\Phi_1\cdot\boldsymbol{\nabla}\Phi_1 + \frac{1}{2}(\mathcal{L}[\Phi_1])^2. \tag{C.9}$$

After substituting (C.5) into the right-hand sides of (C.8) and (C.9) and imposing the solvability conditions on the right-hand side terms proportional to the first-order (or homogeneous) solutions, one can obtain the expressions of c_1 and a_2 as

$$c_1 = \pm c_0\, (\cos\theta/8)^{1/2} q(\theta)\, K a_1, \qquad a_2 = \pm(\cos\theta/8)^{1/2} a_1. \tag{C.10}$$

Then, the surface elevation $\zeta = \zeta_1 + \zeta_2 + O(\epsilon^3)$ can be written as

$$\zeta = 2a_1 \cos(K_y Y)\cos\psi + 2a_2 \cos(2\psi) + O(\epsilon^3), \tag{C.11}$$

from which (73) can be recovered with $a_0 = 2a_1$ and $\psi = K_x\xi$ given by

$$\psi = K_x X - \Omega\left[1 \pm (\cos\theta/32)^{1/2} q(\theta)\, K a_0\right] T. \tag{C.12}$$