Exact Evolution Equations for Surface Waves

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Abstract: This paper considers surface gravity-capillary waves in an ideal fluid of finite depth and generalizes exact evolution equations for free gravity waves obtained by Dyachenko et al. to those for forced gravity-capillary waves. The model derived here describes the time evolution of the free surface and the velocity potential evaluated at the free surface under external pressure forcing. Two integro-differential equations are written explicitly in terms of these two dependent variables, and no extra step is required to close the system. These equations are solved numerically for the particular case of stationary periodic waves, and the results compared with analogous ones available in the literature.

INTRODUCTION

Highly nonlinear time-dependent wave phenomena at the free surface of a fluid layer have been the focus of many investigations, starting with last century's pioneering work by Stokes (1847). Nevertheless, the understanding of this type of complex motion is, in general, far from complete. Under certain hypotheses, e.g., small wave amplitude or long wavelength, various simplified mathematical models to describe the motion of the free surface are available [see Mei (1989) for a review and Choi (1995) for recent advances]. However, by their very nature these models cannot be consistently used whenever the underlying simplifying assumptions cease to hold during the time evolution of the free surface. A familiar example is that of sea waves running up a beach, where even the smallest waves eventually become steep and break.

A primary difficulty in any theoretical description of the problem lies in the fact that boundary conditions are applied at a boundary (the free surface) that is not known a priori, and needs to be determined from the solution of the problem. This poses a formidable task for analytical investigations. The only practical way of studying the free surface evolution without drastic approximations is that of numerical simulations, and many schemes have been developed for this purpose. Among these, the method of singularity distribution along the boundary of the fluid domain of interest has been quite popular. By placing source- and sink-type singularities on the free surface in combination with Green's theorem, this formulation, known as the boundary integral method, results in an integro-differential system involving time and one spatial variable for a two-dimensional flow field (Longuet-Higgins and Cokelet 1976). Alternatively, Baker et al. (1982) used vortex singularities and managed to improve the efficiency of boundary integral computations. A detailed analysis for stability and convergence can be found in Beale et al. (1996). The drawback of the boundary integral method lies in the extra step that is required to determine the strength of singularities on the free surface. Moreover, the rather awkward form of the equations resulting from this approach constitutes a serious obstacle against any analytical understanding that can be gained in limiting regimes, where asymptotic techniques might apply.

Recently, for two-dimensional deep water, Dyachenko et al. (1996a,b) showed that the Hamiltonian formulation by Zakharov (1968) and the conformal mapping technique used in a novel way can yield a much simpler set of exact evolution equations in the form of integro-differential equations involving the Hilbert transform. Dyachenko et al. (1996b) also extended the deep water formulation to the case of finite depth for free gravity waves. As a first step toward a comprehensive study of the possibilities, including numerical ones, offered by the new formalism, we present here the derivation of these evolutionary equations starting from the original Euler system, and include the effects of pressure forcing and surface tension. The resulting two explicit equations describe, with no simplifying assumptions, the time evolution of the free surface and the velocity potential evaluated at the free surface. For steady wave problems, we show that the current formulation can be reduced to the classical conformal map method used by Byatt-Smith (1970), Schwartz (1974), Cokelet (1977), and many others, including a recent work by Vanden-Broeck and Miloh (1995). We compute free traveling wave solutions of the exact set of equations and compare them with previous results.

MATHEMATICAL FORMULATION

Governing Equations

We consider surface gravity-capillary waves in an ideal fluid of finite depth. Introducing the two-dimensional Cartesian coordinates (X, Y), the velocity potential $\Phi(X, Y, t)$ is governed by the Laplace equation

$$\bar{\Phi}_{XX} + \bar{\Phi}_{YY} = 0 \quad \text{for } -\bar{h} \le Y \le \bar{y}(X, t) \tag{1}$$

with the kinematic boundary condition at the bottom given by

$$\bar{\Phi}_Y = 0 \quad \text{at } Y = -\bar{h} \tag{2}$$

On the free surface at $Y = \bar{y}(X, t)$, the following kinematic and dynamic boundary conditions need to be imposed:

$$\bar{y}_t + \bar{\Phi}_X \bar{y}_X = \bar{\Phi}_Y \quad \text{at } Y = \bar{y}(X, t) \tag{3}$$

$$\bar{\Phi}_t + \frac{1}{2}(\bar{\Phi}_X^2 + \bar{\Phi}_Y^2) + g\bar{y} + \bar{S} + \bar{P}_E/\rho = C(t) \quad \text{at } Y = \bar{y}(X, t) \tag{4}$$

where g = gravitational acceleration; $\bar{P}_E(X, t)$ = known external pressure applied on the free surface; ρ = fluid density; and $\bar{S}(X, t)$ is given by

$$\bar{S} = -\left(\frac{\gamma}{\rho}\right) \frac{\bar{y}_{XX}}{(1 + \bar{y}_X^2)^{3/2}} \tag{5}$$

with the surface tension γ. In (4), an arbitrary function of time $C(t)$ can be absorbed into $\bar{\Phi}_t$.

Derivation of Exact Evolution Equations

Here, we present the derivation of evolution equations based on the method of series solutions. Our approach differs slightly

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Note. Associate Editor: Michelle H. Teng. Discussion open until December 1, 1999. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on September 24, 1998. This paper is part of the ***Journal of Engineering Mechanics***, Vol. 125, No. 7, July, 1999. ©ASCE, ISSN 0733-9399/99/0007-0756–0760/$8.00 + $.50 per page. Paper No. 19320.

from that of Dyachenko et al. (1996b), in that we manipulate the Euler equations directly and not their variational principle. First, we find a transformation that maps the physical domain bounded by the free surface and the flat bottom into a strip of uniform thickness h, which might depend on time. The required transformation from (X, Y) to (ξ, η) can be found by solving the Dirichlet boundary-value problem given by

$$Y_{\xi\xi} + Y_{\eta\eta} = 0 \quad \text{for } -h \le \eta \le 0 \tag{6}$$

$$Y = y(\xi, t) \quad \text{at } \eta = 0 \tag{7}$$

$$Y = -\bar{h} \quad \text{at } \eta = -h \tag{8}$$

We assume that the displacement of the free surface $y(\xi, t) \equiv \bar{y}[X(\xi, 0, t), t]$, parametrically written in terms of ξ, can be expanded as

$$y(\xi, t) = \sum_{n=-\infty}^{n=\infty} a_n(t) e^{ink\xi} \tag{9}$$

where $k = 2\pi/l$; and l = period in the transformed plane. For real y, $a_{-n} = a_n^*$ has been assumed, where a_n^* is the complex conjugate of a_n. The solution of (6)–(8) can be easily obtained as

$$Y(\xi, \eta, t) = \left(\frac{\bar{h}}{h}\right)\eta + a_0\left(\frac{\eta + h}{h}\right) + \sum{}' \, a_n \frac{\sinh nk(\eta + h)}{\sinh nkh} e^{ink\xi} \tag{10}$$

where Σ' denotes summation over n, except $n = 0$. By using the Cauchy-Riemann relations, $X_\xi = Y_\eta$ and $X_\eta = -Y_\xi$, we can find X_ξ as

$$X_\xi = 1 + \frac{(\bar{h} - h) + a_0}{h} + \sum{}' \, nka_n \frac{\cosh nk(\eta + h)}{\sinh nkh} e^{ink\xi} \tag{11}$$

In (11), the choice of $h(t)$ is arbitrary. If we take

$$h(t) = \bar{h} + a_0(t) \tag{12}$$

the solutions of the Euler system [(1)–(4)] will be periodic in X, with the same (constant) period l of the transformed plane. Then, as $\eta \to 0$, (10) and (11) can be written as

$$x_\xi - 1 = -\mathcal{T}_c[y_\xi]; \quad y_\xi = -\mathcal{T}_x[x_\xi - 1] \tag{13}$$

where $x(\xi, t) = X(\xi, 0\ t)$. The two integral operators, $\mathcal{T}_c$ and $\mathcal{T}_s$, are

$$\mathcal{T}_c(f) = \frac{1}{2h} ⨍_{-\infty}^{\infty} f(\xi')\coth[\pi(\xi' - \xi)/2h]\, d\xi' \tag{14}$$

$$\mathcal{T}_s(f) = -\frac{1}{2h} ⨍_{-\infty}^{\infty} f(\xi')\operatorname{cosech}[\pi(\xi' - \xi)/2h]\, d\xi' \tag{15}$$

$$\mathcal{T}_c[\mathcal{T}_s(f)] = f; \quad \mathcal{T}_s[\mathcal{T}_c(f)] = f \tag{16}$$

where $⨍$ represents the Cauchy principal value integral; and, to obtain (13), we have used

$$\mathcal{T}_c(e^{ink\xi}) = i \coth nkhe^{ink\xi}; \quad \mathcal{T}_s(e^{ink\xi}) = -i \tanh nkhe^{ink\xi} \tag{17}$$

Integrating (13) once with respect to ξ, we can determine $x(\xi, t)$ as

$$x(\xi, t) = \xi - \mathcal{T}_c[y] + x_0(t) \tag{18}$$

where the choice of $x_0(t)$, or the choice of the origin of the the X-coordinate in the physical domain, is arbitrary.

The analogous choice for the Y-coordinate defines the total mean depth

$$H = \bar{h} + \frac{1}{l}\int_{-l/2}^{l/2} y\, dx = \bar{h} + M[\bar{y}] \tag{19}$$

By using (13) in (19), the quantity $M[\bar{y}]$, the mean value of the free surface in the physical plane, can be written as

$$M[\bar{y}] = \frac{1}{l}\int_{-l/2}^{l/2} \bar{y}\, dX = m[y] - \frac{1}{l}\int_{-l/2}^{l/2} y\mathcal{T}_c[y_\xi]\, d\xi \tag{20}$$

where $m[y](=a_0)$ = mean value in the transformed plane (i.e., integration over ξ rather than x), given by

$$m[y] = \frac{1}{l}\int_{-l/2}^{l/2} y\, d\xi \tag{21}$$

Notice that the total mean depth H is necessarily independent of time because of mass conservation [(27)].

In the transformed plane, the velocity potential $\Phi(\xi, \eta, t) \equiv \bar{\Phi}[X(\xi, \eta, t), Y(\xi, \eta, t), t]$ and its harmonic conjugate $\Psi(\xi, \eta, t)$ satisfy the Laplace equation. For example the stream function Ψ can be found by solving the following boundary-value problem:

$$\Psi_{\xi\xi} + \Psi_{\eta\eta} = 0 \quad \text{for } -h \le \eta \le 0 \tag{22}$$

$$\Psi = \psi(\xi, t) \quad \text{at } \eta = 0 \tag{23}$$

$$\Psi = -Q_h(t) \quad \text{at } \eta = -h \tag{24}$$

In (23) and (24), $\psi(\xi, t) \equiv \Psi(\xi, 0, t)$ = stream function at the free surface to be determined, while $Q_h(t)$ = function of time only, implying that the bottom is always a streamline. Since only the difference between the stream function at the free surface and that at the bottom is physically meaningful, we may set $Q_h = 0$ without loss of generality and absorb the time dependence into ψ.

Applying the same technique we used for the spatial variables (X, Y) to the dependent variables (Φ, Ψ) yields

$$\phi_\xi - m[\psi]/h = -\mathcal{T}_s(\psi_\xi); \quad \psi_\xi = -\mathcal{T}_s[\phi_\xi - m(\psi)/h] \tag{25a,b}$$

where $\phi(\xi, t) \equiv \Phi(\xi, 0, t)$; and $m[\psi]$ = mean value of ψ in the transformed plane, as defined by (21), with ψ in place of y. The freedom of choice introduced by $m[\psi]$ reflects the Galilean invariance of Euler equations in physical space.

Once the time evolution of y and ϕ is known, x and ψ can be readily determined from (13) and (25) (or vice versa).

The evolution equations for y and ϕ can be found from the two boundary conditions at the free surface [(3) and (4)] as follows. The kinematic boundary condition [(3)] can be written, by using the chain rule for differentiation, as

$$y_t x_\xi - x_t y_\xi = -\psi_\xi \tag{26}$$

where we have used $\bar{y}_t = y_t - \bar{y}_X x_t$ and $\bar{y}_X = y_\xi/x_\xi$. Integrating (26) with respect to ξ yields

$$\frac{d}{dt}\int_{-l/2}^{l/2} yx_\xi\, d\xi = 0 \tag{27}$$

which implies the conservation of mass.

Similarly, the dynamic equation [(4)] can be written as

$$\phi_t + \frac{1}{J}\left[-\phi_\xi(y_t y_\xi + x_t x_\xi) + \psi_\xi(y_t x_\xi - x_t y_\xi) + \frac{1}{2}(\phi_\xi^2 + \psi_\xi^2)\right] + gy = -P_E/\rho + S + C(t) \tag{28}$$

with the external pressure $P_E(\xi, t) = \bar{P}_E[X(\xi, 0, t), t]$; from (5), the surface tension term S is

$$S = -\left(\frac{\gamma}{\rho}\right)\frac{x_\xi y_{\xi\xi} - y_\xi x_{\xi\xi}}{(x_\xi^2 + y_\xi^2)^{3/2}} \tag{29}$$

As in Dyachenko et al. (1996b), we can further simplify these implicit equations [(26) and (28)] by isolating the time-

derivatives of the dependent variables. We first notice that (26) can be written as

$$\mathrm{Im}\left(\frac{z_t}{z_\xi}\right) = -\frac{\psi_\xi}{J} \tag{30}$$

where $z = x + iy$; and the Jacobian J is given by $J = x_\xi^2 + y_\xi^2$. Second, real and imaginary parts of analytic functions evaluated at the free surface satisfy relations of the same type as (13) or (25). Since (z_t/z_ξ) is harmonic, we can determine the real part of (z_t/z_ξ) by

$$\mathrm{Re}\left(\frac{z_t}{z_\xi}\right) = -\mathcal{T}_c\left[\mathrm{Im}\left(\frac{z_t}{z_\xi}\right)\right] + q(t) = \mathcal{T}_c\left[\frac{\psi_\xi}{J}\right] + q(t) \tag{31}$$

where Re and Im = real and imaginary parts, respectively. To obtain (31), we have used, from the analytic function theory, $\dot{h} = -m[\psi_\xi/J]$, where $\dot{h}$ represents the time derivative of $h(t)$. Since $q(t)$ in (31) is related to $x_0(t)$ in (18), we can fix one of these time-dependent functions and hence choose $q(t) = 0$. Then (31) can be written as

$$y_t y_\xi = x_t x_\xi = J\mathcal{T}_c(\psi_\xi/J) \tag{32}$$

By solving (26) and (32) for x_t and y_t, we have explicit evolution equations for x and y as

$$x_t = x_\xi \mathcal{T}_c[\psi_\xi/J] + y_\xi(\psi_\xi/J) \tag{33}$$

$$y_t = -x_\xi(\psi_\xi/J) + y_\xi \mathcal{T}_c[\psi_\xi/J] \tag{34}$$

By use of (26) and (32), the dynamic equation [(28)] can be written as

$$\phi_t + \frac{1}{J}\left[\frac{1}{2}\phi_\xi^2 - \frac{1}{2}\psi_\xi^2 - J\phi_\xi\mathcal{T}_c(\psi_\xi/J)\right] + gy = -P_E/\rho + S + C(t) \tag{35}$$

Eqs. (34) and (35) form a complete set of evolution equations for y and ϕ when (13) and (25) are used for expressing x and ψ in terms of y and ϕ. Eqs. (34) and (35) with $P_E = 0$ and $\gamma = 0$ coincide with those obtained by Dyachenko et al. (1996b) through the Hamiltonian formulation of the free water-wave problem. As $h \to \infty$, $\mathcal{T}_c$ and $\mathcal{T}_s$ become $\mathfrak{H}$ and $-\mathfrak{H}$, respectively, where $\mathfrak{H}$ is the Hilbert transform given by

$$\mathfrak{H}(f) = \fint_{-\infty}^{\infty} \frac{f(\xi')}{\xi' - \xi}\, d\xi \tag{36}$$

PERIODIC STEADY GRAVITY WAVES

Governing Equation

For steady waves ($\partial/\partial t = 0$), the kinematic equation [(26)] reduces to

$$\psi_\xi = 0 \tag{37}$$

We can take $\psi(\xi) = Q_0$, where Q_0 is a constant, implying that the free surface is another streamline for steady flow. Then (25) yields

$$\phi_\xi = (Q_0/h) = c \tag{38}$$

where c = wave speed defined by the following integration carried out for a fixed Y:

$$c = \frac{1}{l}\int_{-l/2}^{l/2} \bar{\Phi}_X\, dX \tag{39}$$

By substituting (37) and (38) into the dynamic boundary condition [(35)] with $P_E = S = 0$, we finally have

$$\frac{1}{2}\frac{c^2}{x_\xi^2 + y_\xi^2} + gy = \frac{1}{2}U^2 \tag{40}$$

where we have written $C = U^2/2$; and the arbitrary constant U can be determined uniquely, once the origin is fixed, as we shall see presently. Since x_ξ can be expressed in terms of y by (13), (40) is a single equation for y parameterized by ξ. When we use $\phi = c\xi$ as an independent variable instead of ξ, we can recover the equation derived by Byatt-Smith (1970).

To compare the previous results by Cokelet (1977), we now choose the origin of our coordinate system so that the mean value $m(y)$ is zero and $h = \bar{h}$. Then U becomes, by integrating (40) with respect to ξ and imposing $m(y) = 0$

$$U^2 = c^2\left(\frac{1}{l}\int_{-l/2}^{l/2} \frac{1}{x_\xi^2 + y_\xi^2}\, d\xi\right) \tag{41}$$

Eq. (40) can be written as

$$x_\xi^2 + y_\xi^2 = \frac{c^2}{U^2 - 2gy} \tag{42}$$

For small-amplitude waves, we can expand y and c as

$$y = a_1 \cos k\xi + a_2 \cos 2k\xi + \cdots \tag{43}$$

$$c^2 = c_0^2(1 + \alpha_1 + \alpha_2 + \cdots) \tag{44}$$

where k = wave number defined by $k = 2\pi/l$. In (43) and (44), we have assumed that $ka_n = O(\varepsilon^n)$ and $\alpha_n = O(\varepsilon^n)$, where ε is the wave slope. By substituting (43) and (44) into (42) and using (13) and (41), we find that

$$c_0^2 = (g/k)\tanh(kh); \quad ka_2 = \frac{3 + \tanh^2 kh}{4\tanh^3 kh}(ka_1)^2 \tag{45}$$

$$\alpha_1 = 0; \quad \alpha_2 = \frac{9 - 6\tanh^2 kh + 5\tanh^4 kh}{4\tanh^4 kh}(ka_1)^2 \tag{46}$$

When y is written in terms of x using (18) and h is replaced by H using (19), (43) becomes identical to the solutions found by Stokes (1847).

Numerical Solutions

To compute wave profiles, we look for solutions of (42) by using the Newton-Raphson method. We take wave speed c, or wave amplitude a defined by $2a = y(0) - y(l/2)$, as a parameter for a given wavelength l so that the wave amplitude, or wave speed, is determined by the solution. For example, when we choose c as a parameter, we can write

$$x = x_0 + x'; \quad y = y_0 + y'; \quad U = U_0 + U' \tag{47}$$

where (x_0, y_0, U_0) = initial guess; and (x', y', U') = correction to be found. Then, from (42), the linearized equation for the correction is

$$-2x_{0\xi}x'_\xi - 2y_{0\xi}y'_\xi + \frac{c^2}{(U_0^2 - 2y_0)^2}(2y' - U') = x_{o\xi}^2 + y_{o\xi}^2 - \frac{c^2}{U_0^2 - 2gy_0} \tag{48}$$

where, from (13) and (41), x' and U' can be written as

$$x'_\xi = -\mathcal{T}_c[y'_\xi]; \quad U' = -\frac{2c^2}{l}\int_{-l/2}^{l/2} \frac{x_{0\xi}x'_\xi + y_{0\xi}y'_\xi}{(x_{o\xi}^2 + y_{o\xi}^2)^2}\, d\xi \tag{49}$$

Substituting (49) into (48) yields a single equation for y'

$$M[x_0, y_0, U_0; c]y' = R[x_0, y_0, U_0; c] \tag{50}$$

With the representation of y' in terms of discrete Fourier (cosine) series of N-modes, we can write (48) evaluated at $\xi = \xi_i = il/N$ ($i = 1, \cdots, N$) as

$$M_{ij}y'_j = R_i \tag{51}$$

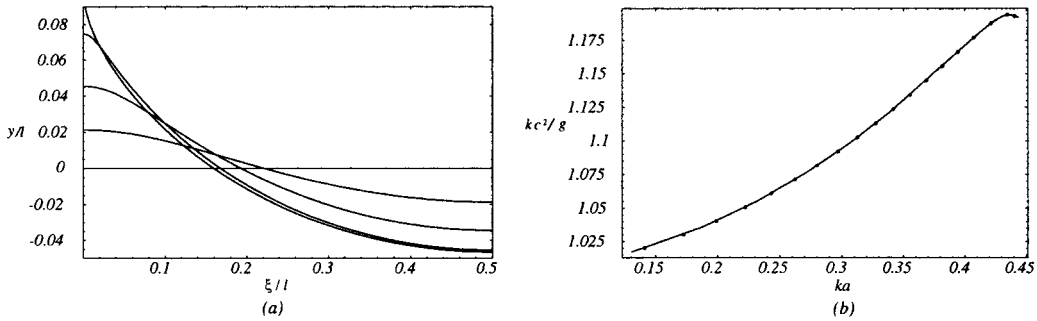

FIG. 1. Graph of: (a) Periodic Wave Profiles in Deep Water for *a/l* = 0.02, 0.04, 0.06, and 0.701; (b) Wave Speed versus Wave Amplitude [Present Calculations (· · ·) Are Compared with Calculations (—) by Cokelet (1977)]

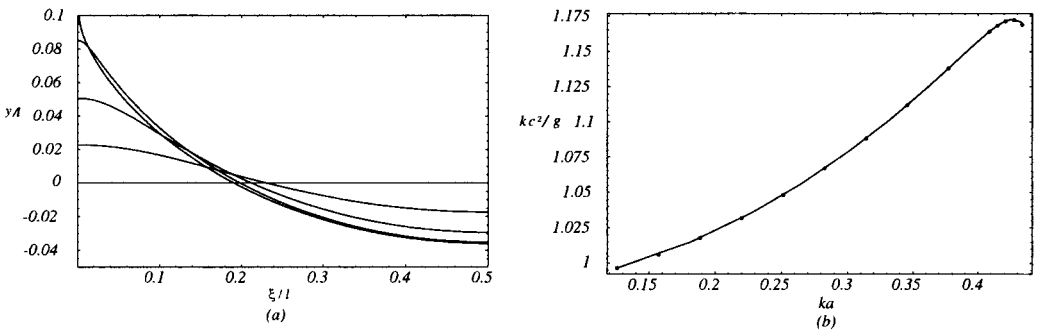

FIG. 2. Graph of: (a) Periodic Wave Profiles in Water of Finite Depth of $\bar{h}/l$ = 0.366 for *a/l* = 0.02, 0.04, 0.06, and 0.069; (b) Wave Speed versus Wave Amplitude [Present Calculations (· · ·) Are Compared with Calculations (—) by Cokelet (1977)]

and solve the algebraic equations [(51)] iteratively until $\max_j |y_j'|$ becomes smaller than an assigned error bound (typically 10^{-6}). For c close to c_0, we choose the initial guess for the wave amplitude a from (44) and (46). Then we proceed to find the solution for larger c by taking the previous result for smaller c as the initial guess.

In Fig. 1 and 2, we show numerical results with $N = 2^8$ (for wave amplitudes near the maximum value, we choose $N = 2^9$). The relationship between wave speed and wave amplitude shows good agreement with Cokelet (1977) for both deep and finite-depth water. The maximum speed $c^2_{\max}$ and the maximum wave amplitude a_{max} are found to be $c^2_{\max} = 1.1946g/k$ (at $ka = 0.435$) and $ka_{\max} = 0.441$ for deep water, and $c^2_{\max} = 1.1722g/k$ (at $ka = 0.427$) and $ka_{\max} = 0.434$ for water of finite depth of $h/l = 0.366$. These results compare favorably with the results of Cokelet (1977): $c^2_{\max} = 1.1945g/k$ (at $ka = 0.436$) and $ka_{\max} = 0.443$ for deep water, and $c^2_{\max} = 1.1725g/k$ (at $ka = 0.426$) and $ka_{\max} = 0.433$ for water of finite depth of $h/l = 0.366$.

DISCUSSION

The major difference between the current formulation originally proposed by Dyachenko et al. (1996b) and others is that the evolution equations are written explicitly in terms of two dependent variables, and no intermediate steps are required to close the system. This offers a good starting point for further analysis and is advantageous for numerical simulations of time-dependent problems. Here we have considered only the simplest case of a traveling wave solution. The study of this set of equations for highly nonlinear time-dependent wave phenomena will be reported in future publications.

ACKNOWLEDGMENTS

We would like to thank Yi Li for useful comments. This work was in part supported by the U.S. Department of Energy under the CHAMMP program, the W-7405-ENG-36 contract, and the Applied Mathematical Sciences contract KC-07-01-01.

APPENDIX I. REFERENCES

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APPENDIX II. NOTATION

The following symbols are used in this paper:

a = wave amplitude;
c = wave speed;

g = gravitational acceleration;
H = total mean depth;
$\mathfrak{H}[f]$ = Hilbert transform of f;
$\underline{h}$ = thickness of transformed plane;
h = water depth;
k = wave number;
l = wavelength;
$m[f]$ = mean value of f in transformed plane;
P_E = external pressure applied on free surface;
$\mathcal{T}_c[f]$ = integral transform of f with coth function as kernel;
$\mathcal{T}_s[f]$ = integral transform of f with cosech function as kernel;
y = surface elevation;
γ = surface tension;
ρ = density;
ϕ = velocity potential at free surface; and
ψ = stream function at free surface.