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Modeling of Strongly Nonlinear Internal Gravity Waves

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Abstract

We consider strongly nonlinear internal gravity waves in a multilayer fluid and propose a mathematical model to describe the time evolution of large amplitude internal waves. Model equations follow from the original Euler equations under the sole assumption that the waves are long compared to the undisturbed thickness of one of the fluid layers. No small amplitude assumption is made. Both analytic and numerical solutions of the new model exhibit all essential features of large amplitude internal waves, observed in the ocean but not captured by the existing weakly nonlinear models. Differences between large amplitude surface and internal solitary waves are addressed.

1 Introduction

Two fundamental physical mechanisms, nonlinearity and dispersion, play important roles in the evolution of long internal gravity waves commonly observed in the ocean, more generally in any stratified fluids. Under the assumption of weakly nonlinear and weakly dispersive, the balance between these two effects leads to a remarkable phenomenon of solitons, very localized disturbances propagating without any change in form. However there have been a number of observations of large amplitude internal waves for which the weakly nonlinear assumption is far from being realistic. For example, in the northwestern coastal area of the Unites States, Stanton & Ostrovsky (1998) observed the train of tidally-generated internal waves in the form of solitary waves depressing the 7m deep pycnocline up to 30m. Evidently the weakly nonlinear models like the Korteweg-de Vries (KdV) equation commonly used in the community of geophysical fluid dynamics are completely irrelevant in such cases and new models need to be developed. The theory to be desired is of course one being able to account for full nonlinearity of the problem but this simple idea has never been successfully realized. Recently Choi and Camassa (1996, 1999) have derived various new models for strongly nonlinear dispersive waves in a simple two-layer system, using a systematic asymptotic expansion method for a natural small parameter in the ocean, that is the aspect ratio between vertical and horizontal length scales. Analytic and numerical solutions of the new models describe strongly nonlinear phenomena which have been observed but not been explained by using any weakly nonlinear models. This indicates the importance of strongly nonlinear aspects in the internal wave propagation. An interesting point to make is that the including strong nonlinearity does not simply add new nonlinear terms but change dispersive characters, from linear to nonlinear.

Despite of recent progress, the two-layer system is often too simplified to be useful for practical applications. For instance, the two-layer model has not only its limited despcription of smoothly stratified ocean but also the lack of higher-order baroclinic wave modes. Therefore the theory needs be extended to the more general situation relevant for real oceanic applications. Here we will study a continuously stratified fluid approximated by a stack of several homogeneous layers.

2 Governing Equations

For an inviscid and incompressible fluid of density ρ_i, the velocity components in Cartesian coordinates (u_i, w_i) and the pressure p_i $(i = 1, \cdots, N)$ satisfy the continuity equation and the Euler equations:

$$u_{ix} + w_{iz} = 0\,, \tag{1}$$

$$u_{it} + u_i u_{ix} + w_i u_{iz} = -p_{ix}/\rho_i\,, \tag{2}$$

$$w_{it} + u_i w_{ix} + w_i w_{iz} = -p_{iz}/\rho_i - g\,, \tag{3}$$

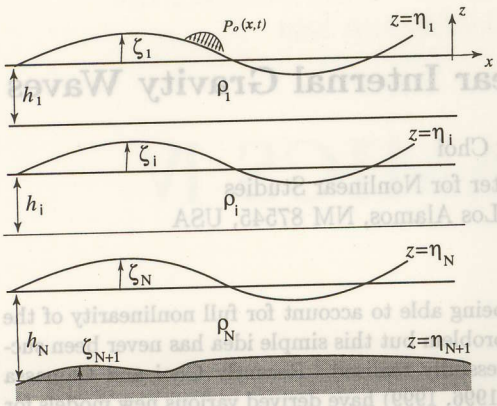

Figure 1: A multi-layer system

where g is the gravitational acceleration and subscripts with respect to space and time represent partial differentiation. For a stable stratification, $\rho_i < \rho_{i+1}$ is assumed . The boundary conditions at the upper and lower interfaces of the i-th layer require the continuity of normal velocity and pressure:

$$\eta_{it} + u_i\eta_{ix} = w_i, \quad \text{at} \quad z = \eta_i(x,t)\,, \tag{4}$$

$$\eta_{i+1t} + u_i\eta_{i+1x} = w_i, \quad \text{at} \quad z = \eta_{i+1}(x,t)\,, \tag{5}$$

$$p_i = p_{i-1} \quad \text{at} \quad z = \eta_i(x,t)\,. \tag{6}$$

In (4)–(6), $\eta_i(x,t)$ $(i = 1, \cdots, N)$ is the location of the upper interface of the i-th layer given by

$$\eta_i = \zeta_i - \sum_{j=1}^{i-1} h_j\,, \tag{7}$$

where h_i is the undisturbed thickness of the i-th layer and $\zeta_i(x,t)$ is the interfacial displacement. The position of the bottom, $\eta_{N+1}(x,t)$ can be written as

$$\eta_{N+1} = \zeta_{N+1} - \sum_{j=1}^{N} h_j = \zeta_{N+1} - h\,, \tag{8}$$

where $\zeta_{N+1}(x,t)$ is the bottom topography and h is the total depth.

3 Nonlinear Dispersive Model

Under the assumption that the thickness of each layer is much smaller than the characteristic wavelength, we have the following scaling relation between u_i and w_i, from the continuity equation (1),

$$w_i/u_i = O(h_i/L) = O(\epsilon) \ll 1\,, \tag{9}$$

where L is a typical wavelength. For finite-amplitude waves, we also assume that

$$u_i/U_0 = O(\zeta_i/h_i) = O(1)\,, \tag{10}$$

where U_0 is a characteristic speed chosen as $U_0 = (gh)^{1/2}$. Based on the scalings in (9)–(10), we non-dimensionalize all physical variables as

$$\left.\begin{array}{l} x = Lx^*, \quad z = hz^*, \quad t = (L/U_0)t^*, \\ \zeta_i = h\zeta_i^*, \quad p_i = (\rho_i U_0^2)p_i^*, \\ u_i = U_0 u_i^*, \quad w_i = \epsilon U_0 w_i^*, \end{array}\right\} \tag{11}$$

and assume that all variables with asterisks are $O(1)$ in ϵ.

By integrating (1)–(2) across the i-th layer $(\eta_{i+1} \le z \le \eta_i)$ and imposing the boundary conditions (4)–(6), we obtain the layer-mean equations (Wu 1981):

$$H_{it} + \left(H_i\overline{u}_i\right)_x = 0, \quad H_i = \eta_i - \eta_{i+1}\,, \tag{12}$$

$$(H_i\overline{u}_i)_t + \left(H_i\overline{u_iu_i}\right)_x = -H_i\overline{p_{ix}}\,, \tag{13}$$

where the layer-mean quantity $\bar{f}$ is defined as

$$\bar{f}(x,t) = \frac{1}{H_i}\int_{\eta_{i+1}}^{\eta_i} f(x,z,t)\,\mathrm{d}z\,, \tag{14}$$

and we have dropped the asterisks for simplicity. The quantities $\overline{u_iu_i}$ and $\overline{p_{ix}}$ prevent closure of the system of layer-mean equations (12)–(13). The following analysis will therefore focus on expressing these quantities in terms of H_i and $\overline{u}_i$ to close the system.

The vertical momentum equation (3) for the i-th layer can be written as

$$p_{iz} = -1 - \epsilon^2\left[w_{it} + u_iw_{ix} + w_iw_{iz}\right]. \tag{15}$$

Hence, we can seek an asymptotic expansion of $f = (u_i, w_i, p_i)$ in powers of ϵ^2,

$$f(x,z,t) = f^{(0)} + \epsilon^2 f^{(1)} + O(\epsilon^4)\,, \tag{16}$$

where $f^{(m)} = O(1)$ for $m = 0, 1, \ldots$.

From (15)–(16), after imposing the pressure continuity across the interface given by (6), the leading-order pressure $p_i^{(0)}$ is

$$p_i^{(0)} = -(z - \eta_i) + P_i(x,t)\,, \tag{17}$$

where $P_i(x,t) = p_{i-1}(x,\eta_i,t)$ is the pressure at the upper surface of the i-th layer. For $i = 1$, since the upper boundary is free, we impose $p_1 = P_1(x,t)$ at $z = \eta_1$, instead of (6), where P_1 is

the known atmospheric pressure. By substituting (16)–(17) into (2), one obtains

$$u_i^{(0)} = u_i^{(0)}(x,t) \quad \text{if} \quad u_{iz}^{(0)} = 0 \quad \text{at} \quad t = 0\,. \tag{18}$$

Notice that condition (18) is automatically satisfied if we assume that the flow is initially irrotational (Choi & Camassa 1996). From (1), we can now obtain the leading-order vertical velocity $w_i^{(0)}$ satisfying the kinematic boundary condition (5) at the interface as

$$w_i^{(0)} = -(u_{ix}^{(0)})(z - \eta_{i+1}) + D_i\eta_{i+1}\,, \tag{19}$$

where D_i stands for material derivative,

$$D_i = \partial_t + u_i^{(0)}\partial_x\,. \tag{20}$$

From (16) and (18), it can be easily shown that

$$H_i\overline{u_i u_i} = H_i\overline{u}_i\,\overline{u}_i + O(\epsilon^4)\,, \tag{21}$$

so that the layer-mean horizontal momentum equation (13) in dimensionless form becomes

$$\overline{u}_{it} + \overline{u}_i\,\overline{u}_{ix} = -\overline{p_{ix}} + O(\epsilon^4)\,. \tag{22}$$

At $O(\epsilon^2)$, from (15)–(16) and (19), the equation for $p_i^{(1)}$ is

$$\begin{aligned} p_{iz}^{(1)} &= -\left[w_{it}^{(0)} + u_i^{(0)}w_{ix}^{(0)} + w_i^{(0)}w_{iz}^{(0)}\right] \\ &= a_i(x,t)(z - \eta_{i+1}) + b_i(x,t)\,. \end{aligned} \tag{23}$$

In equation (23), a_i and b_i are defined as

$$a_i(x,t) = \overline{u}_{ixt} + \overline{u}_i\,\overline{u}_{ixx} - (\overline{u}_{ix})^2\,, \tag{24}$$

$$b_i(x,t) = -D_i^2\eta_{i+1}\,, \tag{25}$$

where we have used $\overline{u_i} = u_i^{(0)} + O(\epsilon^2)$. After integrating (23) with respect to z and imposing the dynamic boundary condition in (6), we obtain $p_i^{(1)}$:

$$\begin{aligned} p_i^{(1)} = \tfrac{1}{2}a_i\Big[(z - \eta_{i+1})^2 - (\eta_i - \eta_{i+1})^2\Big] \\ + b_i\Big[(z - \eta_{i+1}) - (\eta_i - \eta_{i+1})\Big]\,. \end{aligned} \tag{26}$$

From (17) and (26), we can find $\overline{p_{ix}}$, the right-hand-side term of the horizontal momentum equation (22), and the final set of equations for the i-th layer can be found, in dimensional form, as

$$H_{it} + \left(H_i\overline{u}_i\right)_x = 0, \qquad H_i = \eta_i - \eta_{i+1}\,, \tag{27}$$

$$\begin{aligned} &\overline{u}_{it} + \overline{u}_i\overline{u}_{ix} + g\eta_{ix} + P_{ix}/\rho_i \\ &= \frac{1}{H_i}\left(\tfrac{1}{3}H_i^3 a_i + \tfrac{1}{2}H_i^2 b_i\right)_x + \left(\tfrac{1}{2}H_i a_i + b_i\right)\eta_{i+1x}\,. \end{aligned} \tag{28}$$

For $1 \le i \le N$, (27)–(28) determines the evolution of $2N$-unknowns, $\overline{u}_i$ and η_i, while P_i is given by the following recursion formula obtained from (17) and (26):

$$P_{i+1} = P_i + \rho_i\left(gH_i - \tfrac{1}{2}a_i H_i^2 - b_i H_i\right)\,. \tag{29}$$

These equations have been derived under the sole assumption that the waves are long compared with the water depth and we have imposed no assumption on wave amplitude. The kinematic equation (27) is exact, while the dynamic equations (28)–(29) have non-hydrostatic contributions correct up to $O(\epsilon^2)$.

Even for rigid-lid approximation ($\zeta_1 = 0$), the system of (27)–(28) is still valid if P_1 is regarded as unknown pressure at the top boundary.

4 Hydrostatic Approximation

When we neglect dispersive effects (say $a_i = b_i = 0$), we recover the hydrostatic equations for multilayer system ($i = 1, \cdots, N$):

$$H_{it} + \left(H_i\overline{u}_i\right)_x = 0, \qquad H_i = \eta_i - \eta_{i+1}\,, \tag{30}$$

$$\begin{aligned} &\overline{u}_{it} + \overline{u}_i\overline{u}_{ix} + g\sum_{j=1}^{i}\left(\frac{\rho_j}{\rho_i}\right)H_{jx} + g\eta_{i+1x} \\ &\qquad = -P_{1x}/\rho_i\,, \end{aligned} \tag{31}$$

where (29) has been used.

5 One-layer System

For the case of a homogeneous layer ($N = 1$), equations (27)–(28) reduce to the system of equations derived by Green & Naghdi (1976) by using the director sheet theory. Without any topography at the bottom and external pressure at the free surface ($\zeta_2 = P_1 = 0$), equations (27)–(28) possess the solitary wave solution given by

$$\zeta_1(X) = a\,\mathrm{sech}^2(kX)\,, \quad X = x - ct\,, \tag{32}$$

$$k^2h_1^2 = \frac{3a}{4(h_1 + a)}, \quad \frac{c^2}{gh_1} = 1 + \frac{a}{h_1}\,, \tag{33}$$

where a and c are wave amplitude and speed, respectively.

For weakly nonlinear unidirectional waves of $a/h_1 = O(\epsilon^2)$, (27)–(28) can be reduced to the KdV equation for ζ_1:

$$\frac{1}{\sqrt{gh_1}}\zeta_{1t} + \zeta_{1x} + \frac{3}{2h_1}\zeta_1\zeta_{1x} + \frac{h_1^2}{6}\zeta_{1xxx} = 0\,. \tag{34}$$

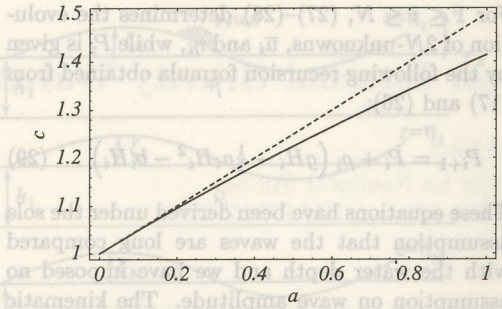

Figure 2: Wave speed ($c/\sqrt{gh_1}$) versus wave amplitude (a/h_1) curves for surface solitary waves: ——, strongly nonlinear theory given by (33); – – –, weakly nonlinear (KdV) theory given by (35).

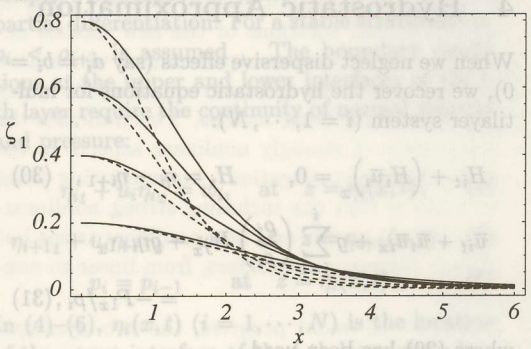

Figure 3: Surface solitary wave solutions (——-) given by (32) for a/h_1 =(0.2, 0.4, 0.6, 0.8) compared with KdV solitary waves (– – –) of the same amplitude. Here we show only half of the wave profile.

It is interesting to notice that the solitary wave solution of the KdV equations has the same form as that of the GN equations give by (32) with

$$k^2h_1^2 = \frac{3a}{4h_1}, \quad \frac{c}{\sqrt{gh_1}} = 1 + \frac{a}{2h_1}, \tag{35}$$

which is the small amplitude limit of (33).

As shown in figure 2, the solitary wave speed from the strongly nonlinear theory increases with wave amplitude at a much slower rate than that from the KdV theory. In figure 3, one can see that the strongly nonlinear solitary wave solution given by (32) is certainly wider than the KdV soliton of the same amplitude. In §6, we will make the similar observations for interfacial solitary waves in a two-layer system and these features seem to be generic in strongly nonlinear waves.

6 Two-layer System

As shown in Choi & Camassa (1999), with rigid lid approximation ($\zeta_1 = 0$), the coupled system of (27)–(28) for $i = 1, 2$ can be reduced, for traveling waves at the interface, to

$$(\zeta_{2X})^2 = \frac{3\zeta_2^2\left[c^2(\rho_1 H_2 + \rho_2 H_1) - g(\rho_2 - \rho_1)H_1H_2\right]}{c^2(\rho_1 h_1^2 H_2 + \rho_2 h_2^2 H_1)}, \tag{36}$$

where $H_1 = h_1 - \zeta_2$ and $H_2 = h_2 + \zeta_2$. This particular form of equation has been obtained earlier by Miyata (1985) using conservation laws, for steay flows.

When replacing ζ_2 by wave amplitude a in (36), the numerator has to vanish and this gives wave speed c in terms of wave amplitude a as

$$\frac{c^2}{{c_0}^2} = \frac{(h_1 - a)(h_2 + a)}{h_1h_2 - ({c_0}^2/g)\,a}, \tag{37}$$

where c_0 is the linear first baroclinic wave speed given by

$${c_0}^2 = \frac{gh_1h_2(\rho_2 - \rho_1)}{\rho_1 h_2 + \rho_2 h_1}. \tag{38}$$

For uni-directional weakly nonlinear waves of $a/h_2 = O(\epsilon^2)$, (27)–(28) for $i = 1, 2$ can be further simplified to the KdV equation for ζ_2 as before:

$$\zeta_{2t} + c_0\zeta_{2x} + c_1\zeta_2\zeta_{2x} + c_2\zeta_{2xxx} = 0, \tag{39}$$

where

$$c_1 = -\frac{3c_0}{2}\frac{\rho_1 h_2^2 - \rho_2 h_1^2}{\rho_1 h_1 h_2^2 + \rho_2 h_1^2 h_2}, \tag{40}$$

$$c_2 = \frac{c_0}{6}\frac{\rho_1 h_1^2 h_2 + \rho_2 h_1 h_2^2}{\rho_1 h_2 + \rho_2 h_1}. \tag{41}$$

The solitary wave solution of (39) is given by (32) with

$$k^2 = \frac{ac_1}{12c_2}, \quad c = c_0 + \frac{c_1}{3}a. \tag{42}$$

Figures 4 and 5 clearly show that strongly nonlinear solitary waves are slower and broader than weakly nonlinear waves of the same amplitude. In fact Choi & Camassa (1999) have demonstrated that the strongly nonlinear theory yields excellent agreement with available expeimental data or numerical solutions of the Euler equations.

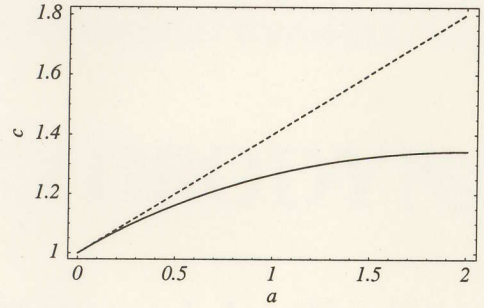

Figure 4: Wave speed (c/c_0) versus wave amplitude (a/h_2) curves for internal solitary wave ($\rho_1/\rho_2 = 0.99$, $h_1/h_2 = 5$): ——, strongly nonlinear theory given by (37); - - -, weakly nonlinear (KdV) theory given by (42).

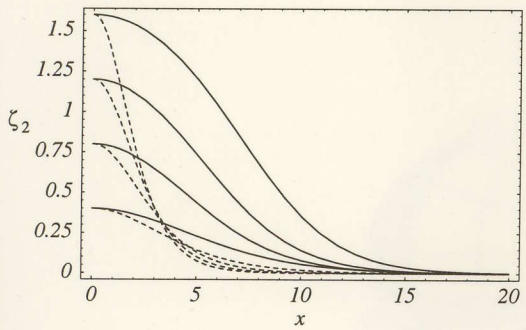

Figure 5: Internal solitary wave solutions (——) of (36) for $\rho_1/\rho_2 = 0.99$, $h_1/h_2 = 5$ and a/h_2 =(0.4, 0.8, 1.2, 1.6) compared with KdV solitary waves (- - -) of the same amplitude given by (42).

For a two-layer system, since wave amplitude cannot be greater than the total depth, there is a maximum wave amplitude beyond which no solitary wave solution exists. As wave amplitude approaches to the limiting value, the solitary wave becomes broader and finally degenerates into a front-like, or internal bore, solution, as shown in figure 6. In this interesting limit, the amplitude and speed can be found, from (36) and (37), as

$$a_m = \frac{h_1 - h_2(\rho_1/\rho_2)^{\frac{1}{2}}}{1 + (\rho_1/\rho_2)^{\frac{1}{2}}}, \qquad (43)$$

$$c_m{}^2 = g(h_1 + h_2)\frac{1 - (\rho_1/\rho_2)^{\frac{1}{2}}}{1 + (\rho_1/\rho_2)^{\frac{1}{2}}}. \qquad (44)$$

For an inviscid homogeneous layer, front solutions are impossible without the loss of energy

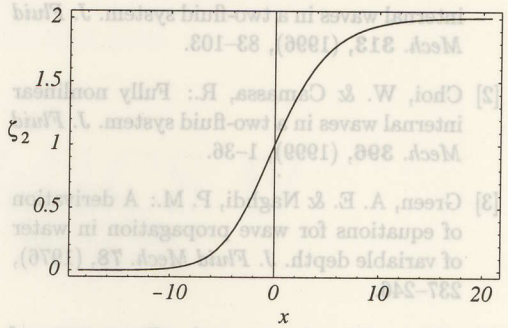

Figure 6: A front solution of (36) for maximum wave amplitude given by (43) for $\rho_1/\rho_2 = 0.99$ and $h_1/h_2 = 5$.

which is often contributed to the effects of viscosity. On the other hand, for internal waves, the front-like solutions of (36) hold the conservation law for energy.

7 Discussion

For small aspect ratio of the thickness of each fluid layer to typical wavelength, we have derived a strongly nonlinear model to describe the evolution of finite amplitude long internal waves in a multi-layer system.

The limiting behaviours of highest waves are completely different between surface and internal waves. As shown by Stokes (1880), surface waves of maximum amplitude have a sharp peak of 120°, which cannot be captured by the present long wave theory. Therefore one can see that the GN theory for long surface waves ceases to be valid when the waves become too high. On the other hand, as wave amplitude increases, internal solitary waves becomes broader and broader, leading to internal bore. Therefore the long wave theory for internal waves is expected to be valid even for large wave amplitude. This explains why the strongly nonlinear theory for internal waves gives much better agreement with experimental data or numerical solutions of the Euler equations (Choi & Camassa 1999) compared to that for surface waves.

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