St. John's, Newfoundland and Labrador, CANADA, 8-13 August 2004

A pseudo-spectral method for non-linear wave hydrodynamics

Wooyoung Choi and Christopher Kent
(University of Michigan, USA)

ABSTRACT

We present a new hybrid asymptotic-numerical method to study nonlinear wave-body interaction in three-dimensional water of arbitrary depth. After solving a simplified body problem numerically using a distribution of singularities along the body surface, we reduce the nonlinear free surface problem to a closed system of two nonlinear evolution equations, using a systematic asymptotic expansion, for the free surface elevation and the velocity potential at the free surface. The system, correct up to third order in wave steepness, is then solved using a pseudo-spectral method based on Fast Fourier Transform. We study the evolution of unstable Stokes waves and the generation of nonlinear surface waves by translating two- and three-dimensional dipoles. In order to validate our numerical method, a translating circular cylinder is also considered and our numerical solution is compared with the fully nonlinear numerical solution.

1 INTRODUCTION

The nonlinear wave-body interaction problem is one of the most exciting and challenging problems in ship hydrodynamics. With the advent of faster computers, there has been remarkable progress made over the last decade in computational hydrodynamics. Recently various numerical methods including Smooth Particle Hydrodynamics (Colagrossi *et al*. 2000) and the Level Set Method (Iafrati *et al*. 2001) as well as finite difference/volume methods have been widely used but, due to the uncertainty of turbulence modeling for free surface flows and the high computational cost, it is still problematic to solve the fully nonlinear, three-dimensional, time-dependent viscous hydrodynamic equations in the presence of a body near or on the free surface. These numerical tools are therefore rarely used in industry for preliminary ship/offshore structure design and optimization.

Though the potential flow assumption is often violated in many flow situations, the computation of potential flows still serves as a powerful tool for practical applications to predict hydrodynamic wave forces on ships and offshore structures. These *outer* potential flow solutions also play an important role of providing the boundary condition for *inner* viscous problems around the body.

The mixed Eulerian-Lagrangian method based on a boundary integral formulation, originally developed by Longuet-Higgins & Cokelet (1976) for free surface waves, has been the most widely-used computational method for inviscid free surface problems. See, for example, the review by Beck & Reed (2001) for its application to wave-body interaction problems. A major difficulty of this method lies in the free surface representation using a distribution of a large number of singularities whose location and strength must be computed at every time step. Erroneous results are sometimes produced by numerical errors introduced when approximating the singular integrals and redistributing singularities on the free surface, and care must be taken when carrying out these steps.

An alternative numerical approach to solve inviscid free surface problems has been proposed by Fenton & Rienecker (1982) using the Fourier-series expansion. For free wave problems, Dommermuth & Yue (1987) have further improved the method by expanding the nonlinear free surface boundary conditions about the mean free surface and by solving the resulting boundary value problems for each order using a pseudo-spectral method based on Fast Fourier Transform (FFT). Liu *et al*. (1992) then modified this approach for wave-body interaction problems. Although these methods have been adopted for practical applications (Lin & Kuang 2004), it is still a nontrivial task to study unsteady wave-body interactions since they are computationally expensive and numerical implementation is rather complex.

Here we present a relatively simple third-order nonlinear formulation which can be solved quickly and accurately by using a surface singularity distribution method for the body problem and a pseudo-spectral method for the free surface problem.

We first decompose the total velocity potential into the velocity potential for the body problem and that for the free surface problem. The body velocity potential satisfying the body boundary condition and the zero boundary condition at the mean free surface can be found, for example, in terms of a distribution of singularities. Using an asymptotic expansion similar to that used by Choi (1995) for free waves, the free surface problem is reduced to two nonlinear evolution equations for two physical variables defined on the free surface: the free surface elevation and the velocity potential. The resulting evolution equations are then solved by using a pseudo-spectral method based on the FFT. Since we only distribute singularities along the body surface and we solve a closed set of the evolution equations in the horizontal plane, our numerical method is substantially faster than other fully nonlinear numerical methods.

This paper is organized in the following order. After introducing the governing equations in §2, the detailed formulation is described in §3 and §4. With the numerical method described in §5, some numerical solutions for infinitely deep water are presented in §6.

2 GOVERNING EQUATIONS

For an ideal fluid we can introduce the velocity potential $\phi(\boldsymbol{x}, z, t)$ satisfying the Laplace equation:

$$\nabla^2 \phi + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \text{for} \quad -h \le z \le \zeta(\boldsymbol{x}, t), \tag{1}$$

where $\zeta(\boldsymbol{x}, t)$ is the free surface elevation, $\boldsymbol{x} = (x, y)$, and the horizontal gradient $\boldsymbol{\nabla}$ is defined by

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right). \tag{2}$$

The body boundary condition can be written as

$$\frac{\partial \phi}{\partial n} = \mathbf{V} \cdot \mathbf{n} \quad \text{on } S_B(t), \tag{3}$$

where $\mathbf{V}$ is the body velocity, $\mathbf{n}$ is the normal vector directed into the body, and $S_B(t)$ represents the instantaneous body position. The bottom boundary condition to be imposed at $z = -h$ is given by

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{at} \quad z = -h. \tag{4}$$

At the free surface, the velocity potential ϕ and the surface elevation ζ satisfy the kinematic and dynamic free surface boundary conditions:

$$\zeta_t + \boldsymbol{u} \cdot \boldsymbol{\nabla} \zeta = w \qquad \text{at} \quad z = \zeta(\boldsymbol{x}, t), \tag{5}$$

$$p = 0 \qquad \text{at} \quad z = \zeta(\boldsymbol{x}, t). \tag{6}$$

The pressure p can be found by using the Bernoulli equation:

$$\phi_t + \tfrac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \tfrac{1}{2}{\phi_z}^2 + gz + p/\rho = 0, \tag{7}$$

where g is the gravitational acceleration and ρ is the fluid density.

By substituting $z = \zeta$ into (5) and (6), and using the following chain rule for differentiation:

$$\left.\frac{\partial \phi}{\partial t}\right|_{z=\zeta} = \frac{\partial \Phi}{\partial t} - \left.\frac{\partial \phi}{\partial z}\right|_{z=\zeta} \left(\frac{\partial \zeta}{\partial t} \right), \tag{8}$$

$$\left.\frac{\partial \phi}{\partial x_j}\right|_{z=\zeta} = \frac{\partial \Phi}{\partial x_j} - \left.\frac{\partial \phi}{\partial z}\right|_{z=\zeta} \left(\frac{\partial \zeta}{\partial x_j} \right), \tag{9}$$

the free surface boundary conditions can be written in terms of the surface elevation ζ and the free surface velocity potential Φ as

$$\frac{\partial \zeta}{\partial t} + \boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla}\zeta = \left(1 + |\boldsymbol{\nabla}\zeta|^2\right) W, \tag{10}$$

$$\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}(\boldsymbol{\nabla}\Phi)^2 + g\zeta = \tfrac{1}{2}\left(1 + |\boldsymbol{\nabla}\zeta|^2\right) W^2, \tag{11}$$

where $\Phi(\boldsymbol{x}, t)$ is defined as

$$\Phi(\boldsymbol{x}, t) = \phi(\boldsymbol{x}, \zeta, t), \tag{12}$$

and W is the vertical velocity evaluated at the free surface defined as

$$W = \left.\frac{\partial \phi}{\partial z}\right|_{z=\zeta}. \tag{13}$$

If we can find the expression for W in terms of ζ and Φ, equations (10)–(11) will be a closed system for ζ and Φ. In order to write W in terms of ζ and Φ, we first expand Φ and W in Taylor series

$$\Phi(\boldsymbol{x}, t) = \phi_0 + \sum_{n=0}^{\infty} (-1)^n \frac{\zeta^{2n+1}}{(2n+1)!} \triangle^n w_0 + \sum_{n=1}^{\infty} (-1)^n \frac{\zeta^{2n}}{(2n)!} \triangle^n \phi_0, \tag{14}$$

$$W(\boldsymbol{x},t) = w_0 + \sum_{n=1}^{\infty}(-1)^n \frac{\zeta^{2n-1}}{(2n-1)!}\triangle^n \phi_0 + \sum_{n=1}^{\infty}(-1)^n \frac{\zeta^{2n}}{(2n)!}\triangle^n w_0, \tag{15}$$

where we have used $\partial^2\phi/\partial z^2 = -\triangle\phi$, $\triangle = \nabla^2$ and

$$\phi_0 = \phi(\boldsymbol{x},0,t), \quad w_0 = w(\boldsymbol{x},0,t)\,. \tag{16}$$

With the relationship between ϕ_0 and w_0 to be found under the assumption of small wave slope, we can write W, from (14)–(15), in terms of Φ and ζ successively up to any order of wave slope but in this paper we will consider the third-order expression for W.

3 FORMULATION

In this section we will describe how to find the relationship between ϕ_0 and w_0 defined in (16). We first decompose the velocity potential ϕ into

$$\phi = \phi^F + \phi^B, \tag{17}$$

where ϕ^F and ϕ^B are the solutions of the free surface and body problems, respectively, as shown in figure 1.

3.1 FREE SURFACE VELOCITY POTENTIAL

The relationship between two physical variables at the mean free surface, ϕ_0 and w_0, can be found by formally solving the following linear Dirichlet-Neumann boundary value problem for ϕ^F:

$$\left(\nabla^2 + \frac{\partial^2}{\partial z^2}\right)\phi^F = 0 \quad \text{for} \quad -h < z < 0, \tag{18}$$

$$\phi^F = \phi_0(\boldsymbol{x},t) \qquad \text{at} \quad z = 0, \tag{19}$$

$$\frac{\partial \phi^F}{\partial z} = 0 \qquad \text{at} \quad z = -h. \tag{20}$$

By using the Fourier transform defined by

$$\overline{\phi^F}(\boldsymbol{k},z,t) \equiv \int_{-\infty}^{\infty} \phi^F(\boldsymbol{x},z,t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}, \tag{21}$$

where $\boldsymbol{k} = (k_1,k_2)$, the solution of (18)–(20) in the transformed plane can be found as

$$\overline{\phi^F} = \overline{\phi_0}(\boldsymbol{k},t)\,\frac{\cosh[k(z+h)]}{\cosh(kh)}, \tag{22}$$

where $k^2 = {k_1}^2 + {k_2}^2$. Then the Fourier Transform of the total vertical velocity at the mean position $w_0(\boldsymbol{x},t)$ can be written as

$$\overline{w_0}(\boldsymbol{k},t) = \left.\overline{\frac{\partial \phi^F}{\partial z}}\right|_{z=0} + \left.\overline{\frac{\partial \phi^B}{\partial z}}\right|_{z=0} = k\,\tanh(kh)\,\overline{\phi_0} + \overline{f^B}, \tag{23}$$

where f^B represents the vertical velocity at the mean free surface induced by the body:

$$f^B(\boldsymbol{x},t) = \left.\frac{\partial \phi^B}{\partial z}\right|_{z=0}, \tag{24}$$

which can be computed after solving the body velocity potential described below. After taking the inverse Fourier Transform, we have the relationship between w_0 and ϕ_0 as

$$w_0(\boldsymbol{x},t) = -\mathrm{L}[\phi_0] + f^B(\boldsymbol{x},t). \tag{25}$$

Since we are using a pseudo-spectral numerical method, the linear integral operator L (Choi 1995) acting on a Fourier component is of interest and is defined as

$$\mathrm{L}\left[\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\right] = -k\tanh(kh)\,\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{26}$$

3.2 BODY VELOCITY POTENTIAL

The body effect on the free surface denoted by $f^B(\boldsymbol{x},t)$ can be found by solving the following boundary value problem:

$$\left(\nabla^2 + \frac{\partial^2}{\partial z^2}\right)\phi^B = 0 \qquad \text{for} \quad -h \le z \le 0, \tag{27}$$

$$\phi^B = 0 \qquad \text{at} \quad z = 0, \tag{28}$$

$$\frac{\partial \phi^B}{\partial z} = 0 \qquad \text{at} \quad z = -h, \tag{29}$$

$$\frac{\partial \phi^B}{\partial n} = \mathbf{V}^B\!\cdot\mathbf{n} - \frac{\partial \phi^F}{\partial n} \quad \text{on } S_B(t), \tag{30}$$

where ϕ^F is given by (22) for known $\phi_0(\boldsymbol{x},t)$ (see also equation (46)). Compared with classical linear formulations for wave-body interaction problems, the boundary condition at the free surface is very simple and there is no need to introduce complicated free surface Green's functions satisfying the physical linear free surface boundary condition, in particular, for infinitely deep water. Notice that the body boundary condition in (30) is imposed at the instantaneous body position. The solution of (27)–(30) can

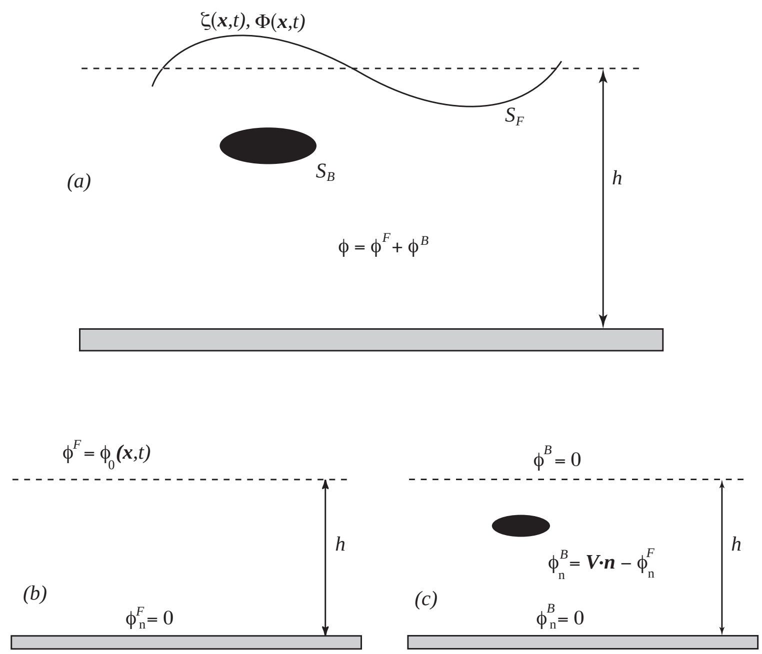

Figure 1: Decomposition of the original problem into the free surface and body problems: (a) original problem, (b) free surface problem, (c) body problem.

be found by various methods including a distribution of singularities, the multipole expansion method, Green's identity, etc. In this paper, ϕ^B is represented by a distribution of singularities as

$$\phi^B(\boldsymbol{x}, z, t) = \int_{S_B} \sigma(\boldsymbol{x}', t)\, G(\boldsymbol{x}, \boldsymbol{x}', z, z')\, \mathrm{d}S', \qquad (31)$$

where σ is determined by imposing the body boundary condition (30) and Green's function $G(\boldsymbol{x}', \boldsymbol{x})$ satisfies

$$\left(\nabla^2 + \frac{\partial^2}{\partial z^2}\right) G = \triangle(\boldsymbol{x} - \boldsymbol{x}', z - z') \quad \text{for } -h \le z \le 0, \qquad (32)$$

$$G = 0 \qquad \text{at} \quad z = 0, \qquad (33)$$

$$\frac{\partial G}{\partial z} = 0 \qquad \text{at} \quad z = -h. \qquad (34)$$

The solution of (32)–(34) can be easily found, for infinitely deep water, as

$$G = -\frac{1}{4\pi} \frac{1}{\left[|\boldsymbol{x} - \boldsymbol{x}'|^2 + (z - z')^2\right]^{1/2}} + \frac{1}{4\pi} \frac{1}{\left[|\boldsymbol{x} - \boldsymbol{x}'|^2 + (z + z')^2\right]^{1/2}}, \qquad (35)$$

where the first term represents the velocity potential for a source located at $(\boldsymbol{x}', z')$ for $z' < 0$ and the second term is for the image potential for the zero free surface boundary condition. For finite water depth, Green's function is more complicated due to the additional bottom boundary condition and is given (Wehausen & Laitone 1960, §13) by

$$G = -\frac{1}{4\pi} \frac{1}{\left[|\boldsymbol{x} - \boldsymbol{x}'|^2 + (z - z')^2\right]^{1/2}} - \frac{1}{4\pi} \frac{1}{\left[|\boldsymbol{x} - \boldsymbol{x}'|^2 + (z + z' + 2h)^2\right]^{1/2}} + ⨍_0^\infty \frac{2\mathrm{e}^{-kh} \cosh[k(z' + h)] \cosh[k(z + h)]}{\cosh(kh)} J_0(k|\boldsymbol{x} - \boldsymbol{x}'|)\, \mathrm{d}k, \qquad (36)$$

where $⨍$ represents a principal value integral and J_0 is the zeroth-order Bessel function of the first kind. In our numerical computations, we need to use doubly periodic Green's functions, as described in Xue *et al.* (2001).

After solving the body problem for ϕ^B and evaluating $f^B(\boldsymbol{x}, t) = \left.\frac{\partial \phi^B}{\partial z}\right|_{z=0}$, we have the relationship between ϕ_0 and w_0 from (25) and then, with using (14)–(15), we can close the system of evolution equations given by (10)–(11).

4 EVOLUTION EQUATIONS

4.1 LINEAR APPROXIMATION

For the leading-order approximation for $\zeta/\lambda = \epsilon \ll 1$ and $\Phi/(c\lambda) = O(\epsilon)$, from (14), (15) and (25), W can be found as

$$W \simeq w_0 = -\mathrm{L}[\phi_0] + f^B \simeq -\mathrm{L}[\Phi] + f^B \,, \qquad (37)$$

and, after dropping nonlinear terms, (10)–(11) can be reduced to

$$\frac{\partial \zeta}{\partial t} + \mathrm{L}[\Phi] = f^B, \qquad \frac{\partial \Phi}{\partial t} + g\,\zeta = 0, \qquad (38)$$

which can be combined into a single equation for Φ:

$$\frac{\partial^2 \Phi}{\partial t^2} - g\,\mathrm{L}[\Phi] = -f^B. \qquad (39)$$

For free waves ($f^B = 0$), by substituting into (38)

$$\zeta = a\,\mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \quad \Phi = b\,\mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}\,,$$

and, using (26), we have

$$-i\omega\, a - k\tanh(kh)\, b = 0, \quad -i\omega\, b + g\, a = 0\,,$$

which gives the dispersion relation for surface gravity waves in water of finite depth:

$$\omega^2 = gk\tanh(kh), \qquad (40)$$

where $k = |\boldsymbol{k}|$.

4.2 THIRD-ORDER APPROXIMATION

For the third-order approximation, equations (14) and (15) can be approximated by

$$\Phi \simeq \phi_0 + \zeta w_0 - \tfrac{1}{2}\zeta^2\nabla^2\phi_0,$$
$$W \simeq w_0 - \zeta\nabla^2\phi_0 - \tfrac{1}{2}\zeta^2\nabla^2 w_0,$$

whose solutions can be found, by succession, as

$$\phi_0 = \Phi + \zeta\Psi + \zeta\mathrm{L}[\zeta\Psi] + \tfrac{1}{2}\zeta^2\nabla^2\Phi, \qquad (41)$$

$$\begin{aligned} W =& -\Psi - \mathrm{L}[\zeta\Psi] - \zeta\nabla^2\Phi - \tfrac{1}{2}\nabla^2(\zeta^2\Psi) \\ & -\mathrm{L}\left[\zeta\mathrm{L}[\zeta\Psi] + \tfrac{1}{2}\zeta^2\nabla^2\Phi\right] + |\nabla\zeta|^2\Psi, \end{aligned}$$

where we have used (25) and Ψ is defined by

$$\Psi(\boldsymbol{x},t) = \mathrm{L}[\Phi] - f^B(\boldsymbol{x},t). \qquad (42)$$

Then, from (10)–(11), the evolution equations correct to third order in wave steepness can be written as

$$\begin{aligned} &\frac{\partial \zeta}{\partial t} + \Psi + \boldsymbol{\nabla}\cdot(\zeta\boldsymbol{\nabla}\Phi) + \mathrm{L}\left[\zeta\Psi\right] \\ &+\nabla^2\left(\tfrac{1}{2}\zeta^2\Psi\right) + \mathrm{L}\Big[\zeta\mathrm{L}\Big[\zeta\Psi\Big] + \tfrac{1}{2}\zeta^2\nabla^2\Phi\Big] = 0, \qquad (43) \end{aligned}$$

$$\begin{aligned} &\frac{\partial \Phi}{\partial t} + g\zeta + \tfrac{1}{2}\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi - \tfrac{1}{2}\Psi^2 \\ &\qquad -\Psi\Big(\zeta\nabla^2\Phi + \mathrm{L}[\zeta\Psi]\Big) = 0. \qquad (44) \end{aligned}$$

Notice that, using the formal expression for the vertical velocity in terms of ζ and Φ, the system of equations (43)–(44) is closed. This is the unique feature of our formulation different from other formulations for nonlinear free surface problems. In the absence of the body, the system can be reduced to that derived by Choi (1995).

5 NUMERICAL METHODS

To solve the evolution equations (43)–(44), we adopt a pseudo-spectral method. First we approximate the free surface elevation ζ and the free surface velocity potential Φ using a truncated Fourier series:

$$\zeta(\boldsymbol{x},t) = \sum_{n=-\frac{N_x}{2}}^{\frac{N_x}{2}} \sum_{m=-\frac{N_y}{2}}^{\frac{N_y}{2}} a_{nm}(t)\,\mathrm{e}^{ink_1x + imk_2y},$$

$$\Phi(\boldsymbol{x},t) = \sum_{n=-\frac{N_x}{2}}^{\frac{N_x}{2}} \sum_{m=-\frac{N_y}{2}}^{\frac{N_y}{2}} b_{nm}(t)\,\mathrm{e}^{ink_1x + imk_2y},$$

where N_x and N_y are the numbers of Fourier modes in the x- and y-directions, respectively, and $k_1 = 2\pi/L_1$ and $k_2 = 2\pi/L_2$ with L_1 and L_2 being the computational domain lengths in the x- and y-directions, respectively. The Fourier coefficients $a_{nm}(t)$ and $b_{nm}(t)$ can be computed by the double Fast Fourier Transform (FFT). All the linear operations are evaluated in the Fourier space, while the product between two functions for the nonlinear terms in the evolution equations are computed in the physical space. For example, the two linear operators ($\boldsymbol{\nabla}$ and L[ ]) are evaluated in the Fourier space as

$$\begin{aligned} \boldsymbol{\nabla}\left(a_{nm}(t)\,\mathrm{e}^{i\boldsymbol{K}\cdot\boldsymbol{x}}\right) &= \mathrm{i}\,\boldsymbol{K}\,a_{nm}(t)\,\mathrm{e}^{i\boldsymbol{K}\cdot\boldsymbol{x}}, \\ \mathrm{L}\Big[a_{nm}(t)\,\mathrm{e}^{i\boldsymbol{K}\cdot\boldsymbol{x}}\Big] &= \\ & \mathrm{i}\,|\boldsymbol{K}|\tanh(|\boldsymbol{K}|h)\,a_{nm}(t)\,\mathrm{e}^{i\boldsymbol{K}\cdot\boldsymbol{x}}, \qquad (45) \end{aligned}$$

where $\boldsymbol{K} = (nk_1, mk_2)$ and $|\boldsymbol{K}| = (n^2{k_1}^2 + m^2{k_2}^2)^{1/2}$.

In order to integrate the evolution equations in time, we use the explicit third-order Adams-Bashforth method. We also use a pressure beach to absorb waves propagating toward the computational boundaries similar to that used by Cao *et al.* (1998). The details of the numerical method to solve the evolution equations including the pressure beach and a numerical filter to eliminate instability due to aliasing error can be found in Kent & Choi (2004a).

Now we summarize our numerical method. For given $\Phi(\boldsymbol{x}, t_n)$ and $\zeta(\boldsymbol{x}, t_n)$, we

1. compute $\phi_0(\boldsymbol{x}, t_n)$ from (41) and find its Fourier coefficients using FFT,
2. compute $\frac{\partial \phi^F}{\partial n}$ on the body surface (S_B) (see equation (46)),
3. solve the linear boundary value problem for $\phi^B(\boldsymbol{x}, z, t_n)$ given by (27)–(30),
4. solve the evolution equations (43)–(44) for $\Phi(\boldsymbol{x}, t_{n+1})$ and $\zeta(\boldsymbol{x}, t_{n+1})$ after computing Ψ from (42).

In order to compute $\frac{\partial \phi^F}{\partial n}$ in step 2, we need to know the gradient of $\phi^F(\boldsymbol{x}, z, t)$ which can be found from the Fourier series of ϕ^F given by

$$\phi^F = \sum_n \sum_m A_{nm}(t) \frac{\cosh[|\boldsymbol{K}|(z+h)]}{\cosh(|\boldsymbol{K}|h)} \, \mathrm{e}^{i\boldsymbol{K}\cdot\boldsymbol{x}}, \quad (46)$$

where A_{nm} is the Fourier coefficients of ϕ_0 found in step 1.

Initial conditions at $t = 0$ for the case of a body or singularity in motion are $\Phi(\boldsymbol{x}, 0) = 0$ and $\zeta(\boldsymbol{x}, 0) = 0$, since its motion starts from rest, while those for the free wave case are Stokes wave solutions.

The hydrodynamic force on the body is calculated by numerically integrating, using the trapezoidal rule, the dynamic pressure on the body $S_B(t)$ given by:

$$\begin{aligned} p =& -\rho \left(\frac{\partial \phi}{\partial t} + \tfrac{1}{2} |\boldsymbol{\nabla}_3 \phi|^2 \right) \\ =& -\rho \left[\frac{\partial \Big(\phi|_{S_B} \Big)}{\partial t} - \boldsymbol{V}^B \cdot \boldsymbol{\nabla}_3 \phi + \tfrac{1}{2} |\boldsymbol{\nabla}_3 \phi|^2 \right], \end{aligned}$$

where $\phi|_{S_B}$ is the total velocity potential evaluated at the instantaneous body position, $\boldsymbol{V}^B$ is the velocity of the body, $\boldsymbol{\nabla}_3 = \left(\boldsymbol{\nabla}, \frac{\partial}{\partial z}\right)$. The chain rule for differentiation has been used to rewrite the time derivative of the velocity potential, which is evaluated using a backward finite difference scheme, while other terms are calculated at the present time step.

6 EXAMPLES FOR INFINITELY DEEP WATER

For numerical solutions shown here, we consider the case of infinitely deep water ($h \to \infty$), for which the linear operator L defined in the Fourier space by (45) becomes

$$\mathrm{L}\Big[a_{nm}(t) \, \mathrm{e}^{i\boldsymbol{K}\cdot\boldsymbol{x}} \Big] = \mathrm{i} \, |\boldsymbol{K}| a_{nm}(t) \, \mathrm{e}^{i\boldsymbol{K}\cdot\boldsymbol{x}}.$$

6.1 STOKES GRAVITY WAVES

In order to validate our computer code, we first study the evolution of free gravity waves in the absence of a body ($f^B \equiv 0$). It is well-known that Stokes waves are unstable to both two-dimensional (Benjamin & Feir 1967) and three-dimensional perturbations (McLean 1982). To simulate three-dimensional instability of Stokes waves, we adopt the following initial conditions for ζ and Φ:

$$\begin{aligned} \zeta(\boldsymbol{x}, 0) &= \zeta_s(x) \\ &\quad + \alpha a_0 \cos(k_0(1+p)x) \cos(k_0 q y), \qquad (47) \\ \Phi(\boldsymbol{x}, 0) &= \Phi_s(x) \\ &\quad + \alpha c a_0 \sin(k_0(1+p)x) \cos(k_0 q y), \qquad (48) \end{aligned}$$

where ζ_s and Φ_s are the surface elevation and the velocity potential, respectively, for one-dimensional Stokes waves of wave slope $k_0 a_0 = 0.314$ traveling in the positive $x-$direction. For small two-dimensional perturbations, we have chosen $\alpha = 0.16$, $p = 0.5$ and $q = 1.22$, for which the linear growth rate is close to its maximum value (McLean 1982). As shown in figure 2, our numerical solutions of the evolution equations (43)–(44) show the development of crescent wave patterns first observed experimentally by Su (1982) and simulated by Xue *et al.* (2001) using a boundary integral method to solve the fully nonlinear Euler equations. In our simulation, the total energy is conserved typically to 0.01% or less, which demonstrates that our numerical method described in §5 introduces no artificial energy source or sink.

6.2 TRANSLATING SINGULARITIES

Next we consider a steadily translating three-dimensional dipole located at $z = -D$. For infinitely deep water, the

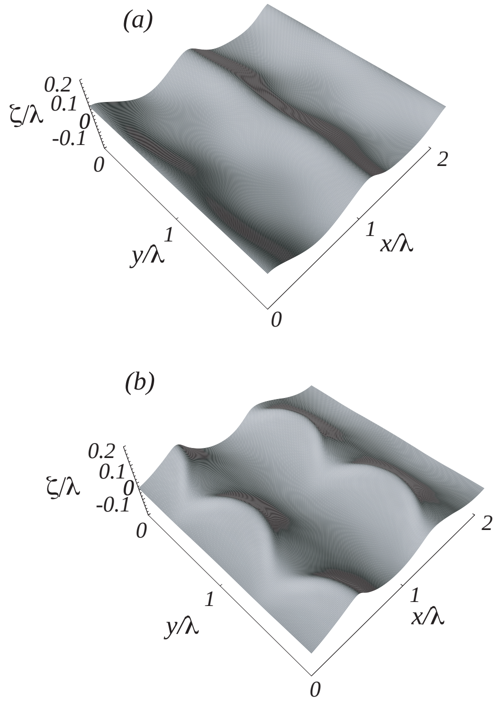

Figure 2: Development of crescent waves from initial Stokes waves of wave slope $k_0a_0 = 0.314$ subject to a perturbation give by (47)–(48) with $\alpha = 0.16$, $p = 0.5$ and $q = 1.22$. The numbers of Fourier modes for this simulation are N_x =128 and N_y=128. (a) $t/T = 0$, (b) $t/T = 4.338$, with T being the wave period.

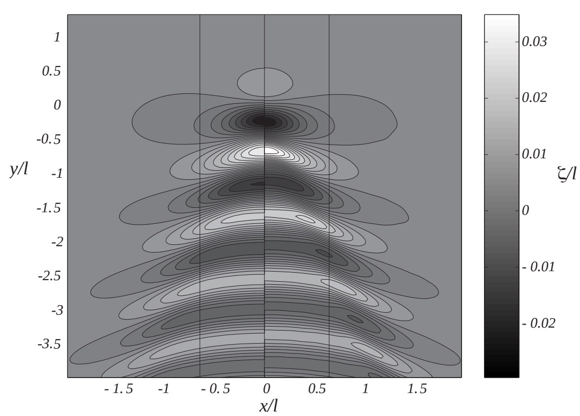

Figure 3: Comparison of wave pattern between first-order (left half) and third-order (right half) solutions for a translating three-dimensional dipole of $U/\sqrt{gl} = 1$ and D/l=2.5, where $l = (\mu/U)^{1/3}$.

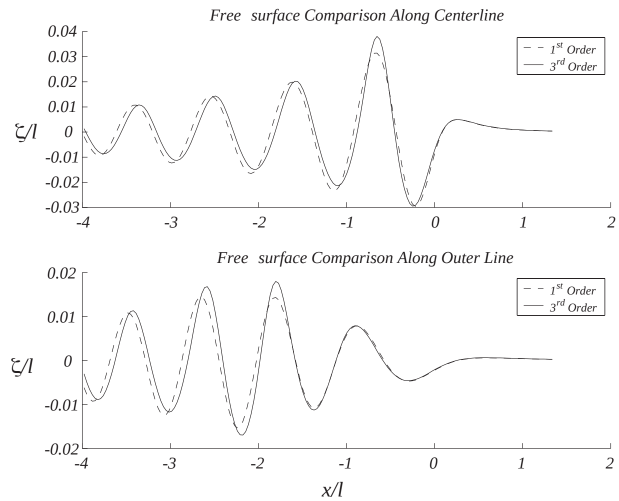

Figure 4: Comparison of the free surface elevation along the straight lines at two different transverse locations shown in figure 3: first-order (- - -) and third-order (——) solutions.

velocity potential (or Green's function for the body problem) for a three-dimensional dipole of strength μ is given, in a frame of reference moving with speed U, by

$$\phi^B(\boldsymbol{x}, z) = \frac{\mu x}{{r_+}^3} - \frac{\mu x}{{r_-}^3}, \tag{49}$$

where $r_\pm$ is defined as

$$r_\pm = \left[|\boldsymbol{x}|^2 + (z \pm D)^2\right]^{1/2}. \tag{50}$$

Notice that the velocity potential in (49) satisfies the free surface boundary condition ($\phi^B = 0$ at $z = 0$) for the body problem shown in figure 1(c).

Numerical results for $D/l = 2.5$ and $U/\sqrt{gl} = 1$ where $l = (\mu/U)^{1/3}$ are shown in figures 3 and 4. The right half of figure 3 shows the third-order nonlinear solution, while the left half represents the linear solution. It can be seen that the diverging waves are more pronounced in the nonlinear solution.

Similarly a translating two-dimensional dipole of strength μ with $U/\sqrt{gl} = 1$ and $D/l = 5$, where $l = (\mu/U)^{1/2}$, is considered and the result is shown in figure 5. Our solution of the linear evolution equations shows good agreement with the linear analytic solution of Havelock (1926). For this two-dimensional case, the difference between nonlinear (both second- and third-order) and linear solutions is greater than that for three-dimensional case with a smaller submergence depth.

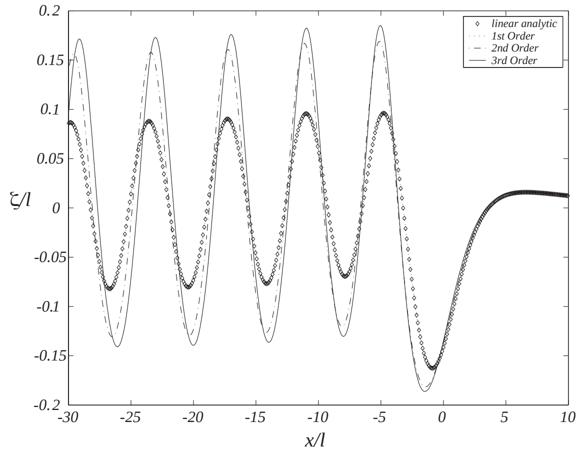

Figure 5: Free surface elevation (ζ) for a two-dimensional dipole located at $z = -5l$ in uniform stream of $U/\sqrt{gl} = 1$, where $l = (\mu/U)^{1/2}$.

6.3 TWO-DIMENSIONAL BODY

For a two-dimensional body in infinitely deep water, Green's function satisfying (32)–(34) is given by

$$G(x, x', z, z') = \frac{1}{4\pi} \log\left[(x - x')^2 + (z - z')^2\right] - \frac{1}{4\pi} \log\left[(x - x')^2 + (z + z')^2\right].$$

As explained in §5, we have used a periodic array of this Green's function in our computation.

To approximate the velocity potential ϕ^B in (31) numerically, we use a desingularized method (Beck 1999): Point sources are distributed at discrete points (x'_j, z'_j) displaced slightly into the body in the normal direction from collocation points on the body surface, so that

$$\phi^B(x, z) = \sum_{j=1}^{N_B} \sigma_j\, G(x, x'_j, z, z'_j), \tag{51}$$

where N_B is the number of sources to represent the body. Imposing the body boundary condition (30) at the collocation points, the source strength σ_j is found by solving a system of linear algebraic equations. Figure 6 shows the free surface elevation due to a circular cylinder of radius R moving with speed $U/(gR)^{1/2} = 1.79$. It is interesting to notice that our numerical solution shows excellent agreement with the earlier fully nonlinear numerical solution of Scullen & Tuck (1995) using a boundary integral method. For this simulation we have used the number of

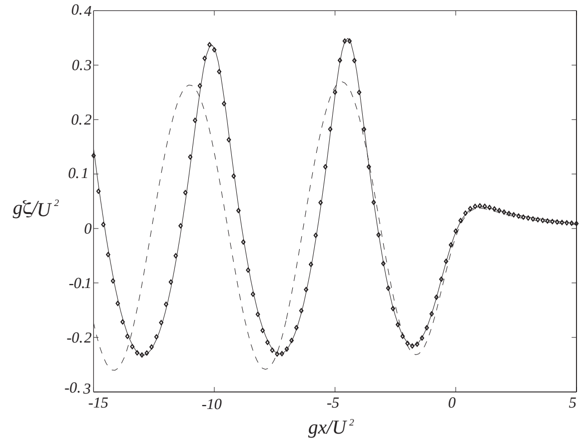

Figure 6: Free surface elevation of a steadily translating circular cylinder of radius $D/R = 5$ with $U/\sqrt{gR} = 1.79$. The third-order solution (——) is compared with the firs-order solution and the fully nonlinear numerical solution of Scullen & Tuck (1995) using a boundary integral formulation.

Fourier modes $N_x = 15$/wavelength to solve the evolution equations (43)–(44) and the number of sources for the body problem $N_B = 40$. See Kent & Choi (2004*b*) and Kent & Choi (2004*c*) for more numerical results for two-dimensional translating and oscillating bodies.

7 DISCUSSION

We propose a new third-order nonlinear formulation to solve unsteady wave-body interaction problems in water of arbitrary depth.

We first find numerically the body velocity potential (ϕ^B) satisfying the exact body boundary condition and the simplified free surface boundary condition, which can be easily solved using, for example, a singularity distribution method. Since the boundary condition at the mean free surface is simple ($\phi^B = 0$ at $z = 0$), it is no longer necessary to evaluate complicated free surface Green's functions containing multiple integrals which appear in classical linear free surface formulations.

After solving the body problem at the instantaneous body position, it is required to solve the system of coupled nonlinear evolution equations to update the free surface elevation (ζ) and the velocity potential at the free surface (Φ). It has been shown that the system can be effectively solved by using the pseudo-spectral method described in §5.

Advantages of our formulation include: (1) compared with other spectral/pseudo-spectral methods, since our system of evolution equations is closed, no intermediate steps to close the system are required and therefore our numerical method is more effective; (2) as in the boundary integral formulation, we solve only two-dimensional equations for three-dimensional problems; (3) compared with the mixed Eulerian-Lagrangian method, no distribution of singularities along the free surface is necessary. On the other hand, the limitations of the method are that (1) being perturbation-based, the method does not allow wave breaking to occur, and (2) the method is valid up to third order in wave slope, though a test with traveling wave solutions has shown promising results even at relatively high wave amplitudes.

Here we present numerical solutions of a translating two- and three-dimensional dipole and a translating two-dimensional submerged body in this paper, more general three-dimensional submerged or floating body problems are under investigation.

REFERENCES

Beck, R.F., "A fully nonlinear water wave computations using a desingularized Euler-Lagrange time-domain approach," Nonlinear Water Wave Interaction, WIT Press, 1999, pp. 1-58.

Beck, R.F. and Reed, A. M., "Modern computational methods for ships in a seaway," Transaction SNAME, Vol. 109, 2001, pp. 1–52.

Benjamin, T.B. and Feir, J.E., "The disintegration of wave-trains on deep water. Part 1. Theory," J. of Fluid Mech. Vol. 27, 1967, pp. 417–430.

Cao, Y., Beck, R.F. and Schultz, W., "An Absorbing Beach for Numerical Simulations of Nonlinear Waves in a Wave Tank," Proceedings of 13th International Workshop on Water Waves and Floating Bodies, Delft, Holland, March 1998.

Choi, W., "Nonlinear evolution equations for two-dimensional surface waves in a fluid of finite depth," J. of Fluid Mech., Vol. 295, 1995, pp. 381–394.

Colagrossi, A., Landrini, M. and Tulin, M. P., "Near shore bore propagation and splashing processes: gridless simulations," Proceedings of 6th International Workshop on Wave Hindcasting and Forecasting, Monterey, 2000.

Dommermuth, D. G. and Yue, D. K. P., "A higher-order spectral method for the study of nonlinear gravity waves," J. of Fluid Mech. Vol. 184, 1987, pp. 267–288.

Fenton ,J. D. & Rienecker, M. M. A Fourier method for solving nonlinear water-wave problems: application to solitary-wave interactions. J. Fluid Mech. Vol. 118, 1982 pp. 411–443.

Havelock, T., "The method of images in some problems of surface waves," Proc. Roy. Soc. A. Vol. 115, 1926, pp. 265–277.

Iafrati, A., Mascio, A. D., and Campana, E. F., "A level set technique applied to unsteady free surface flows," Int. J. of Num. Methods in Fluids, Vol. 35, 2001, pp. 281–297.

Kent, C. and Choi, W., "A fully nonlinear theory for surface wave hydrodynamics. Part 1. Pressure forcing in deep water," Submitted for publication, 2004a.

Kent, C. and Choi, W., "A fully nonlinear theory for surface wave hydrodynamics. Part 2. Interaction with a submerged body," Submitted for publication, 2004b.

Kent, C. and Choi, W., "A new numerical method to compute nonlinear hydrodynamic forces on a submerged cylinder," Proceedings of 19th International Workshop on Water Waves and Floating Bodies, Cortona, Italy, 2004c.

Lin, R.-Q. and Kuang, W., "Nonlinear waves of a steadily moving ship in environmental waves," J. Mar. Sci. Technol. Vol. 8, 2004, pp. 109-116.

Liu, Y., Dommermuth, and D. G. Yue, D. K. P., "A higher-order spectral method for nonlinear wave-body interactions," J. Fluid Mech. Vol. 245, 1992, pp. 115–136.

Longuet-Higgins, M. S. and Cokelet, E. D., "The deformation of steep surface waves on water. I. A numerical method of computation," Proc. Roy. Soc. Lond. A Vol. 350, 1976, pp. 1–26.

McLean, J. W., "Instabilities of finite amplitude water waves," J. of Fluid Mech. Vol. 114, 1982, pp. 315–330.

Scullen, D. and Tuck, E. O., "Nonlinear free-surface flow computations for submerged cylinders," J. of Ship Res. Vol. 39, 1995, pp. 185–193.

M.-Y. Su "Three-dimensional deep-water waves. Part 1. Experimental measurement of skew and symmetric wave patterns," J. of Fluid Mech. Vol. 124, 1982, pp. 73–108.

Wehausen, J. V. & Laitone, E. V., "Surface waves," Encyclopedia of physics Vol. IX, 1960, pp. 446-778.

Xue, M., Xu, H. B., Liu, Y. and Yue, D. K. P., "Computations of fully nonlinear three-dimensional wave-wave and wave-body interaction. Part 1. Three-dimensional steep waves," J. of Fluid Mech. Vol. 438, 2001, pp. 11-39.