On the fission of algebraic solitons

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The fission of an algebraic internal soliton climbing onto a shelf in a two-fluid system of infinite depth is investigated. It is shown that, by using conservation laws of the Benjamin–Ono equation with variable coefficients, we can predict the number of solitons emerging from an incident solitary wave and their amplitudes. These predictions are also verified with numerical solutions.

1. Introduction

When a solitary wave in shallow water of uniform density propagates over a topography varying slowly from one constant depth to another (smaller) constant depth, it splits into a finite number of solitons after reaching a shelf. This phenomenon was first studied by Madsen & Mei (1969) both numerically and experimentally. By making use of the Korteweg–de Vries (KdV) equation with variable coefficients, Tappert & Zabusky (1971), Ono (1972) and Johnson (1972, 1973) provided analytical descriptions on the phenomenon. Also Djordjevic & Redekopp (1978) examined the disintegration of internal solitary waves in (stratified) shallow water by using the similar KdV model.

In this paper, we study the fission of an internal solitary wave climbing onto a shelf in a two-fluid system of infinite depth as illustrated schematically in figure 1.

Without any topographical disturbance in the lower fluid layer, the unidirectional propagation of weakly nonlinear long waves in this system is described by the Benjamin–Ono (BO) equation (Benjamin 1967; Davis & Acrivos 1967; Ono 1975), which admits a solitary wave solution decaying algebraically at infinity. In addition to those in the laboratory experiments (Davis & Acrivos 1967; Koop & Butler 1981), a number of observations of internal solitary waves in the lower atmosphere have been reported in recent years (Christie *et al.* 1978, 1979; also see the review by Smith 1988) and these waves have been identified as algebraic solitary waves governed by the BO equation. But the effects of topographical disturbances such as hills or mountains, which are essential in understanding the long-time evolution of internal waves in the atmosphere, have been neglected in the literature for algebraic solitary waves. In spite of its physical importance, the main reason for this neglect is that no simple mathematical model appropriate to this circumstance has been available.

Here we propose a simple model equation and consider the fission phenomenon as the first step in understanding the transformation of algebraic solitary waves propagating in a non-uniform medium. It can be easily conjectured that a similar fission phenomenon as for the KdV solitary waves may occur when an algebraic solitary wave propagates over a shelf of decreasing depth. This conjecture, however, has not been confirmed by any means and no definite results have been reported.

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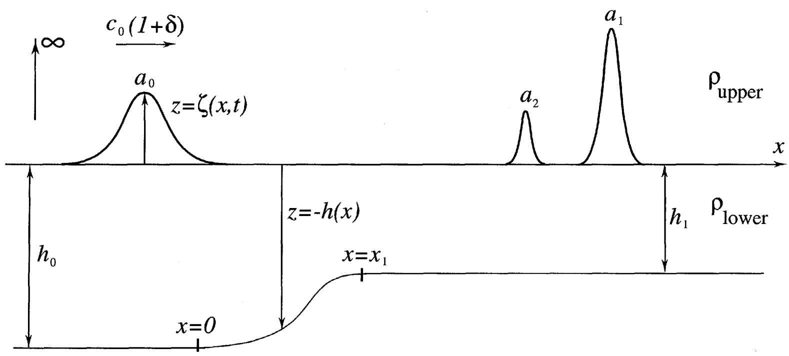

Figure 1. Schematic illustration of the problem.

From the Euler equations, by using a systematic asymptotic expansion method, Choi & Camassa (1996*a*) recently have derived various nonlinear evolution equations for long internal (bidirectional) waves including the effects of non-uniform boundaries. One of their models appropriate for the current problem can be written, in terms of the displacement of the interface ζ and the layer-mean velocity across the lower layer u, as

$$\zeta_t + [(h+\zeta)u]_x = 0, \quad u_t + uu_x + (1-\rho_{\mathrm{r}})g\zeta_x = \rho_{\mathrm{r}}\mathcal{H}[(hu_t)_x], \tag{1.1}$$

where $h(x)$ is the thickness of the lower fluid layer perturbed from h_0 by a stationary topography, the density ratio between two fluids is assumed to be $\rho_{\mathrm{r}}(=\rho_{\text{upper}}/\rho_{\text{lower}}) < 1$ for stable stratification and g is the gravitational acceleration. The non-local operator $\mathcal{H}$ in (1.1) is the Hilbert transform defined by

$$\mathcal{H}[f] = \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{f(x')}{x'-x}\,\mathrm{d}x', \tag{1.2}$$

where $⨍$ stands for the integration as Cauchy principal value. In order to derive (1.1)†, it was assumed that the thickness of the lower fluid layer is much smaller than a characteristic wavelength and nonlinear and dispersive effects balance each other, which gives

$$\epsilon = h_0/L \ll 1, \quad u/c_0 = O(\zeta/h_0) = O(\epsilon), \tag{1.3}$$

where L is a typical wavelength and c_0 is the linear long wave speed given by ${c_0}^2 = gh_0(1-\rho_{\mathrm{r}})$. Therefore we have, for weakly nonlinear long waves, $(\zeta_t, \zeta_x, u_t, u_x) = O(\epsilon^2)$ and, for a slowly varying topography, $h_x = O(\epsilon)$. While the first equation in (1.1) implying conservation of mass is exact, the second equation from conservation of horizontal momentum has an error of $O(\epsilon^4)$.

From the bidirectional model (1.1), we derive a simpler model equation in § 2 for unidirectional waves in a slowly varying medium, which is the BO equation with variable coefficients. By using conservation laws of the model, it is shown in § 3 that we can predict the number of solitons emerging from an incident solitary wave on a shelf and their amplitudes. Numerical solutions of the model are obtained and compared with our analytical predictions in § 4.

† The bidirectional model (1.1) can be obtained from (5.40)–(5.41) in Choi & Camassa (1996*a*), where the upper (lower) fluid layer is thin (deep), after replacing (ζ_2, g, $\mathcal{T}^{-1}$, ρ_{r}) by ($-\zeta$, $-g$, $\mathcal{H}$, $1/\rho_{\mathrm{r}}$) and neglecting the higher-order dispersive terms of $O(\epsilon^4)$.

2. The unidirectional model

First we non-dimensionalize all physical variables in (1.1) as

$$x = \rho_{\mathrm{r}} h_0 x^*, \quad t = (\rho_{\mathrm{r}} h_0 / c_0) t^*, \quad \zeta = h_0 \zeta^*, \quad h = h_0 h^*, \quad u = c_0 u^*, \tag{2.1}$$

by which h_0, ρ_{r} and the effective gravity $g_{\mathrm{e}} \equiv (1-\rho_{\mathrm{r}})g$ can be scaled out from (1.1). To derive a model for unidirectional waves in a slowly varying medium, a characteristic length for the variation of a topography L_h is assumed to be much greater than a typical wavelength L, so that $L/L_h = O(\epsilon)$ and $h_x = O(h_0/L_h) = O(\epsilon^2)$ while $h_x = O(\epsilon)$ in the bidirectional model (1.1). We also adopt the following stretched coordinates

$$\xi = \epsilon \left[\int^x \frac{\mathrm{d}\sigma}{c(\epsilon^2 \sigma)} - t \right], \quad \tau = \epsilon^2 x, \tag{2.2}$$

where the local wave speed, $c = c(\tau)$ to be determined, is assumed to be a function of τ only, signifying the slow variation of depth $h = h(\tau)$. After dropping asterisks for dimensionless variables, we expand the physical variables $f = (\zeta, u)$ as

$$f(\xi, \tau) = \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \cdots, \tag{2.3}$$

and assume that the amplitude of the stationary topography can be $O(1)$, in other words $h(\tau) = O(1)$, while the wave amplitude is $O(\epsilon)$. Substituting (2.3) into (1.1) and using

$$\partial_t = -\epsilon \partial_\xi, \quad \partial_x = (\epsilon / c(\tau)) \partial_\xi + \epsilon^2 \partial_\tau, \tag{2.4}$$

at first order, we have

$$u^{(1)} = (1/c(\tau)) \zeta^{(1)}, \quad c(\tau) = \pm \sqrt{h(\tau)}, \tag{2.5}$$

where $\zeta^{(1)} = u^{(1)} = 0$ is imposed at infinity and $c(\tau)$ is taken as positive (negative) for right-going (left-going) waves.

At second order, (1.1) yields

$$\zeta_\xi^{(2)} - \frac{h(\tau)}{c(\tau)} u_\xi^{(2)} = h u_\tau^{(1)} + h_\tau u^{(1)} + \frac{1}{c} (\zeta^{(1)} u^{(1)})_\xi, \tag{2.6}$$

$$u_\xi^{(2)} - \frac{1}{c(\tau)} \zeta_\xi^{(2)} = \zeta_\tau^{(1)} + \frac{1}{c} u^{(1)} u_\xi^{(1)} + \frac{h}{c} \mathcal{H}[u_{\xi\xi}^{(1)}], \tag{2.7}$$

from which the evolution equation for $\zeta^{(1)}$ can be obtained, by use of (2.5), as

$$\zeta_\tau^{(1)} + \frac{3}{2c^3} \zeta^{(1)} \zeta_\xi^{(1)} + \tfrac{1}{2} \mathcal{H}[\zeta_{\xi\xi}^{(1)}] + \frac{1}{2} \frac{c_\tau}{c} \zeta^{(1)} = 0, \tag{2.8}$$

where $\mathcal{H}$ is understood as the Hilbert transform in ξ.

In terms of the original physical variables (x, t) with $\zeta = \epsilon \zeta^{(1)} + O(\epsilon^2)$, (2.8) can be written, with the same order of approximations, as

$$\frac{1}{c} \zeta_t + \zeta_x + \frac{3}{2h} \zeta \zeta_x + \tfrac{1}{2} h \mathcal{H}[\zeta_{xx}] + \frac{h_x}{4h} \zeta = 0, \tag{2.9}$$

where $c = c(\epsilon^2 x)$, $h = h(\epsilon^2 x)$ and we have used

$$\epsilon \partial_\xi = -\partial_t, \quad \epsilon^2 \partial_\tau = \partial_x + (1/c) \partial_t, \tag{2.10}$$

the leading-order equation $\zeta_t = -c \zeta_x [1 + O(\epsilon)]$ to obtain the higher-order (nonlinear and dispersive) terms and $\mathcal{H}[h(\epsilon^2 x)\zeta_{xx}] = h(\epsilon^2 x)\mathcal{H}[\zeta_{xx}][1 + O(\epsilon^2)]$. Equation (2.9) is the BO equation with variable coefficients which represents the effects of a non-uniform bed. It is analogous to the KdV equation with variable coefficients, for surface waves in shallow water of slowly varying depth, derived by Kakutani (1971) and Johnson (1973).

From now on, we only consider the right-going waves and choose $c(\epsilon^2 x) = +\sqrt{h(\epsilon^2 x)}$.

Without any topographical disturbance ($h = 1$), (2.9) can be reduced to the BO equation:

$$\zeta_t + \zeta_x + \tfrac{3}{2}\zeta\zeta_x + \tfrac{1}{2}\mathcal{H}[\zeta_{xx}] = 0, \tag{2.11}$$

the solitary wave solution of which (Benjamin 1967) is given by

$$\zeta_0(X) = \frac{a_0 {b_0}^2}{{b_0}^2 + X^2}, \tag{2.12}$$

where $a_0 > 0$ and

$$X = x - (1+\delta)t, \quad b_0 = \frac{4}{3a_0}, \quad \delta = \tfrac{3}{8}a_0. \tag{2.13}$$

The BO equation, (2.11), is known to have an infinite number of conservation laws and the first four conserved quantities (Ono 1975; Meiss & Pereira 1978) are given by

$$\mathcal{I}_1 = \int_{-\infty}^{\infty} \zeta \, \mathrm{d}x, \quad \mathcal{I}_2 = \int_{-\infty}^{\infty} \zeta^2 \, \mathrm{d}x, \tag{2.14}$$

$$\mathcal{I}_3 = \int_{-\infty}^{\infty} \tfrac{1}{3}(\zeta^3 + \zeta\mathcal{H}[\zeta_x]) \, \mathrm{d}x, \quad \mathcal{I}_4 = \int_{-\infty}^{\infty} (\tfrac{1}{4}\zeta^4 + \tfrac{1}{2}\zeta^2\mathcal{H}[\zeta_x] + \tfrac{2}{9}{\zeta_x}^2) \, \mathrm{d}x. \tag{2.15}$$

For the case of a non-uniform depth, $h = h(\epsilon^2 x)$, these quantities are no longer conserved and, for example, the conservation laws for the first two quantities in (2.14) (corresponding to mass and energy, respectively) have to be replaced, from (2.9), by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} \zeta \, \mathrm{d}x = \frac{1}{4}\int_{-\infty}^{\infty} \left(\frac{h_x}{h^{1/2}}\right) \zeta \, \mathrm{d}x + O(\epsilon^3), \tag{2.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} \zeta^2 \, \mathrm{d}x = O(\epsilon^3). \tag{2.17}$$

Although the unidirectional model (2.9) fails to conserve mass because the reflected waves are neglected, wave energy is adiabatically invariant as shown in (2.16), (2.17). The same conclusion has been drawn for the KdV equation with variable coefficients by Miles (1979). In addition to energy, the evolution equation (2.9) has another quantity conserved adiabatically, which is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} h^{-1/4}\zeta \, \mathrm{d}x = O(\epsilon^3). \tag{2.18}$$

On the other hand, it can be shown that the bidirectional model (1.1) with (2.1) conserves mass exactly and energy adiabatically

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} \zeta \, \mathrm{d}x = 0, \tag{2.19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} \tfrac{1}{2}(\zeta^2 + hu^2)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} \zeta^2\,\mathrm{d}x + O(\epsilon^3) = O(\epsilon^3), \tag{2.20}$$

where, from (2.5), $\zeta^2 = hu^2 + O(\epsilon^3)$ has been used.

3. The disintegration of a solitary wave

To provide an analytic description of the development of a solitary wave moving over a shelf, we make a further analysis of the unidirectional model (2.8) for a topography shown in figure 1. By substituting the following expression:

$$\zeta^{(1)}(\tau,\xi) = [h(\tau)]^{3/2}\psi(\tau,\xi), \tag{3.1}$$

(2.8) for the right-going waves can be written as

$$\psi_\tau + \tfrac{3}{2}\psi\psi_\xi + \tfrac{1}{2}\mathcal{H}[\psi_{\xi\xi}] + \nu(\tau)\psi = 0, \tag{3.2}$$

where

$$\nu(\tau) = \frac{7}{4}\frac{h_\tau(\tau)}{h(\tau)}. \tag{3.3}$$

For (3.2), we can show that

$$\frac{\mathrm{d}\mathcal{J}_1}{\mathrm{d}\tau} = 0, \quad \mathcal{J}_1 = \exp\left(\int_0^\tau \nu(\tau)\,\mathrm{d}\tau\right)\int_{-\infty}^{\infty}\psi\,\mathrm{d}\xi, \tag{3.4}$$

$$\frac{\mathrm{d}\mathcal{J}_2}{\mathrm{d}\tau} = 0, \quad \mathcal{J}_2 = \exp\left(2\int_0^\tau \nu(\tau)\,\mathrm{d}\tau\right)\int_{-\infty}^{\infty}\psi^2\,\mathrm{d}\xi, \tag{3.5}$$

where, from (3.3), we have

$$\exp\left(\int_0^\tau \nu(\tau)\,\mathrm{d}\tau\right) = h^{7/4}. \tag{3.6}$$

These conservation laws in (3.4), (3.5) for (3.2) are related to two approximate conservation laws in (2.17), (2.18) for the evolution equation (2.9) written in terms of the physical variables.

When $\nu(\tau) = 0$ (or $h_\tau = 0$), (3.2) has a solitary wave solution, from (2.12), (2.13), given by

$$\psi_0 = \frac{\alpha_0{\beta_0}^2}{{\beta_0}^2 + (\xi - \gamma_0\tau)^2}, \tag{3.7}$$

where

$$\beta_0 = 4/(3\alpha_0), \quad \gamma_0 = \tfrac{3}{8}(\alpha_0). \tag{3.8}$$

Owing to the slow variation of topography, we assume that the profile of the incident wave transformed from (3.7) upon reaching the shelf (at $x = x_1$ or $\tau = \tau_1$, see figure 1) is still approximated to the shape of the solitary wave

$$\psi = \frac{\alpha\beta^2}{\beta^2 + (\xi - \gamma\tau)^2}, \tag{3.9}$$

and then we can find the expressions for the local wave amplitude and wavelength, $\alpha(\tau)$ and $\beta(\tau)$, respectively, from the conservation laws in (3.4), (3.5), as

$$\alpha = \alpha_0 h^{-7/4}, \quad \beta = \beta_0. \tag{3.10}$$

On the shelf where $\nu(\tau) = 0$ in (3.2), the BO equation describes the subsequent development of the incident wave given by (3.9). First notice that the area of an algebraic solitary wave (3.7) is independent of wave amplitude such that

$$A_0 = \int_{-\infty}^{\infty} \psi_0 \,\mathrm{d}\xi = \pi\alpha_0\beta_0 = \tfrac{4}{3}\pi, \tag{3.11}$$

where (3.8) has been used for the last expression. If the incident wave of area A given by (3.9) splits into several solitons (possibly plus a small dispersive wave tail) on the shelf, we can see that the number of solitons N (Ono 1975) is the greatest integer satisfying the following inequality:

$$N \leqslant A/A_0, \quad A = \int_{-\infty}^{\infty} \psi \,\mathrm{d}\xi = \pi\alpha\beta. \tag{3.12}$$

By substituting (3.10), (3.11) into (3.12), the number of solitons emerging from an initial solitary wave can be obtained from

$$N \leqslant h^{-7/4}. \tag{3.13}$$

In other words, for given h_1 which is the thickness of the lower fluid layer on the shelf, N can be determined by

$$(N+1)^{-4/7} < h_1 \leqslant N^{-4/7}. \tag{3.14}$$

Suppose the incident wave given by (3.9) breaks into N solitons of amplitude α_i, $(i = 1, \ldots, N)$ on the shelf without radiation, so that the four conservation laws in (2.14)–(2.15) for ψ yield the following relationships between α and α_i (Ono 1975)

$$N = \tfrac{3}{4}\alpha\beta, \tag{3.15}$$

$$\sum_{n=1}^{N} \alpha_n = N\alpha, \quad \sum_{n=1}^{N} (\alpha_n)^2 = (2N-1)\alpha^2, \quad \sum_{n=1}^{N} (\alpha_n)^3 = \left(\frac{5N^2 - 6N + 2}{N}\right)\alpha^3. \tag{3.16}$$

Equation (3.15) from mass conservation gives (3.12) and three relationships in (3.16) determine the wave amplitudes up to three solitary waves. Generally, in order to determine the wave amplitudes of N solitons, we need $(N+1)$ successive conservation laws of the BO equation.

For the formation of two solitons (for $0.534 < h_1 \leqslant 0.673$), the first two equations in (3.16) yield the algebraic equation for α_i, $(i = 1, 2)$, as

$$\left(\frac{\alpha_i}{\alpha}\right)^2 - 2\left(\frac{\alpha_i}{\alpha}\right) + \tfrac{1}{2} = 0, \tag{3.17}$$

and its solutions are given by

$$\alpha_1 = (1 + \tfrac{1}{2}\sqrt{2})\alpha \approx 1.707\alpha_0 {h_1}^{-7/4}, \quad \alpha_2 = (1 - \tfrac{1}{2}\sqrt{2})\alpha \approx 0.293\alpha_0 {h_1}^{-7/4}, \tag{3.18}$$

where we have used (3.10), and then the remaining relationship in (3.16) is automatically satisfied. For three solitons ($0.453 < h_1 \leqslant 0.534$), the wave amplitude α_i, $(i = 1, 2, 3)$, can be determined from the solution of the following cubic equation

$$\left(\frac{\alpha_i}{\alpha}\right)^3 - 3\left(\frac{\alpha_i}{\alpha}\right)^2 + 2\left(\frac{\alpha_i}{\alpha}\right) - \tfrac{2}{9} = 0, \tag{3.19}$$

$$\alpha_1 \approx 2.097\alpha_0 {h_1}^{-7/4}, \quad \alpha_2 \approx 0.765\alpha_0 {h_1}^{-7/4}, \quad \alpha_3 \approx 0.139\alpha_0 {h_1}^{-7/4}. \tag{3.20}$$

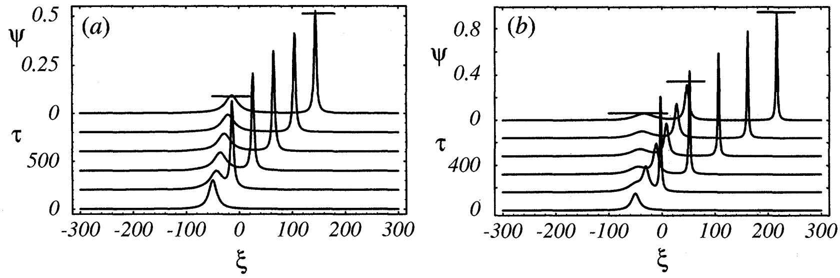

Figure 2. Numerical solutions of (3.2) for an incident solitary wave of $\alpha_0 = 0.15$. The shape of the topography is given by (4.5) with $\tau_1 = 50$ and (*a*) $h_1 = 0.673$, (*b*) $h_1 = 0.534$. Horizontal bars indicate the predicted wave amplitudes by (3.18) and (3.20).

From (3.1), notice that the physical wave amplitude, a_i, for ζ can be found as $a_i = h_1{}^{(3/2)}\alpha_i$. To verify these predictions, we proceed to obtain numerical solutions of (3.2).

4. Numerical solutions

In solving (3.2) numerically, we adopt a finite difference scheme (using central differencing in space and the leap-frog method in time) with periodic boundary conditions in space. For any periodic function of period λ, the Hilbert transform $\mathcal{H}$ can be written, by use of the partial fraction expansion of $\cot(\xi)$, as

$$\mathcal{H}[f] = \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} [f(\xi') - f(\xi)] \cot[\pi(\xi' - \xi)/\lambda] \, \mathrm{d}\xi', \tag{4.1}$$

where we have regularized the integration by use of

$$\frac{1}{\lambda} ⨍_{-\lambda/2}^{\lambda/2} \cot[\pi(\xi' - \xi)/\lambda] \, \mathrm{d}\xi' = 0, \tag{4.2}$$

and the integration in (4.1) is performed using the trapezoidal rule. Also the periodic wave solution of wavelength λ (Benjamin 1967) of the BO equation, (3.2) with $\nu(\tau) = 0$, is given by

$$\psi(\theta) = \frac{A}{1 - B\cos(2\pi\theta/\lambda)}, \tag{4.3}$$

where $\theta = \xi - \delta\tau$ and

$$\delta = \tfrac{3}{8}\alpha_c\left(\frac{\alpha_0}{\alpha_c}\right), \quad A = \tfrac{1}{2}\alpha_c\left(\frac{\alpha_c}{\alpha_0}\right), \quad B = \left[1 - \left(\frac{\alpha_c}{\alpha_0}\right)^2\right]^{1/2}, \quad \alpha_c = \frac{8\pi}{3\lambda}, \tag{4.4}$$

which reduces to the solitary wave solution (2.12) as $\lambda \to \infty$.

To test our numerical code, we solve (3.2) with $\nu(\tau) = 0$ by taking as the initial condition the periodic wave solution (4.3) of $\alpha_0 = 0.15$ with large $\lambda(= 600)$ for which numerical solutions for periodic waves can be regarded as those for solitary waves. When we choose increments for space- and time-like variables as $\Delta\xi = 0.5$ and $\Delta\tau = 0.2$, respectively, the maximum local error, relative to the wave amplitude, of numerical solutions from (4.3) is less than 1% at $\tau = 10^3$ and mass and energy are conserved up to $O(10^{-6})$.

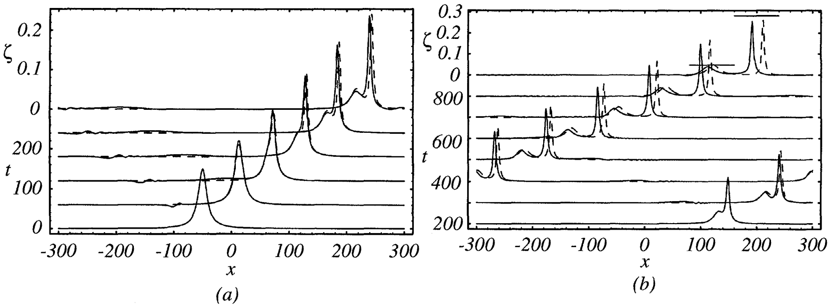

Figure 3. Comparison of numerical solutions between the bidirectional model (—) given by (1.1) and the unidirectional model (- - -) given by (2.9): (*a*) transformation of an incident solitary wave of $a_0 = 0.15$ propagating over the topography given by (4.5) with $h_1 = 0.673$ and $\tau_1 = 50$; (*b*) disintegration into two solitons over the uniform shelf of $h = 0.673$ from the numerical solution of the unidirectional model at $t = 200$ shown in (*a*). Horizontal bars indicate the predicted wave amplitudes $a_i = h^{3/2}\alpha_i$, $(i = 1, 2)$, where α_i is given by (3.18).

For a topography in the form of

$$h(\tau) = \begin{cases} 1, \quad \text{for } \tau \leqslant 0, \\ 1 - (1 - h_1)\cos^2(\pi(\tau - \tau_1)/2\tau_1), \quad \text{for } 0 \leqslant \tau \leqslant \tau_1, \\ h_1, \quad \text{for } \tau_1 \leqslant \tau, \end{cases} \tag{4.5}$$

the length of the transition between two uniform regions (from $h = 1$ to h_1) is chosen to be $\tau_1 = 50$ so that its slope is $O(10^{-2})$ in the computations. We choose the amplitude of an incident solitary wave $\alpha_0 = 0.15$ and the depth of the lower fluid over the shelf $h_1 = 0.673$ and $h_1 = 0.534$ for the formation of two and three solitons, respectively. As indicated by horizontal bars in figure 2, the predicted wave amplitudes ($\alpha_1 \approx 0.512$, $\alpha_2 \approx 0.088$ for $h_1 = 0.673$ and $\alpha_1 \approx 0.944$, $\alpha_2 \approx 0.344$, $\alpha_3 \approx 0.063$ for $h_1 = 0.534$) by (3.18) and (3.20) are in good agreement with the numerical results. For values of h_1 somewhat lower than the upper values predicted by (3.14), numerical solutions show small dispersive tails in addition to solitary waves whose amplitudes are slightly greater than those predicted by (3.18) or (3.20).

We also use the same numerical scheme to solve the bidirectional model (1.1) and the unidirectional model (2.9) in the physical space and time (x, t). The comparison between numerical solutions of the two different models for $h_1 = 0.673$ is made in figure 3.

As shown in figure 3*a*, the numerical solutions of the bidirectional model exhibit small reflected waves which have been neglected in the unidirectional model and there is a slight disagreement in phase between the two solutions which increases in time. Although the unidirectional model is unsatisfactory in conserving mass and energy (with 6.7% loss of mass and 6.1% gain of energy at $t = 300$) compared with the bidirectional model (with 2.2% loss of energy), it still captures all salient features of the initial fission process. The phase difference between two solutions decreases as the slope of a topography becomes smaller, since reflected waves neglected in the unidirectional model are insignificant in this case. A similar observation has been made for surface waves in comparison of numerical solutions between the uni- and bidirectional models (the KdV equation and the Boussinesq equations, respectively) by Teng & Wu (1993). To carry out numerical simulations beyond $t = 300$, a larger domain of computation is required for bidirectional waves, as indicated in figure 3*a*. Instead of increasing a computational domain to examine the fission process over a longer time, we obtain numerical solutions for the uniform depth of $h = 0.673$ by taking as the initial condition the numerical solution of the unidirectional model (2.9) at $t = 200$. As shown in figure 3*b*, numerical solutions of the bidirectional model also support the predictions in § 3, except a phase shift, based on the unidirectional model for a slowly-varying topography.

5. Concluding remarks

We have studied the disintegration of an algebraic solitary wave propagating over a shelf by use of the BO equation with variable coefficients. Contrary to the case of the KdV equation with variable coefficients for which theoretical predictions rely on the information of eigenvalues of the related linear operator (which is the Schrödinger operator), it is shown that all predictions here are made by simply imposing an adiabatic approximation and using conservation laws of the BO equation (with variable coefficients). These predictions are confirmed by numerical solutions of both the unidirectional and bidirectional models.

Since both models proposed here are valid only for weakly nonlinear waves, an important question is the effect of finite wave amplitude. A theoretical investigation on the fully nonlinear effects has been initiated recently by Choi & Camassa (1996*b*) by deriving a simple set of equations and we may have a more complete explanation on the phenomenon of interest in the near future. Despite the fact that we neglect the compressibility of air and the background shear due to wind, the unidirectional model derived here seems to be still useful for more general problems of the propagation of internal waves over a topography in the atmosphere. In fact, without any topographical disturbance, it has been shown that the governing equation is still the BO equation even when these two effects (compressibility and shear) are taken into account (Miessen *et al.* 1990; Rottman & Einaudi 1993).

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