Long Internal Waves of Finite Amplitude

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We derive new nonlinear evolution equations for long internal waves in a two-fluid system, when the thickness of the lower layer is effectively infinite, by making the only assumption that the thickness of the upper layer is small compared with the characteristic wavelength. The resulting equations have the full nonlinearity of the original problem and retain the leading-order dispersive effects. For large amplitude internal solitary waves, we show that our model captures the scaling relation between the amplitude and the characteristic wavelength observed experimentally by Koop and Butler. [S0031-9007(96) 01019-8]

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Solitary waves are perhaps the most remarkable manifestation of the balance between two fundamental mechanisms of gravity wave propagation: dispersion and nonlinearity. The relative strength of these two mechanisms can be measured by two independent nondimensional parameters, the nonlinearity ratio $\alpha = a/h_1$ of wave amplitude a and fluid layer thickness h_1 and the aspect ratio $\beta = h_1/L$ of h_1 and typical wavelength L. In the case of waves at the free surface of a homogeneous fluid layer, when viscosity is negligible, the balance between nonlinearity and dispersion often occurs in regimes for which these parameters are small and it is expressed by a scaling relation between α and β. Thus $\alpha = O(\beta^2)$ for small β results in weakly nonlinear long solitary waves. It turns out that this relation does extremely well in describing most observable solitary waves in homogenous layers, and so a "universal" model based on this scaling, like the Korteweg–de Vries (KdV) equation, captures most of the relevant features of solitary wave propagation in a homogenous fluid.

The case of internal waves at the interface between two fluids of different densities can be expected to offer a richer class of phenomena. This is reflected by the need of introducing at least another independent nondimensional parameter, the thickness ratio $\gamma = h_2/h_1$ for an upper lighter fluid layer of thickness h_1 (and density ρ_1) and a lower heavier fluid of thickness h_2 (and density ρ_2), when the density ratio $\rho_2/\rho_1 = \rho_r > 1$ is considered as fixed. Various regimes are now possible, even within the long wave limit $\beta = h_1/L \ll 1$, ranging between the two extremes of lower fluid thickness that is also small compared to wavelength to the case of a lower fluid layer that can be considered effectively infinite.

The relative magnitude of α and β representing the balance between nonlinearity and dispersion can now be expected to vary according to the depth ratio γ. For shallow water of $\gamma = O(1)$, the scaling $\alpha = O(\beta^2)$, $\beta \ll 1$, leads again to the KdV equation for unidirectional wave propagation [1]. When the depth of the lower layer is much greater than that of the upper layer ($\gamma \gg 1$), the scaling $\alpha = O(\beta)$ leads to the intermediate long wave (ILW) equation for unidirectional waves. It reduces to the Benjamin-Ono (BO) equation in the limit of $\gamma \longrightarrow \infty$ [2–6].

The common feature of all these models is that, by taking advantage of the small parameters introduced for looking at certain regimes, the dependence on the vertical coordinate is eliminated, thereby simplifying the problem considerably. However, there is no *a priori* guarantee that these asymptotic models address a sufficiently broad class of physical situations. As already mentioned, this seems to be the case for the KdV equation, but the situation is much different for the ILW or BO models. Indeed, the experimental study by Koop and Butler [7] shows that the domain of applicability of the ILW equation and its limiting form, the BO equation, is rather narrow [8]. In particular, the weak nonlinearity (small amplitude) assumption seems inappropriate for a large set of their experimental data.

In this paper, we show that the assumption of weak nonlinearity can be removed while still being able to derive a simple model for long waves at the interface with a deep lower layer. The model retains the leading order dispersive effects and has no vertical coordinate dependence. The flows we consider are characterized by $\alpha = a/h_1 = O(1)$, $\beta = h_1/L \ll 1$, $\gamma = h_2/h_1 \longrightarrow \infty$. In these regimes, our (bidirectional) model can be written as

$$\zeta_t - [(h_1 - \zeta)\overline{u}_1]_x = 0,$$
$$\overline{u}_{1t} + \overline{u}_1\overline{u}_{1x} + (1 - \rho_r)g\zeta_x = \rho_r\mathcal{H}[\zeta_{tt}], \qquad (1)$$

where h_1 is the undisturbed upper fluid thickness, ζ is the displacement of the interface away from the vertical coordinate origin $z = 0$, and $\overline{u}_1$ is the upper layer mean velocity. Here the operator $\mathcal{H}[f]$ acting on a function $f(x)$ denotes the Hilbert transform $\mathcal{H}[f] \equiv (1/\pi)\mathcal{P}\int_{-\infty}^{\infty} f(x')/(x' - x)\,dx'$, with $\mathcal{P}$ standing for Cauchy principal value of integration. We have not been able to find closed form traveling wave solutions for this model and had to resort to numerical solutions, found via a Newton-Raphson technique. As shown

below, these are in good agreement with the solitary wave scaling law between amplitude and effective wavelength exhibited by the data [7] over a wide range of *large* amplitude waves where the BO model can be off by up to a factor of two. System (1) reduces to the BO equation for small amplitude unidirectional wave propagation. Thus it can be expected to share the same limitations as the BO model in the weakly nonlinear, very long wave regimes (see the discussion in [7]).

Derivation of the model equations.—For inviscid and incompressible fluids, the velocity components in Cartesian coordinates (u_i, w_i) and the pressure p_i satisfy the continuity equation and the Euler equations

$$u_{ix} + w_{iz} = 0, \tag{2}$$

$$u_{it} + u_i u_{ix} + w_i u_{iz} = -p_{i_x}/\rho_i, \tag{3}$$

$$w_{it} + u_i w_{ix} + w_i w_{iz} = -p_{i_z}/\rho_i - g, \tag{4}$$

where, in a two-fluid system, $i = 1$ for the upper fluid and $i = 2$ for the lower fluid. The constant acceleration of gravity g is directed along the negative z axis. The boundary conditions at the interface are

$$\left.\begin{array}{c}\zeta_t + u_1\zeta_x = w_1, \quad \zeta_t + u_2\zeta_x = w_2,\\ p_1 = p_2\end{array}\right\} \quad \text{at } z = \zeta(x,t). \tag{5}$$

Since we want to focus on the interface wave motion only, we assume that there exists a (flat) rigid lid at top of the upper fluid layer, so that the kinematic boundary condition at $z = h_1$ is simply $w_1 = 0$. With an infinitely deep lower fluid layer the other boundary condition is $(u_2, w_2) \longrightarrow 0$ as $z \longrightarrow -\infty$.

We will now implement an asymptotic expansion of the equations of motion (3) and (4) in the small parameter β. Throughout the derivation we assume that the flow in each layer is irrotational, as it is the case for motion starting from rest. The assumption $\beta \ll 1$ signifies that the upper layer thickness is much smaller than the characteristic wavelength of the motion we are interested in. We will first derive evolution equations correct up to order $O(\beta^2)$ for the upper layer based on this assumption. This part of the derivation mirrors closely the derivation of the classical Airy's shallow water model [9]. We will then derive a model equation for the lower fluid based on an asymptotic expansion where the relative ordering of terms is dictated by that of the upper fluid through the interface boundary conditions (5).

We first nondimensionalize the physical variables of the upper fluid via

$$\begin{gathered} x = Lx^*, \quad z = h_1 z^*, \quad t = (L/U_0)t^*, \\ \zeta = h_1\zeta^*, \quad p_1 = (\rho_1 U_0^2)p_1^*, \\ (u_1, w_1) = U_0(u_1^*, \beta w_1^*), \end{gathered} \tag{6}$$

where all variables with asterisks and their derivatives are assumed to be $O(1)$. Here we have chosen the typical velocity scale U_0 as $U_0 = (gh_1)^{1/2}$. The scaling $w_1/u_1 = O(h_1/L) = O(\beta)$ in (6) is imposed by the continuity equation (2). After dropping the asterisks for dimensionless variables, the vertical momentum equation for the upper fluid (4) can be written as

$$p_{1z} = -1 - \beta^2[w_{1t} + u_1 w_{1x} + w_1 w_{1z}]. \tag{7}$$

As to the horizontal momentum equation, it is convenient to look at two of the layer-mean equations [10] that can be obtained from (2) and (3) by vertically integrating across the layer $\zeta \le z \le 1$ and taking into account the kinematic boundary condition in (5),

$$\eta_t + (\eta\overline{u}_1)_x = 0, \quad (\eta\overline{u}_1)_t + (\eta\overline{u_1 u_1})_x = -\eta\overline{p_{1_x}}. \tag{8}$$

Here $\eta = 1 - \zeta$ is the thickness of the upper fluid layer and the layer mean quantity $\bar{f}$ is defined as $\bar{f}(x,t) = \eta^{-1}\int_\zeta^1 f(x,z,t)\,dz$. From (7) we see that the pressure is hydrostatic at leading order. Thus by taking into account the interface boundary conditions (5), we have $p_1(x,z,t) = -z + \zeta(x,t) + P(x,t) + O(\beta^2)$, where $P(x,t) \equiv p_2(x,\zeta,t)$. Hence $p_{1x} = \zeta_x + P_x + O(\beta^2)$, that is, the horizontal gradient of the pressure is independent of z at leading order. Accordingly, Eq. (3) for $i = 1$ shows that at leading order the horizontal velocity can be taken to be z independent $u_1(x,z,t) = u_1^{(0)}(x,t) + O(\beta^2)$, if the initial conditions are so chosen, in agreement with the irrotationality assumption. Substituting the expressions for p_1 and u_1 in (8) and noticing that $\overline{u_1 u_1} = \overline{u}_1\overline{u}_1 + O(\beta^4)$ yields the equations for the upper fluid in the form

$$\begin{gathered} \eta_t + (\eta\overline{u}_1)_x = 0, \\ \overline{u}_{1t} + \overline{u}_1\overline{u}_{1x} + \zeta_x = -P_x + O(\beta^2). \end{gathered} \tag{9}$$

The correction terms at order $O(\beta^2)$, which add dispersion to this system, can easily be found (see [11]). However, we shall see presently that the leading order dispersion, coming through P from the presence of the lower fluid, enters at order β, so that we can neglect $O(\beta^2)$ terms in this analysis.

We now look at the lower fluid. First, we nondimensionalize the independent variables by

$$x = Lx^*, \quad z = Lz^*, \quad t = (L/U_0)t^*, \tag{10}$$

as is natural due to the assumption that the depth of the lower fluid be much larger than h_1. The domain occupied by the lower fluid in the rescaled variables is therefore (dropping asterisks) $-\infty \le z \le \beta\zeta(x,t)$, for $-\infty \le x \le \infty$. We have already seen that the continuity equation for the upper fluid, once it is scaled through (6), requires $w_1/u_1 = O(\beta)$. Under the scaling of z in (10), however, the continuity equation for the lower fluid suggests $w_2/u_2 = O(1)$. The continuity of normal velocity at $z = \zeta$ given by (5) and $\zeta_x = O(h_1/L) = O(\beta)$ points to $w_2/w_1 = O(1)$. This in turn implies $u_2/u_1 = O(\beta)$.

Hence we nondimensionalize the dependent variables for the lower fluid as

$$p_2 = (\rho_1 U_0^2) p_2^*, \quad u_2 = \beta U_0 u_2^*, \quad w_2 = \beta U_0 w_2^* .$$

The irrotationality assumption allows us to introduce a velocity potential $\phi(x,z,t)$ for the lower fluid $(\phi_x, \phi_z) = (u_2, w_2)$. The potential ϕ is determined by solving Laplace equation (again, dropping asterisks) $\phi_{xx} + \phi_{zz} = 0$ in $-\infty \le z \le \beta\zeta(x,t)$ with the kinematic boundary conditions $\phi_z = \zeta_t + \beta\zeta_x\phi_x$ at $z = \beta\zeta(x,t)$ and $\phi_z = 0$ at $z = -\infty$. The equations of motion (3) and (4) reduce to the Bernoulli equation, which at $z = \beta\zeta(x,t)$ is $\beta\phi_t + \beta^2(\phi_x^2 + \phi_z^2)/2 + \zeta + P/\rho_r = 0$. This provides the pressure derivative $P_x(x,t)$, the term needed to couple the upper fluid evolution to that of the lower fluid in (9). We have, up to order $O(\beta^2)$,

$$P_x(x,t) = -\rho_r[\zeta_x + \beta\phi_{xt}(x,0,t)] + O(\beta^2) . \quad (11)$$

Solving the Laplace equation using Fourier transforms and taking into account the boundary condition at $z = -\infty$ yields $\phi_x(x,0,t) = \mathcal{H}[\phi_z(x,0,t)]$. The other boundary condition $\phi_z = \zeta_t + O(\beta)$ then gives $P_x(x,t) = -\rho_r\{\zeta_x + \beta\mathcal{H}[\zeta_{tt}]\} + O(\beta^2)$. When used in (9), this expression produces the nondimensional version of our model (1). We remark that the notation ζ_{tt} in the dynamic equation (1) is just shorthand notation for $(\eta\overline{u}_1)_{xt}$, since only first order derivatives with respect to time should enter the equations of motion.

System (1) has two obvious conserved quantities $\int_{-\infty}^{\infty}\zeta\,dx$, and $\int_{-\infty}^{\infty}\overline{u}_1\,dx$, which represent mass and vorticity (or irrotationality) conservation, respectively. In addition, the horizontal momentum $\mathcal{M} = \int_{-\infty}^{\infty}dx\{\rho_1(h_1 - \zeta)\overline{u}_1 + \rho_2\zeta\mathcal{H}[\zeta_t]\}$ and the total energy $\mathcal{E} = (1/2)\int_{-\infty}^{\infty}dx\{(\rho_2 - \rho_1)g\zeta^2 + \rho_1(h_1 - \zeta)\overline{u}_1\overline{u}_1 - \rho_2(h_1 - \zeta)\overline{u}_1\mathcal{H}[\zeta_t]\}$ are also conserved. In fact, it can be shown that $\mathcal{E}$ becomes the Hamiltonian for system (1) with the appropriate Hamiltonian operator. We can also see that, by use of $\mathcal{H}[e^{ikx}] = i\,\mathrm{sgn}(k)\,e^{ikx}$, the linear dispersion relation of (1) is given by $\omega^2 = gh_1k^2(\rho_r - 1)/(1 + \rho_r|k|h_1)$, which is the correct limit for small kh_1 of the full linear dispersion relation ([12], Sec. 231). For weakly nonlinear unidirectional waves, our model (1) reduces to the BO equation [2]

$$\zeta_t + c_0\zeta_x - \frac{3c_0}{2h_1}\zeta\zeta_x + \frac{\rho_r c_0 h_1}{2}\mathcal{H}[\zeta_{xx}] = 0 , \quad (12)$$

where $c_0^2 = gh_1(\rho_r - 1)$. This equation admits the family of solitary wave solutions [13] parametrized by the amplitude a, $\zeta_s(X) = a/[1 + (X/l)^2]$, with $X = x - c_0(1 + \delta)t$, $\delta = -3a/(8h_1)$, and $|l| = -(4\rho_r/3)(h_1^2/a)$.

Traveling wave solution.—For waves of finite amplitude traveling with constant speed U, we substitute $\zeta = \zeta(X)$ and $\overline{u}_1 = \overline{u}_1(X)$ with $X = x - Ut$ into (1) and integrate once. After taking the integration constants to be zero, thereby fixing the mean level, the first (kinematic) equation gives $\overline{u}_1 = -U\zeta/(1 - \zeta)$. Substituting this into the second (dynamic) equation yields

$$G[\zeta] \equiv -\frac{1}{U^2}\zeta + \frac{1}{2}\frac{1}{(1-\zeta)^2} - \rho_r\mathcal{H}[\zeta_X] - \frac{1}{2} = 0 , \quad (13)$$

where we have taken $h_1 = (\rho_r - 1)g = 1$ for normalization. We look for a solution of (13) by using the Newton-Raphson method. We substitute $\zeta(X) = \zeta^0(X) + \Delta(X)$ into (13) and derive the linearized equation for Δ,

$$M[\zeta^0; U]\Delta \equiv \left[\frac{1}{U^2} - \frac{1}{(1-\zeta^0)^3} + \rho_r\partial_X\mathcal{H}\right]\Delta = G[\zeta^0] . \quad (14)$$

Thus ζ^0 is the initial guess (or the result from the previous iteration) and Δ is the correction to be found. By using a finite difference method for N points with trapezoidal rule for the Hilbert transform, (14) evaluated at $X = X_i = i\lambda/N$ $(i = 1, \ldots, N)$ can be written as $M_{ij}\Delta_j = G_i$, where M_{ij} is an element of $N \times N$ matrix resulting from the discretization of the operator $M[\zeta^0; U]$ in (14) and $\Delta_j = \Delta(X_j)$. By taking the wave speed U as a parameter for given wavelength λ and choosing the periodic wave solution [13] of the BO equation as the initial guess for small $U - 1$, we solve the linear algebraic equation for Δ_j iteratively until $\max(\Delta_j)$ is smaller than the error bound e. Then we proceed to find the solution for larger $U - 1$ by taking the previous results for smaller $U - 1$ as the initial guess. In the computations, we choose a large wavelength λ (typically $\lambda = 400$) for traveling waves close to solitary waves, $N = 800$, and an error bound $e = 10^{-6}$. We also take $\rho_r = \rho_2/\rho_1 = 1.58$ for comparison with the experimental data of Koop and Butler [7]. In Fig. 1, the solutions obtained by the Newton-Raphson method are compared with periodic waves of the BO equation [13] of the same speed and wavelength for four different values of $U - 1$. The finite amplitude traveling waves in our model are wider and taller as $U - 1$ increases when compared with the BO weakly nonlinear waves [14].

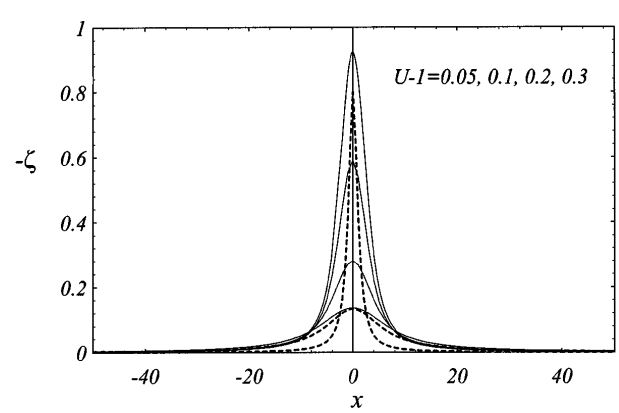

FIG. 1. Solitary wave solutions (——) of (1) for $U - 1 = 0.05, 0.1, 0.2, 0.3$ compared with those (– – –) of the Benjamin-Ono equation for $U - 1 = 0.05, 0.3$.

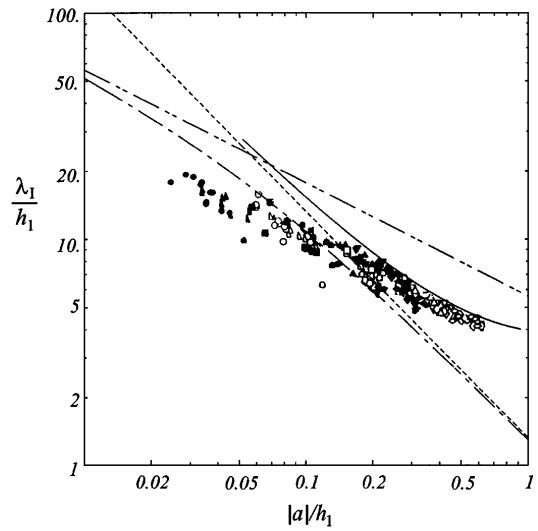

FIG. 2. λ_I/h_1 vs $|a|/h_1$ curves compared with the experimental data (symbols, reproduced with permission from Cambridge University Press) [7]: ——, model (1); – – –, BO model; — - —, ILW model; — - - —, KdV model.

An extensive experimental investigation of solitary waves at the interface of two immiscible fluids was carried out by Koop and Butler [7]. They summarized their findings by plotting the effective wavelength $\lambda_I \equiv (1/a)\int_0^\infty \zeta(X)\,dX$ versus the wave amplitude a. For the data corresponding to the experiment when the thickness of the upper fluid layer is 35.05 times that of the lower one ($\gamma = 35.05$), the data intersect the theoretical curves provided by the ILW equation and its limiting form, the BO equation, around the amplitude $|a|/h_1 = \alpha = 0.1$ (see Fig. 2). For both smaller $0.02 < \alpha < 0.1$ and larger $0.15 < \alpha < 0.65$ amplitudes the ILW curves fail to represent the data, and clearly have the wrong slope throughout. However, the regimes for small amplitude data should fulfill the assumptions for the asymptotic derivation of the ILW model based on weak nonlinearity with the scaling $\alpha = O(\beta)$. Hence the discrepancy between model and data cannot be immediately attributed to limitations of the ILW model. However, the large amplitudes waves $0.15 < \alpha < 0.65$ could be outside the domain of asymptotic validity of the ILW (and BO) equations. As the wave amplitude increases, the effective wavelength becomes much shorter than the lower fluid thickness h_2, so that h_2 can be considered infinite, and our model (1) applies. In Fig. 2 we compare the data from [7] with the curve (solid) for the solitary waves of (1), as well as those for ILW and BO. The agreement for $0.2 < \alpha < 0.65$ is good, with the data being slightly overpredicted, but clearly with the right trend for increasing α, thereby showing that the weak nonlinearity assumption $\alpha = O(\beta)$ is the principal cause of discrepancy in these regimes. As α decreases, the solid curve of model (1) limits onto the BO curve, as anticipated. Hence in this limit our model can be expected to suffer from the same limitations of the BO model, which are mainly due to the wavelength becoming comparable to the total fluid depth, thereby making the infinite depth assumption invalid. By replacing the operator $\mathcal{H}$ in (1) with its equivalent for finite lower layer depth, we can introduce a model which is the finite amplitude bidirectional counterpart of the ILW model. Agreement with the data by using this finite depth model could be further improved, especially for amplitudes $\alpha < 0.2$, with the curve λ_I/h_1 vs $|a|/h_1$ approaching that of the ILW model for lower amplitudes.

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[1] T. B. Benjamin, J. Fluid Mech. **25**, 241 (1966).
[2] T. B. Benjamin, J. Fluid Mech. **29**, 559 (1967).
[3] R. E. Davis and A. Acrivos, J. Fluid Mech. **29**, 593 (1967).
[4] H. Ono, J. Phys. Soc. Jpn. **39**, 1082 (1975).
[5] R. I. Joseph, J. Phys. A **10**, L225 (1977).
[6] T. Kubota, D. R. S. Ko, and L. D. Dobbs, AIAA J. Hydrodyn. **12**, 157 (1978).
[7] C. G. Koop and G. Butler, J. Fluid Mech. **112**, 225 (1981).
[8] H. Segur and J. L. Hammack, J. Fluid Mech. **118**, 285 (1982).
[9] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).
[10] T. Y. Wu, J. Eng. Mech. Div. ASCE **107**, 501 (1981).
[11] W. Choi and R. Camassa, J. Fluid Mech. **313**, 83 (1996).
[12] H. Lamb, *Hydrodynamics* (Dover, New York, 1932).
[13] This algebraic soliton solution is obtained as the limiting form for large wavelength of the two-parameter family (λ, a) of periodic wave solutions [2]: $\zeta_p(X) = A/[1 - B\cos(2\pi X\lambda)]$, where $A = (a_c^2/2a)$, $B^2 = 1 - (a_c/a)^2$, and $a_c = -(8\pi\rho_r/3)(h_1^2/\lambda)$.
[14] This trend is in agreement with that observed experimentally by Davis and Acrivos [3] for continuously stratified fluid.