ON THE HYPERBOLICITY OF TWO-LAYER FLOWS

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We consider the two-layer shallow water equations in the presence of the top free surface and find explicit conditions for which the system is hyperbolic. It is commonly believed that, analogously to the rigid-lid case, this can only happen for small relative speeds. Using both the root location criteria for a quartic equation and a geometrical approach, it is shown that hyperbolicity is held for not only small, but also large relative speeds.

Keywords: Hyperbolicity; Two-layer flows; Shallow water; Quartic equation.

1. Introduction

We consider a two-layer system composed of two immiscible fluids of different *constant densities* ρ_1 and ρ_2 confined between the upper free surface and the lower rigid boundary in which the flow is governed by the two-layer shallow water equations (e.g., see Baines[1]):

$$\begin{aligned} (h_i)_t + (h_i\, u_i)_x &= 0, \\ (u_i)_t + u_i(u_i)_x + g\big(h_1 + h_2 + \delta_{i1}(\rho - 1)h_2\big)_x &= 0. \end{aligned} \tag{1}$$

In these equations, u_1 and u_2 are the depth-averaged velocities, h_1 and h_2 are the layer thicknesses, δ_{ij} is the Kronecker delta, g is the gravitational acceleration, and $\rho < 1$ is defined by $\rho = \rho_2/\rho_1$, with 1 and 2 associated with the lower and upper layer, respectively. By defining $U = (h_1, h_2, u_1, u_2)^T$, the quasilinear system can be written in the form $U_t + A\,U_x = 0$, for which the characteristic polynomial $P(\lambda) = \det(A - \lambda I)$ is given by

$$\begin{aligned} P(\lambda) = \lambda^4 - 2(u_1 + u_2)\lambda^3 + (u_1^2 + 4u_1u_2 + u_2^2 - gh_1 - gh_2)\lambda^2 + 2\,(gh_1u_2 + \\ + gh_2u_1 - u_1u_2^2 - u_2u_1^2)\lambda + u_1^2u_2^2 - gh_1u_2^2 - gh_2u_1^2 + g^2h_1h_2(1-\rho). \end{aligned} \tag{2}$$

The system (1) is hyperbolic if $P(\lambda) = 0$ admits only real roots. It seems to be widely accepted that this can happen only if the relative speed be-

tween the two layers is small (see Refs. 2–7). Approximate expressions for the eigenvalues of A valid in the Boussinesq limit ($\rho \approx 1$) were first obtained by Schijf & Schönfeld[2] while the exact expressions for the eigenvalues were found by Lawrence[4] without relying on the Boussinesq approximation. This result has been recently extended in Ref. 6 with including the effects of topography. It is possible to present the eigenvalues (or characteristic speeds) in two distinct sets corresponding to the *external* and *internal* wave motions, respectively. All these authors seem to agree that the eigenvalues corresponding to the internal wave mode become complex when the *Froude number* F defined by $F = (u_2 - u_1)/\sqrt{gh_1}$ exceeds a critical value. However, this seems to be in contradiction with the result obtained by Ovsyannikov,[8] who showed, by means of a geometrical representation of the characteristics, that the model can still be hyperbolic for large relative speeds. In this paper, by examining carefully the criteria for the quartic equation (2) to have real roots, we validate the result of Ref. 8 that the internal wave speeds can indeed be complex only for a bounded range of Froude numbers.

2. Hyperbolicity of the two-layer shallow water model

2.1. *Root location criteria*

Before proceeding further, we summarize below some essentials of the root distribution for the quartic equation.

2.1.1. *Preliminaries*

Consider the quartic equation

$$f(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0, \tag{3}$$

whose coefficients a_i ($i = 0, \cdots, 4$) are all real with $a_0 > 0$. It is known that the *discriminant* Δ_f of (3) is given by

$$\Delta_f = {a_0}^6 \prod_{i<k} (x_i - x_j)^2,$$

where x_1, x_2, x_3, and x_4 are the roots of the polynomial f. Since Δ_f is a symmetric polynomial, it can be polynomially expressed in terms of the real coefficients a_i (see page 80 of Ref. 9). The discriminant is a powerful tool that can fully describe the structure of the roots for quadratic and cubic equations. However, the same cannot be achieved for the quartic equation since (*i*) $\Delta_f > 0$: four distinct real or four distinct complex roots; (*ii*) $\Delta_f = 0$: at least two equal roots; (*iii*) $\Delta_f < 0$: two distinct real roots and two complex roots.

Several attempts have been made in the past to obtain conditions, in terms of the literal coefficients of a polynomial, concerning a special root distribution (see references therein). Among them, Jury & Mansour[10] presented a series of algorithms involving *characteristic expressions* for a quartic equation, allowing a full characterization of the root distribution in a much more concise form than the one provided by previous approaches. Similar criteria involving only inner determinants were also obtained by Fuller.[11] Following this elegant exposition, when considering the inner determinants Δ_3, Δ_5, Δ_7*:

$$\Delta_7 = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 \\ 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 \\ 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 \\ 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 & 0 \end{vmatrix},$$

with Δ_3 and Δ_5 being defined as the determinants of the inner matrices with dimensions 3×3 and 5×5, respectively (as denoted by the two inner squares in the definition of Δ_7), we have the following result (see page 778 of Fuller[11]):

Theorem 2.1. *Equation* (3) *has its roots all real if and only if one of the two following sets of conditions holds:* (*a*) $\Delta_3 > 0$, $\Delta_5 > 0$, $\Delta_7 \geqslant 0$*;* (*b*) $\Delta_3 \geqslant 0$, $\Delta_5 = 0$, $\Delta_7 = 0$.

2.1.2. *Real characteristic speeds of long waves*

We first rewrite (2) in terms of non-dimensional variables:

$$\Lambda = \frac{\lambda}{\sqrt{gh_1}}, \quad H = \frac{h_2}{h_1}, \quad F = \frac{u_2 - u_1}{\sqrt{gh_1}}$$

and assume without loss of generality that $u_1/\sqrt{gh_1} = 1$ (by choosing a moving reference frame such that this condition is met). Then, the characteristic equation $P(\lambda) = 0$ becomes

$$\Lambda^4 - 2(2+F)\,\Lambda^3 + [(1+F)(5+F) - H]\,\Lambda^2 + 2\left[H - (1+F)^2\right]\Lambda - \rho H = 0,$$

for which the inner determinants are found as

$$\Delta_3 = 2F^2 + 8(H+1),$$

*Notice that Δ_7 is precisely the determinant of the Sylvester matrix, hence it consists on the discriminant of f, *i.e.*, $\Delta_7 = \Delta_f$.

$$\Delta_5 = 8(H{+}1)F^4{-}16\left[H^2-(6+\rho)H+1\right]F^2{+}8(H{+}1)\left[(H-1)^2+4\rho H\right].$$

We will show that, in our particular case, the set of conditions stated in Theorem 2.1 reduces to $\Delta_7 \geqslant 0$, as the conditions $\Delta_3 > 0$ and $\Delta_5 > 0$ are automatically satisfied. First, it is clear that $\Delta_3 > 0$. Second, in order to prove that $\Delta_5 > 0$, we look at Δ_5 as a parabola in terms of the variable $y = F^2$. If its discriminant is less than zero, the parabola has no real roots and, therefore, $\Delta_5 > 0$. On the other hand, if it has real roots, it is sufficient to prove that both roots are negative, which assures that Δ_5 has no real roots for the Froude number F. The discriminant of Δ_5 becomes positive only if

$$6(\rho+2)H^2 - \left[36+(\rho+2)^2\right]H + 6(\rho+2) < 0, \tag{4}$$

which holds for $\frac{\rho+2}{6} < H < \frac{6}{\rho+2}$. For this range of H, however, the coefficient of F^2 in Δ_5 is always positive and so the result follows.

We have shown that, in our particular case, a full description of the root distribution can be achieved by means of its discriminant. Therefore, from Theorem 2.1, we can conclude that $P(\lambda) = 0$ has four real solutions for $\Delta_7 \geqslant 0$, while it has two complex and two real solutions for $\Delta_7 < 0$.

Straightforward calculations reveal that the discriminant Δ_7 depends only on the variable $y = F^2$ and the physical parameters ρ and H:

$$\Delta_7 = 16H\,Q(y), \tag{5}$$

with $Q(y)$ defined by

$$Q(y) = y^4{+}(H{+}1)(\rho{-}4)\,y^3{-}\left[3(\rho-2)-(4-26\rho+\rho^2)H+3(\rho-2)H^2\right]y^2{+}$$
$$(H{+}1)\left[3\rho-4+(8+10\rho-20\rho^2)H+(3\rho-4)H^2\right]y{+}(1{-}\rho)\left((H-1)^2+4\rho H\right)^2.$$

As shown in Fig. 1, for any prescribed values for the physical parameters, the polynomial $Q(y)$ has two positive real roots $(F^-_{\text{crit}})^2$ and $(F^+_{\text{crit}})^2$, and the condition $\Delta_7 \geqslant 0$ is satisfied for $0 \leqslant F^2 \leqslant (F^-_{\text{crit}})^2$ or $F^2 \geqslant (F^+_{\text{crit}})^2$. The first inequality for the Froude number implies that the system (1) is hyperbolic for small relative speeds between the two layers, as noted by several authors. However, the figure shows clearly a new range of Froude numbers characterized by large relative speeds, for which the flow is hyperbolic (*cf.* Ovsyannikov[8]).

2.1.3. *Comparison with the characteristic speeds in Lawrence (1990)*

We find in the Appendix of Ref. 4 an exact derivation of the characteristic speeds of long waves, based on the Descartes-Euler solution expressed by

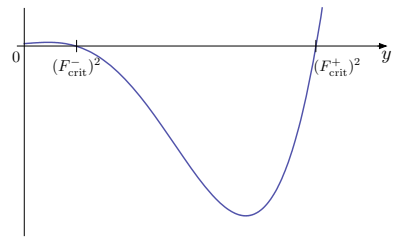

Fig. 1. A sketch of the behavior of the polynomial $Q(y)$

means of the solutions of the cubic resolvent (A.6). The discriminant of this cubic resolvent given by (A.8) is denoted by D in Ref. 4 whose sign is determined by the quantity δ (with the opposite sign of D) defined by

$$\delta = \beta + (1 - F_\Delta^2) \sum_{n=0}^{3} b_n \epsilon^n,$$

or, alternatively, in our notation,

$$\delta = \frac{4}{(1+H^2)^4(1-\rho)}\, Q(y). \tag{6}$$

The characteristic speeds are all real provided that $D \leqslant 0$ and the roots of the cubic resolvent are positive. In his work, Lawrence[4] stated that *the requirement that $D \leqslant 0$ for the solutions to be real restricts the value of the Froude number F_Δ^2 to be less than or equal to a critical value* $(F_\Delta^2)_{\text{crit}}$, which contradicts our finding. We know from (6) that, since D has the opposite sign of δ, $D \leqslant 0$ is equivalent to $Q(y) \geqslant 0$, which implies the statement on hyperbolicity in Ref. 4 is inaccurate.

2.2. *A geometrical approach*

The results presented so far will now become more clear. Using the approach proposed in Ref. 8, we will be able to provide a geometrical interpretation for the roots of Δ_7 in (5). The key step is to rewrite the characteristic polynomial in a simpler form:

$$P(\lambda) = \big((u_1-\lambda)^2 - gh_1\big)\big((u_2-\lambda)^2 - gh_2\big) - g^2\rho h_1 h_2.$$

This form of presenting (2) is not new, but, as shown in Ref. 8, allows us to better understand the structure of the roots for the characteristic equation. If we define

$$\lambda - u_1 = q\sqrt{gh_1}, \quad \lambda - u_2 = p\sqrt{gh_2}, \tag{7}$$

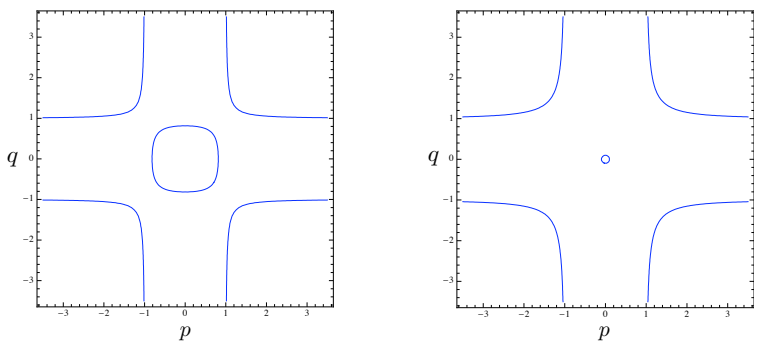

Fig. 2. Plots of the curve (8) for different physical parameters: $\rho = 1/3$ (*left-hand side*) and $\rho = 99/100$ (*right-hand side*).

the characteristic equation yields

$$(p^2 - 1)(q^2 - 1) = \rho. \tag{8}$$

On the (p, q) plane, Eq. (8) describes a fourth-order curve having four axes of symmetry where we can distinguish an inner region (in the interior of the unit square centered at the origin) and an outer region, as shown in Fig. 2. The limit cases correspond to assigning to ρ the values 0 and 1. In the first case ($\rho = 0$), Eq. (8) reduces to the lines $|p| = 1$ and $|q| = 1$. In the second case ($\rho = 1$), the inner region confines to one single point, the origin, confirming its tendance to shrink as the values of ρ approach 1. Both cases reduce to a one-layer flow with a free surface, but the latter allows a velocity discontinuity in the interior of the fluid domain.

As a consequence of (7), p and q are related by

$$q = \sqrt{H}\, p + F. \tag{9}$$

Combining these results, we conclude that the real characteristic speeds correspond to the solutions of the system

$$(p^2 - 1)(q^2 - 1) = \rho, \quad q = \sqrt{H}\, p + F, \tag{10}$$

with p, q all real. More precisely, each intersection point yielding a solution of this system corresponds to a real eigenvalue of A. Additionally, this geometrical interpretation reveals that the system has at least two and a maximum of four real solutions. Hence, the system is of mixed type: it is strictly hyperbolic when we can present four real and distinct characteristics, and a system of composite type when there are two real and two imaginary characteristics, confirming the result obtained by using inner

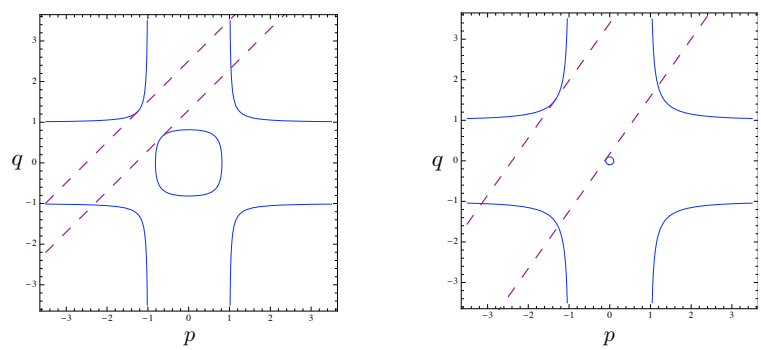

Fig. 3. Tangency condition for different physical parameters: $\rho = 1/3$, $H = 1$ (*left-hand side*) and $\rho = 99/100$, $H = 2$ (*right-hand side*). The solid and dashed lines represent (8) and (9) with $F = F^{\pm}_{crit}$, respectively.

determinants. The passage between the two scenarios happens when the straight line described by (9) becomes tangent to the curve (8), as shown in Fig. 3. Notice that the intersection points with the boundary of the inner region representing the internal wave speeds disappear as the Froude number increases, leaving only the external (or surface) wave speeds, until it reaches F^{+}_{crit} beyond which the internal wave modes reappear.

For prescribed values of ρ and H, the curve (8) and the slope of the line (9) are completely determined. We seek the values of initial ordinates F for which the tangency holds. From (10), it follows

$$H\,p^4 + 2\sqrt{H}F\,p^3 + (F^2 - H - 1)\,p^2 - 2\sqrt{H}F\,p - F^2 + (1-\rho) = 0.$$

Multiple roots for this polynomial arise when its *discriminant* vanishes, leading to the condition $16H\,Q(y) = 0$. Surprisingly, this discriminant is precisely the same as Δ_7 in (5) and we now realize that the roots of $\Delta_7 = 0$ yield nothing but the condition for tangency. The results obtained can be summarized as follows:

Proposition 2.1. *For any physical parameters, there are two distinct positive real numbers F^{-}_{crit} and F^{+}_{crit}, with $F^{-}_{crit} < F^{+}_{crit}$, such that the system* (1) *is hyperbolic if and only if $|F| \leqslant F^{-}_{crit}$ or $|F| \geqslant F^{+}_{crit}$.*

3. Concluding remarks

Without solving the quartic equation, we have found explicit relations for which the system (1) is hyperbolic. In particular, in complete agreement with Ovsyannikov,[8] we perceive the effect caused by the presence of the

free surface: the range of Froude numbers for hyperbolicity is not bounded, which is contrary to the rigid-lid case for which it can be shown[8] that *the characteristics for the rigid-lid system are real if and only if* $|F| \leqslant F_c$, where

$$F_c{}^2 = (1-\rho)(H+\rho)/\rho.$$

Numerical computations show that $F_{\mathrm{c}} > F^-_{\mathrm{crit}}$ and that these two values become almost indistinguishable in the Boussinesq limit. This could justify the common assumption that the difference between the rigid-lid and free-surface cases is insignificant, which we know is valid only for small relative speeds. Moreover, it is worth noticing that for small density ratios, F_c can actually exceed the value F^+_{crit}.

The well-posedness of the system was not addressed in this paper, but is a matter of great importance. It would be reasonable to expect that the system remains hyperbolic for initial data prescribed in the first range of Froude numbers. It is not clear yet if this is the case for the second branch.

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References

1. P. G. Baines, *Topographic Effects on Stratified Flows*. Cambridge Monographs on Mechanics (Cambridge University Press, 1995).
2. J. B. Schijf, J. C. Schönfeld, Theoretical considerations on the motion of salt and fresh water. In *Proc. of the Minn. Intl. Hyd. Conv. Joint meeting of the IAHR and Hyd. Div. ASCE*, pp. 321–333 (1953).
3. L. Armi, The hydraulics of two flowing layers with different densities. *J. Fluid Mech.*, **163**, pp. 27–58 (1986).
4. G. A. Lawrence, On the hydraulics of Boussinesq and non-Boussinesq two-layer flows. *J. Fluid Mech.*, **215**, pp. 457–480 (1990).
5. M. Castro, J. Macías, C. Parés, A Q-Scheme for a class of systems of coupled conservation laws with source term. Application to a two-layer 1-D shallow water system, *Math. Model. Numer. Anal.*, **35**, No. **1** pp. 107–127 (2001).
6. P. J. Montgomery, T. B. Moodie, Two-layer Gravity Currents with Topography. *Stud. Appl. Math.*, **102** pp. 221–266 (1999).
7. R. Barros, Conservation laws for one-dimensional shallow water models for one and two-layer flows. *Math. Models Meth. App. Sci.*, **16**, pp. 119–137 (2006).
8. L. V. Ovsyannikov, Two-layer "shallow water" model. *Journal of Applied Mechanics and Technical Physics*, **20**, Issue **2**, pp. 127–135 (1979).
9. V. V. Prasolov, *Polynomials* (New York: Springer, 2004).
10. E. I. Jury, M. Mansour, Positivity and nonnegativity conditions of a quartic equation and related problems. *IEEE Trans. Automat. Contr.*, **AC-26**, No. **2**, pp. 444–251 (1981).
11. A. T. Fuller, Root location criteria for quartic equations. *IEEE Trans. Automat. Contr.*, **AC-26**, No. **3**, pp. 777–782 (1981).