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Vortex solitons in a rotating fluid within a non-uniform tube

Wooyoung Choi [1], T. Yao-tsu Wu *

Division of Engineering and Applied Sciences, California Institute of Technology, Pasadena, CA 91125, USA

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Abstract

For analyzing forced axisymmetric flow of a non-uniformly rotating, inviscid and incompressible fluid within a long tube of slowly varying radius, a theoretical model called the forced Korteweg–de Vries (fKdV) equation with variable coefficients is derived to calculate the amplitude function of the Stokes stream function. When the fluid system is placed under forcing by axisymmetric disturbance steadily moving with a transcritical velocity, new numerical results of flow streamlines are presented to show that well-defined axisymmetrical recirculating eddies can be periodically produced and sequentially emitted to radiate upstream of the disturbance, becoming permanent in form as a procession of vortex solitons. The Rankine vortex and the Burgers vortex are adopted as two primary flows to exemplify this phenomenon and it is shown that flow with a highly centralized axial vorticity is more effective in producing upstream-radiating vortex solitons.

1. Introduction

The striking phenomena of weakly nonlinear and weakly dispersive waves generated by resonant forcing have recently attracted much attention. One of their remarkable features is that a steadily moving transcritical disturbance can produce, continuously and periodically, a succession of solitons to be radiated upstream of the moving disturbance. This phenomenon was identified first numerically by Wu and Wu [1] and subsequently validated experimentally by Lee et al. [2] for the upstream-radiating solitons generated in a layer of water by a surface pressure distribution or a submerged topography moving with a constant transcritical velocity. Several theoretical models have been used to simulate this phenomenon, among which the forced Korteweg–de Vries (fKdV) model is relatively simple yet requires numerical methods for solution [2–5]. In connection with exploring the basic mechanism underlying this phenomenon, the hydrodynamic instability of several forced steady solitons of the fKdV family has been investigated by Camassa and Wu [6] and Yates and Wu [7].

Analogous phenomena have been found in various physical systems and are in fact deemed possible to occur in all soliton-bearing systems under sustained resonant excitation. Studies have been made, e.g., for forced generation of nonlinear waves in stratified fluids by Grimshaw and Smyth [8] and Zhu et al. [9].

* Corresponding author.

[1] Present address: Theoretical Division and Center for Nonlinear Studies, MS-B258, Los Alamos National Laboratory, Los Alamos, NM 87545, USA.

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Recently, Grimshaw [10] and Hanazaki [11] investigated the problem of periodic production of upstream-radiating solitary waves in a axisymmetric rotating fluid confined in a uniform circular cylinder while being resonantly excited by a body moving along the axis, for which they derived a fKdV equation which has constant coefficients. Here we extend their analyses to axisymmetric rotating flows of inviscid and incompressible fluid bounded by a tube of gradually varying radius, for which a model is derived here to give a fKdV equation with variable coefficients. For specific primary flow subject to sustained critical forcing by an axisymmetric slender body, a parametric domain prevails in which vortex solitary waves can be periodically produced to radiate upstream of the forcing body. Corresponding streamline patterns are important for viewing flow structure, but have not been evaluated in the previous studies. Our new results for the flow streamlines exhibit the process of wave generation with a vivid flow visualization of the upstream-radiating vortex solitons produced, which we call 'vortons'. Each vorton assumes the form of a recirculating eddy, closed and permanent in shape, just as that found by Benjamin [12] and Leibovich [13] for the critical rotating flow in their studies of vortex breakdown. (For axisymmetric rotating flows, the critical state is reached when the ratio of axial velocity to the swirl velocity is such that the axial long-wave velocity vanishes.) In addition, an attempt is made here to explore the basic mechanism underlying the phenomenon in question and further examine the effect of periodic production of vorton on the variation of vorticity distribution and its transport.

This paper first presents the basic equations in Section 2 and derives in Section 3 the fKdV equation for modeling the weakly nonlinear and weakly dispersive long waves propagating in an axisymmetric rotating flow of inviscid and incompressible fluid bounded by a non-uniform tube of slowly varying radius. The flow is sustained under forcing by a slender body and/or a slender ring-shaped topography within the tube, or adjacent to the tube wall, each moving with a transcritical velocity. The original tube non-uniformity is reflected by the variable coefficients of the fKdV equation, much in analogy with the model equations for free gravity waves in open channels of gradually varying depth [14–16] and for free waves in rotating flows bounded by a non-uniform tube [17]. In Section 5, the Rankine vortex and the Burgers vortex are adopted as two primary flows here subjected to a sustained resonant forcing. The Rankine vortex is distinguished in affording analytical results determined in closed form, which is of value as a standard reference for assessing results from the Burgers vortex which can be acquired only by numerical means. The corresponding wave resistance experienced by the forcing agency furnishes a very sensitive measure of the periodicity and rate of growth of the unstable modes that maybe discerned only on a slow timescale.

2. Basic equations

To facilitate analyzing the axisymmetric motion of an inviscid and incompressible fluid in question, we adopt the Stokes stream function $\psi(x, r, t)$ with the cylindrical coordinates (x, r, θ). With the new variable $y = r^2$, the axial and radial velocity components (u, v) and the azimuthal velocity w are given by

$$u = 2\psi_y, \qquad rv = -\psi_x, \tag{2.1}$$

$$rw(x, y, t) = \Gamma(x, y, t), \tag{2.2}$$

where $\Gamma(x, y, t)$ is the circulation around the circle $y =$ const., which is also the angular momentum density. With the fluid assumed inviscid, ψ and Γ satisfy the inviscid vorticity equation and the θ-momentum equation,

$$D^2\psi_t + 2\psi_y D^2\psi_x - 2y\psi_x \frac{\partial}{\partial y}\left(\frac{1}{y}D^2\psi\right) + \frac{2}{y}\Gamma\Gamma_x = 0, \tag{2.3}$$

$$\Gamma_t - 2\psi_x\Gamma_y + 2\psi_y\Gamma_x = 0, \tag{2.4}$$

where

$$D^2 = \frac{\partial^2}{\partial x^2} + 4y\frac{\partial^2}{\partial y^2}. \tag{2.5}$$

Concerning the boundary conditions, the case of interest is that of a forced motion produced by a slender body on the axis, $r = r_0(x, t)$, and a slender radial topography at the tube wall, $r = r_w(x, t)$, so that the inner and outer boundary conditions are

$$b_{0t} + 2\psi_y b_{0x} + 2\psi_x = 0 \quad \text{at } y = r_0^2 = b_0(x, t), \tag{2.6}$$

$$-b_{wt} - 2\psi_y b_{wx} + 2\psi_x = 0 \quad \text{at } y - 1 = r_w^2 - 1 = -b_w(x, t). \tag{2.7}$$

For resonant forcing, the slender bodies will be taken to move with a transcritical velocity in a stationary, non-uniform, axisymmetric long tube (see Fig. 1).

Disregarding any tube radius variations for the moment, the primary flow field is assumed steady and uniform in x and may have both the axial and azimuthal velocity components arbitrarily sheared in the radial direction

$$U(y) = (U_0(y), 0, W_0(y)), \tag{2.8}$$

where $U_0(y)$ and $W_0(y)$ are as specified and assumed to be sufficiently smooth and stable to axisymmetric disturbances, i.e., they satisfy the linear stability criterion of Howard and Gupta [18], $\Gamma\Gamma_y \geq y^2\psi_{yy}^2$ in the present notation. In connection with the manifest of the remarkable phenomenon of resonantly forced rotating flows, it is actually necessary, as will be seen, that W_0/r be sheared because the nonlinear effects disappear, so does the resonant forcing, when W_0/r is constant.

3. The model equation for forced nonlinear waves in rotating fluids

It is well known that weakly nonlinear and weakly dispersive long waves of a typical wavelength λ and amplitude a in a fluid rotating with typical speed W_m within a tube of typical radius R are characterized by two important small parameters:

$$\epsilon = R^2/\lambda^2, \qquad \alpha = a/R. \tag{3.1}$$

This is clear if the radial length is scaled by R, axial length by λ, velocity by W_m and time by λ/W_m. For waves of the Boussinesq family in particular, we have $\alpha = O(\epsilon) \ll 1$.

Guided by these parameters, it is however simpler if the lengths in all directions are scaled by R so as to provide the following dimensionless (primed) variables:

$$x = Rx', \quad y = R^2 y', \quad t = Rt'/W_m, \quad \psi = W_m R^2 \psi', \quad \Gamma = W_m R\Gamma', \tag{3.2}$$

in terms of which (2.3) and (2.4) remain unaltered in form, upon dropping the primes.

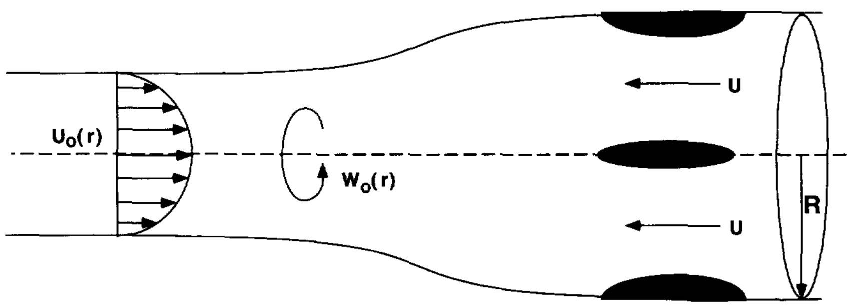

Fig. 1. Schematic view of the problem.

To facilitate modeling evolution of uni-directional (left-going) long waves of the Boussinesq family as specified above, we adopt the following multiscale coordinates [19]:

$$\xi = \epsilon^{1/2}(x + c_0 t), \qquad \eta = \epsilon^{1/2}x, \qquad \tau = \epsilon^{3/2}t, \tag{3.3}$$

where ξ is a slow axial coordinate fixed in the wave frame moving in the negative x-direction with the phase velocity c_0 of infinitesimal long waves which depends on the primary flow field and is yet to be determined. The coordinate η serves to account for the effects due to the fixed slow variations of the tube radius. Here, the various ϵ factors arise in accordance with the dispersion relationship and the condition $\alpha = O(\epsilon)$.

To render the forcing resonant, the slender body and wall topography are assumed to move to the left with a transcritical velocity U,

$$b_0(x + Ut) = \epsilon^2 \tilde{b}_0(x + Ut), \qquad b_w(x, t) = \epsilon \tilde{b}_s(x) + \epsilon^2 \tilde{b}_1(x + Ut) + O(\epsilon^3), \tag{3.4}$$

with U possibly detuned from the critical speed c_0 by a margin of $O(\epsilon)$,

$$U = c_0 + \epsilon\delta + O(\epsilon^2), \qquad \delta = O(1). \tag{3.5}$$

Here $\epsilon\tilde{b}_s(x)$ indicates the radius variation and that the moving disturbances are void of $O(\epsilon)$ terms is dictated by the solvability condition for the solution as will be seen later.

The solutions for ψ and Γ are assumed to consist of two (steady and unsteady) parts,

$$\psi = \psi_0(y) + \psi^U(\xi, \tau; y) + \psi^S(\eta; y), \tag{3.6}$$

$$\Gamma(\xi, \tau; y) = \Gamma_0(y) + \Gamma^U(\xi, \tau; y) + \Gamma^S(\eta; y), \tag{3.7}$$

where

$$\psi_0 = \frac{1}{2}\int_0^y U_0(y)\, dy, \qquad \Gamma_0 = r W_0(y), \tag{3.8}$$

and they can be expanded as

$$\left(\psi^U(\xi, \tau; y), \psi^S(\eta; y)\right) = \epsilon\left(\psi_1^U(\xi, \tau; y), \psi_1^S(\eta; y)\right) + \epsilon^2\left(\psi_2^U(\xi, \tau; y), \psi_2^S(\eta; y)\right) + O(\epsilon^3), \tag{3.9}$$

$$\left(\Gamma^U(\xi, \tau; y), \Gamma^S(\eta; y)\right) = \epsilon\left(\Gamma_1^U(\xi, \tau; y), \Gamma_1^S(\eta; y)\right) + \epsilon^2\left(\Gamma_2^U(\xi, \tau; y), \Gamma_2^S(\eta; y)\right) + O(\epsilon^3). \tag{3.10}$$

In these expansions, the steady part $\psi^S(\eta; y)$ is needed to account for the effects due to small variations of the tube radius and the unsteady component $\psi^U(\xi, \tau; y)$ is due to the moving disturbances.

Substituting (3.4)–(3.10) in (2.6) and (2.7) and expanding these functions into their Taylor series about $y = 0$ and $y = 1$ separately, we find the boundary conditions to the first order as

$$\psi_1^U(\xi, \tau; 0) = 0, \quad \psi_1^S(\eta; 0) = 0 \qquad \text{at } y = 0, \tag{3.11}$$

$$\psi_1^U(\xi, \tau; 1) = 0, \quad \psi_1^S(\eta; 1) = \tfrac{1}{2}U_0(1)\tilde{b}_s \qquad \text{at } y = 1, \tag{3.12}$$

and to the second order,

$$\psi_{2\xi}^U(\xi, \tau; 0) = -\tfrac{1}{2}(U_0(0) + c_0)\tilde{b}_{0\xi}(\xi, \tau) \qquad \text{at } y = 0, \tag{3.13}$$

$$\psi_{2\xi}^U(\xi, \tau; 1) = \tfrac{1}{2}(U_0(1) + c_0)\tilde{b}_{1\xi}(\xi, \tau) + \psi_{1\xi y}^U(\xi, \tau; 1)\tilde{b}_s + \psi_{1y}^U(\xi, \tau; 1)\tilde{b}_{s\eta} \qquad \text{at } y = 1, \tag{3.14}$$

where b_0, b_1 and b_s are taken positive (or negative) when they decrease (or increase) the cross-sectional area of the tube.

Substituting (3.3) and (3.6)–(3.10) into (2.3) and (2.4) yields, for the unsteady component, the first-order equations of the form

$$\psi_1^{\mathrm{U}}(\xi,\tau;y)=\phi_1(y)A_1(\xi,\tau),\qquad \Gamma_1^{\mathrm{U}}=\gamma_1(y)A_1(\xi,\tau), \tag{3.15}$$

$$\gamma_1=\frac{2\Gamma_0'}{U_0+c_0}\phi_1, \tag{3.16}$$

$$L\phi_1=0, \tag{3.17}$$

where

$$L=\frac{\mathrm{d}^2}{\mathrm{d}y^2}+q(y),\qquad q(y)=\frac{1}{(U_0+c_0)}\left[\frac{\Gamma_0\Gamma_0'}{(U_0+c_0)y^2}-U_0''\right], \tag{3.18}$$

and the prime means differentiation with respect to y. The boundary conditions (3.11)–(3.12) give

$$\phi_1(0)=0,\qquad \phi_1(1)=0, \tag{3.19}$$

the corresponding kinematic conditions being $v(\xi,\tau;0)=0$ and $v(\xi,\tau;1)=0$. The system of homogeneous equations (3.17) with (3.19) now constitute an eigenvalue problem for ϕ_1 with the eigenvalue c_0. According to Chandrasekhar [20, Section 78b], there exist at least two eigenvalues, say c_{0m} and c_{0M}, such that

$$-c_{0m}<\min_{y\in(0,1)}U_0(y),\qquad -c_{0M}>\max_{y\in(0,1)}U_0(y), \tag{3.20}$$

provided the flow is stable. Consequently, the singularity of $q(y)$ with $U_0+c_0=0$ will not arise in the interval (0, 1) of interest and the eigenvalue problem is therefore regular (in the Sturm–Liouville sense).

For ψ_1^{S}, the leading order equation becomes

$$\psi_1^{\mathrm{S}}(\eta;y)=\varphi_1(y)\tilde{b}_{\mathrm{s}}(\eta),\qquad L_{\mathrm{s}}\varphi_1=0, \tag{3.21}$$

$$\varphi_1(0)=0,\qquad \varphi_1(1)=\tfrac{1}{2}U_0(1), \tag{3.22}$$

where

$$L_{\mathrm{s}}=\frac{\mathrm{d}^2}{\mathrm{d}y^2}+q_{\mathrm{s}}(y),\qquad q_{\mathrm{s}}(y)=\frac{1}{U_0}\left[\frac{\Gamma_0\Gamma_0'}{U_0y^2}-U_0''\right]. \tag{3.23}$$

Since the differential operator L_{s} in (3.21) becomes identical to L in (3.17) when $c_0=0$, and the boundary conditions (3.22) are inhomogeneous, we tacitly assume that $c_0=0$ is not an eigenvalue of (3.17) for the solution, leaving the critical case of $c_0=0$ to be discussed in Section 4 as a special case. When $c_0=0$ at criticality, the waves and all disturbances will become stationary in the incoming stream so there is no distinction between b_{s} and b_1, which suggests how to recover the critical case from the non-critical case.

To find the evolution equation for $A_1(\xi,\tau)$, it is necessary to find the solvability condition for the next higher-order equations for the unsteady component $\psi_2^{\mathrm{U}}(\xi,\tau;y)$. On the other hand, the higher-order terms of the steady part, i.e., $\psi_n^{\mathrm{S}}(\eta;y)$ $(n>1)$ need not be further pursued since they have no effects on the evolution equation for the first-order amplitude function A_1.

The second-order terms in (2.3) and (2.4) yield for ψ_2^{U} the equation

$$L\psi_{2\xi}^{\mathrm{U}}=f_1(y)\phi_1A_{1\tau}+f_2(y)\phi_1^2A_1A_{1\xi}+f_3(y)\phi_1A_{1\xi\xi\xi}+f_4(y)\tilde{b}_{\mathrm{s}}A_{1\xi}+f_5(y)\tilde{b}_{\mathrm{s}\eta}A_1, \tag{3.24}$$

where

$$f_1(y)=\frac{1}{U_0+c_0}\left(2q+\frac{U_0''}{U_0+c_0}\right), \tag{3.25}$$

$$f_2(y) = -\frac{2}{U_0+c_0}\left[q' + \frac{1}{y^2(U_0+c_0)}\left(\frac{\Gamma_0\Gamma_0'}{U_0+c_0}\right)'\right], \tag{3.26}$$

$$f_3(y) = -1/4y, \tag{3.27}$$

$$\begin{aligned} f_4(y) = &-\frac{2}{U_0+c_0}\left[q_s' + \frac{1}{y^2(U_0+c_0)}\left(\frac{\Gamma_0\Gamma_0'}{U_0}\right)'\right]\varphi_1\phi_1 \\ &-\frac{2}{U_0+c_0}\left[(q_s-q) + \frac{c_0\Gamma_0\Gamma_0'}{y^2U_0(U_0+c_0)^2}\right]\varphi_1'\phi_1, \end{aligned} \tag{3.28}$$

$$\begin{aligned} f_5(y) = &-\frac{2}{U_0+c_0}\left[q' + \frac{\Gamma_0'^2}{y^2U_0(U_0+c_0)} + \frac{\Gamma_0}{y^2(U_0+c_0)}\left(\frac{\Gamma_0'}{U_0+c_0}\right)'\right]\varphi_1\phi_1 \\ &-\frac{2}{U_0+c_0}\left[(q-q_s) - \frac{c_0\Gamma_0\Gamma_0'}{y^2U_0(U_0+c_0)^2}\right]\varphi_1\phi_1'. \end{aligned} \tag{3.29}$$

To obtain the solvability condition for (3.24), we take the inner product of (3.24) with ϕ_1 which satisfies (3.17), where for the inner product of two real functions $f(y)$ and $g(y)$,

$$(f,g) = \int_0^1 f(y)g(y)\,\mathrm{d}y, \tag{3.30}$$

by definition. Thus, the left-hand side of (3.24) gives

$$\begin{aligned} &(\phi_1, L\psi_{2\xi}^{\mathrm{U}}) \\ &= \int_0^1 \phi_1(\partial_y^2+q)\psi_{2\xi}^{\mathrm{U}}\,\mathrm{d}y = (\phi_1\psi_{2\xi y}^{\mathrm{U}} - \phi_{1y}\psi_{2\xi}^{\mathrm{U}})\Big|_0^1 + (L\phi_1, \psi_{2\xi}^{\mathrm{U}}) = -\phi_{1y}\psi_{2\xi}^{\mathrm{U}}|_0^1 \\ &= -\tfrac{1}{2}(U_0(0)+c_0)\phi_{1y}(0)\tilde{b}_{0\xi} - \tfrac{1}{2}(U_0(1)+c_0)\phi_{1y}(1)\tilde{b}_{1\xi} - \left(\phi_1'(1)\right)^2\left(\tilde{b}_sA_{1\xi} + \tilde{b}_{s\eta}A_1\right) \\ &\equiv (f_1,\phi_1^2)\tilde{F}_\xi - \left(\phi_1'(1)\right)^2\left(\tilde{b}_sA_{1\xi} + \tilde{b}_{s\eta}A_1\right) \quad \text{(say)}, \end{aligned} \tag{3.31}$$

on account of (3.17) and conditions (3.13), (3.14) and (3.19), so that, when combined with the contribution from the right-hand side of (3.24), we obtain for A_1 the fKdV equation as

$$A_{1\tau} + c_1A_1A_{1\xi} + c_2A_{1\xi\xi\xi} + c_3\tilde{b}_s(\eta)A_{1\xi} + c_4\tilde{b}_{s\eta}(\eta)A = \tilde{F}_\xi, \tag{3.32}$$

$$c_1 = \frac{(f_2,{\phi_1}^3)}{(f_1,{\phi_1}^2)}, \qquad c_2 = \frac{(f_3,{\phi_1}^2)}{(f_1,{\phi_1}^2)}, \tag{3.33}$$

$$c_3 = \left[(f_4,\phi_1) + \left(\phi_1'(1)\right)^2\right]/(f_1,\phi_1^2), \qquad c_4 = \left[(f_5,\phi_1) + \left(\phi_1'(1)\right)^2\right]/(f_1,\phi_1^2), \tag{3.34}$$

$$\tilde{F} = c_a\tilde{b}_0 + c_w\tilde{b}_1, \qquad c_a = -\frac{\frac{1}{2}(U_0(0)+c_0)\phi_1'(0)}{(f_1,\phi_1^2)}, \qquad c_w = -\frac{\frac{1}{2}(U_0(1)+c_0)\phi_1'(1)}{(f_1,\phi_1^2)}. \tag{3.35}$$

In terms of the original physical variables x and t, we have for the amplitude function $A = \epsilon A_1 + O(\epsilon^2)$ and the forcing function $F = \epsilon^2 \tilde{F}$ the equation

$$A_t - c(x)A_x + c_1 A A_x + c_2 A_{xxx} - (c_4/c_3)c'(x)A = F_x(x + Ut), \tag{3.36}$$

where $c(x) = c_0 - c_3 b_s(x)$ is the local wave velocity.

For free waves ($F = 0$), Eq. (3.36) agrees with the model equation of Leibovich and Randall [17]. Furthermore, for free waves in uniform tube ($b_s \equiv 0$), (3.22) has the classical solitary wave solution

$$A(x,t) = a\,\mathrm{sech}^2\left[\frac{1}{2}\sqrt{\frac{ac_1}{3c_2}}(x + \left(c_0 - \frac{ac_1}{3}\right)t)\right], \tag{3.37}$$

which is a one-parameter family in wave amplitude a, and has also the cnoidal wave solution as waves of permanent form. In particular, when wave amplitude reaches the critical value at which the stagnation point appears on the axis of rotation, the stream surfaces of the free solitary wave take the recirculating formation of an axisymmetrical vortex eddy enclosed by a stream surface [12,13].

For waves in rotating fluids within non-uniform tubes, however, this model has a drawback in not conserving mass because it can admit uni-directional waves but not the reflected ones. Nevertheless, it is known that, like for gravity waves in gradually varying non-uniform water channels, wave energy is adiabatically invariant (see, e.g., [16,21]). To achieve both mass and energy conservation, models of the Boussinesq type as often used in studies of open channel flows would be desirable in this case.

4. The critical case

For the critical case with $c_0 = 0$ in the presence of an axial velocity, the evolution equation (3.36) becomes the fKdV equation of constant coefficients with $b_s = 0$, just as that first derived by Grimshaw [10]. In this case, finite amplitude waves of $O(\epsilon)$ are generated by disturbances of $O(\epsilon^2)$ (b_0 and b_1) since the nonlinear effects become significant.

The fKdV equation can also be expressed with respect to the body frame defined by $X = x + Ut$, $T = t$, in which the steadily moving disturbance appears stationary, so that

$$A_T + (U - c_0)A_X + c_1 A A_X + c_2 A_{XXX} = F_X(X), \tag{4.1}$$

where

$$F(X) = c_a b_0(X) + c_w b_1(X), \tag{4.2}$$

where c_0 is the linear long wave velocity in the fixed frame (in which the forcing disturbances are moving with velocity U). From this expression it is obvious that $| U - c_0 | \le O(\epsilon)$ is required to conform with the order estimate of the other terms of (4.1), thus qualifying the transcritical range.

A body moving in a rotating fluid experiences a resistance due to generation and radiation of waves, which can be determined as follows. First, we assume for the pressure p, non-dimensionalized by ρW_m^2, the same asymptotic expansion as (3.6), (3.7), and (3.9), (3.10) (notice that no steady part is necessary for the solution in the critical case),

$$p(X,y,T) = P_0(y) + \epsilon p_1(X,y,T) + \epsilon^2 p_2(X,y,T) + O(\epsilon^3). \tag{4.3}$$

With (3.6), (3.7), (3.9), (3.10), (4.3) and $\partial(\cdot)/\partial X = O(\epsilon^{1/2})$, we derive from the Euler equation the expressions for P_0 and p_1 in terms of their derivatives as

$$P_{0y} = \Gamma_0^2/2y^2, \tag{4.4}$$

$$p_{1X} = -2\Big((c_0 + U_0)\psi^{\mathrm{U}}_{1yX} - U_0'\psi^{\mathrm{U}}_{1X}\Big), \tag{4.5}$$

$$p_{1y} = \Gamma_0\Gamma_1^{\mathrm{U}}/y^2. \tag{4.6}$$

The equations for the first-order pressure p_1 can be easily integrated, by use of (3.15)–(3.17), to give

$$p_1 = 2\{U_0'\phi_1 - (U_0 + c_0)\phi_1'\}A(X, T). \tag{4.7}$$

The wave resistance D_{w}, non-dimensionalized by $\pi\rho W_{\mathrm{m}}^2 R^2$, is given by

$$D_{\mathrm{w}} = -\int_{-\infty}^{\infty} (r_{\mathrm{w}} r_{\mathrm{w}X} p_{r_1} - r_0 r_{0X} p_{r_0})\,\mathrm{d}X. \tag{4.8}$$

In view of $b_0 = r_0^2 = \mathrm{O}(\epsilon^2)$ and $b_1 = 1 - r_{\mathrm{w}}^2 = \mathrm{O}(\epsilon^2)$, the wave resistance D_{w} can be written as

$$D_{\mathrm{w}} = \int_{-\infty}^{\infty} \Big\{b_{1X}(p)_{y=1} + b_{0X}(p)_{y=0}\Big\}\,\mathrm{d}X + \mathrm{O}(\epsilon^4). \tag{4.9}$$

Using (3.19), (4.2) and (4.7), the wave resistance D_{w} is given, to the leading order, by

$$D_{\mathrm{w}} = 4(f_1, \phi_1^2)\int_{-\infty}^{\infty} F_X(X)A(X, T)\,\mathrm{d}X. \tag{4.10}$$

On physical ground that the fKdV model equation derived here is generic in analogy with the fKdV equation previously obtained for the critical open channel flows, we expect that a process of periodic generation of forward-radiating vortex eddies must be possible in rotating fluids under resonant forcing just as that found theoretically and experimentally for gravity waves in shallow water [2], which is found true.

5. Two forced primary flows

To exemplify the phenomenon of generation of upstream-radiating vortex solitons and their evolution, we choose two relatively simple primary flows, namely the Rankine vortex and the Burgers vortex. The Rankine vortex is perhaps the simplest of the mathematical models with a swirl velocity non-uniform in the radial direction, while still is a good approximation of more realistic vortex flows. Although the preceding analysis cannot be directly applied to the Rankine vortex flow without modification to overcome the difficulties due to the vorticity discontinuity in the basic flow, the model nevertheless has merits that the coefficients of the evolution equation (3.22) can be determined in closed form which can serve as a basic standard for comparison. The other basic flow is the Burgers vortex whose velocity profile well represents what are observed in experiments of swirling flows [22].

5.1. The Rankine vortex

The Rankine-vortex primary flow can be written, for the convenience in this section, in terms of the original coordinate r instead of y as

$$U_0(r) = U_0(= \text{constant}), \tag{5.1}$$

$$W_0(r) = \begin{cases} r/r_c & \text{for } 0 \le r \le r_c, \\ r_c/r & \text{for } r_c \le r \le 1, \end{cases} \tag{5.2}$$

where r_c is the radius of a vortex core of solid-body rotation, which is enclosed by an irrotational potential vortex. Here we use the tube radius R and the maximum swirl velocity Ωr_c to scale all the physical variables as before.

We note that the vortex surface originally at $r = r_c$, being free to move, may take a displacement ζ when set in motion. Inside the vortex core (the inner region, $0 \le r \le r_c + \zeta$) the stream function ψ and the circulation Γ still satisfy Eqs. (2.3) and (2.4), now with r as the independent variable. In the irrotational flow region (the outer region, $r_c + \zeta \le r \le 1$), the stream function Ψ satisfies the irrotationality equation:

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} = 0, \tag{5.3}$$

and the circulation rW_0 is constant (equal to r_c) in the outer region.

The boundary conditions at the interface ($r = r_c + \zeta$) require the continuity of axial, radial and swirl velocities which can be written as

$$\psi = \Psi, \quad \psi_r = \Psi_r, \quad \Gamma = r_c \quad \text{at } r = r_c + \zeta(x,t). \tag{5.4}$$

Under these conditions the surface of discontinuity in vorticity will remain a material surface and the pressure will be continuous across it. With the same forcing excitation as before, the boundary conditions at the body surface ($r = r_0(x + Ut)$) and at the tube wall ($r = r_w(x,t) = 1 - r_s(x) - r_1(x + Ut)$) read

$$(Ur_0 + \psi_r)r_{0x} + \psi_x = 0 \quad \text{at } r = r_0(x + Ut), \tag{5.5}$$

$$r_w r_{wt} + \Psi_r r_{wx} + \Psi_x = 0 \quad \text{at } r = r_w(x,t). \tag{5.6}$$

We adopt again the coordinates (3.3) and the expansions (3.6)–(3.10) for ψ, Γ, and assume that Ψ and ζ can be expanded as

$$\Psi = \Psi_0(r) + \Psi^{\mathrm{U}}(\xi,\tau;r) + \Psi^{\mathrm{S}}(\eta;r), \tag{5.7}$$

$$(\Psi^{\mathrm{U}}, \Psi^{\mathrm{S}}) = \epsilon\Big(\Psi_1^{\mathrm{U}}(\xi,\tau;r), \Psi_1^{\mathrm{S}}(\eta;r)\Big) + \epsilon^2\Big(\Psi_2^{\mathrm{U}}(\xi,\tau;r), \Psi_2^{\mathrm{S}}(\eta;r)\Big) + \mathrm{O}(\epsilon^3), \tag{5.8}$$

where $\Psi_0(r) = \frac{1}{2}U_0 r^2$ and

$$(\zeta^{\mathrm{U}}, \zeta^{\mathrm{S}}) = \epsilon\Big(\zeta_1^{\mathrm{U}}(\xi,\tau), \zeta_1^{\mathrm{S}}(\eta)\Big) + \epsilon^2\Big(\zeta_2^{\mathrm{U}}(\xi,\tau), \zeta_2^{\mathrm{S}}(\eta)\Big) + \mathrm{O}(\epsilon^3), \tag{5.9}$$

now with r_0, $|r_s|$ and $|r_1|$ assumed to be of $\mathrm{O}(\epsilon)$, $\mathrm{O}(\epsilon)$ and $\mathrm{O}(\epsilon^2)$, respectively, for the same reason as stated before. Substituting (5.7) and (5.8) into (5.3) and using (3.15)–(3.17), we have for Γ_1^{U}, ψ_1^{U} and Ψ_1^{U} the first-order equations in the form

$$\Gamma_1^{\mathrm{U}} = \gamma_1(r)A_1(\xi,\tau), \qquad \psi_1^{\mathrm{U}} = \phi_1(r)A_1(\xi,\tau), \qquad \Psi_1^{\mathrm{U}} = \Phi_1(r)A_1(\xi,\tau), \tag{5.10}$$

$$\gamma_1 = \frac{2}{r_c(U_0 + c_0)}\phi_1, \qquad L^-\phi_1 = 0, \qquad L^+\Phi_1 = 0, \tag{5.11}$$

$$L^- = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{4}{r_c^2(U_0 + c_0)^2}, \qquad L^+ = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}. \tag{5.12}$$

Substituting (3.6)–(3.10) and (5.7)–(5.9) into (5.4)–(5.6) yields the boundary conditions to the first order as

$$\phi_1(0) = 0, \qquad \Phi_1(1) = 0, \tag{5.13}$$

$$\phi_1(r_c) - \Phi_1(r_c) = 0, \qquad \phi_1'(r_c) - \Phi_1'(r_c) = 0, \tag{5.14}$$

and, by (5.4), the displacement of interface ζ_1^{U} is given by

$$\zeta_1^{\mathrm{U}} = -\tfrac{1}{2}\gamma_1(r_c)A_1(\xi,\tau). \tag{5.15}$$

From (5.10)–(5.12) with boundary conditions (5.13) and (5.15), we find the first-order unsteady solutions as

$$\psi_1^{\mathrm{U}} = rJ_1(\beta r)A_1(\xi,\tau), \tag{5.16}$$

$$\Psi_1^{\mathrm{U}} = \sigma(r^2-1)A_1(\xi,\tau), \tag{5.17}$$

$$\Gamma_1^{\mathrm{U}} = \frac{2}{r_c(U_0+c_0)} rJ_1(\beta r)A_1(\xi,\tau), \tag{5.18}$$

$$\zeta_1^{\mathrm{U}} = -\frac{J_1(\beta r_c)}{U_0+c_0}A_1(\xi,\tau), \tag{5.19}$$

where J_n denotes the nth order Bessel function and $\beta r_c = |2/(U_0+c_0)|$. Two unknowns, σ and β (or c_0), can be determined by making use of (5.14), giving

$$\frac{J_1(\beta r_c)}{(\beta r_c)J_0(\beta r_c)} = \frac{1}{2}\left(1-\frac{1}{r_c^2}\right), \tag{5.20}$$

$$\sigma = \tfrac{1}{2}\beta J_0(\beta r_c). \tag{5.21}$$

For given r_c, (5.20) has an infinite number of roots for βr_c, with each βr_c lying between $j_{0,n}$ and $j_{1,n}$, the nth zeros of J_0 and J_1, respectively. By choosing the first root βr_c for which the eigenfunction has no zero in $0 < r < 1$,

$$j_{0,1}(=2.4048) < \beta r_c < j_{1,1}(=3.8317), \tag{5.22}$$

$|U_0+c_0|$, non-dimensionalized by Ωr_c, lies in

$$\frac{2}{j_{1,1}}(=0.5220) < |U_0+c_0| < \frac{2}{j_{0,1}}(=0.8317). \tag{5.23}$$

In a similar way, the first-order steady solutions satisfying the non-homogeneous boundary condition (3.22) at the tube wall can be obtained as

$$\psi_1^{\mathrm{S}} = \Theta_s rJ_1(\beta_s r)b_s(\eta), \qquad \Psi_1^{\mathrm{S}} = \left[\sigma_s(r^2-1)+U_0/2\right]b_s(\eta), \tag{5.24}$$

$$\Gamma_1^{\mathrm{S}} = \frac{2\Theta_s}{r_cU_0} rJ_1(\beta_s r)b_s(\eta), \qquad \zeta_1^{\mathrm{S}} = -\frac{\Theta_s J_1(\beta_s r_c)}{U_0}b_s(\eta), \tag{5.25}$$

where

$$b_s(\eta) = 2r_s(\eta), \qquad \beta_s = \frac{2}{r_cU_0}, \tag{5.26}$$

$$\Theta_s = \frac{(\beta/\beta_s)J_0(\beta r_c)}{r_c^2\Big(\beta J_0(\beta r_c)J_1(\beta_s r_c) - \beta_s J_0(\beta_s r_c)J_1(\beta r_c)\Big)}, \tag{5.27}$$

$$\sigma_s = \tfrac{1}{2}\Theta_s\beta_s J_0(\beta_s r_c). \tag{5.28}$$

The above Θ_s and σ_s will assure that the flow velocity remains continuous at $r = r_c$.

As delineated before, the differential equation for the amplitude function $A_1(\xi, \tau)$ is obtained from the solvability condition for the next higher-order problem which gives again (3.36). In the present case, the coefficients in (3.36) can be determined as described before or by directly substituting (5.16)–(5.19) and (5.24), (5.25) into (3.33)–(3.35) (with the second derivative of Γ represented by a Dirac δ-function at $r = r_c$), so that

$$c_1 = \frac{2J_1^3(\beta r_c)}{r_c \Delta}, \quad \Delta = J_0^2(\beta r_c) + J_1^2(\beta r_c) - \frac{2}{\beta r_c} J_0(\beta r_c) J_1(\beta r_c), \tag{5.29}$$

$$c_2 = -\frac{(U_0 + c_0)^3}{8}\left[\frac{2\sigma^2(r_c^2 - \frac{1}{4}r_c^4 - \ln r_c - \frac{3}{4})}{\Delta} + r_c^2\right], \tag{5.30}$$

$$c_3 = \frac{\Theta_s}{r_c^2 \Delta}\left[(2r_c)J_1(\beta_s r_c)J_1^2(\beta r_c) - \frac{4c_0}{U_0}K_1 + \frac{2c_0(3U_0 + c_0)}{U_0^2}K_2\right] + \frac{(U_0 + c_0)}{2}\frac{J_0^2(\beta r_c)}{\Delta r_c^2}, \tag{5.31}$$

$$c_4 = \frac{\Theta_s}{r_c^2 \Delta}\left[(2r_c)J_1(\beta_s r_c)J_1^2(\beta r_c) - \frac{c_0(5U_0 + c_0)}{U_0^2}K_1 + \frac{c_0(3U_0 + c_0)}{U_0^2}K_2\right] + \frac{(U_0 + c_0)}{2}\frac{J_0^2(\beta r_c)}{\Delta r_c^2}, \tag{5.32}$$

$$K_1 = \int_0^{r_c} J_1(\beta_s r)J_1^2(\beta r)\,\mathrm{d}r, \qquad K_2 = \int_0^{r_c} \beta r J_1(\beta_s r)J_0(\beta r)J_1(\beta r)\,\mathrm{d}r, \tag{5.33}$$

$$F = c_a b_0 + c_w b_1 = c_a r_0^2 + c_w(2r_1), \quad c_a = -\frac{2}{r_c^4\beta^3}\frac{1}{\Delta}, \quad c_w = -\frac{2}{r_c^4\beta^3}\frac{J_0(\beta r_c)}{\Delta}, \tag{5.34}$$

where $\Gamma_0(r) = (r^2/r_c)H(r_c - r) + r_c H(r - r_c)$, $H(x)$ being the Heaviside function.

As an example, we choose for the core radius $r_c = \frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ to first obtain β from (5.20) by applying the method of Newton–Rapson then determine σ from (5.21) and hence all the coefficients of the evolution equation (3.36) from (5.29)–(5.34). We also choose the uniform axial velocity $U_0 = 0.3$ to evaluate c_3 and c_4.

The results are shown in Tables 1 and 2. In the following numerical calculations, $r_c = \frac{1}{2}$ is chosen as a representative case.

Table 1
The coefficients of the evolution equation associated with the unsteady components for core radius $r_c = \frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$

r_c	βr_c	σ	$(U_0 + c_0)$	c_1	c_2	c_a	c_w
1/2	2.6411	−.3061	0.7573	1.4636	−.0204	−0.8207	0.0951
1/3	2.5025	−.1862	0.7992	2.7334	−.0146	−1.4252	0.0707
1/4	2.4584	−.1352	0.8135	3.8744	−.0106	−1.9997	0.0550

Table 2
The coefficients of the evolution equation associated with the steady components for core radius $r_c = \frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ and uniform axial velocity $U_0 = 0.3$

r_c	$\beta_s r_c$	Θ_s	σ_s	c_3	c_4
1/2	6.6667	0.1106	0.2078	0.3563	0.2120
1/3	6.6667	0.0601	0.1711	0.2219	0.1527
1/4	6.6667	0.0429	0.1612	0.1839	0.1349

Without external disturbance ($F = 0$ and $b_s = 0$), we may have the classical free solitary wave as a solution (3.37). The r-dependence of the first-order stream function and axial velocity is shown in Fig. 2. To have the streamlines form a recirculating eddy at the axis of rotation, we first find the minimum amplitude of solitary wave a by requiring that the axial velocity at the axis vanish, which gives

$$|a| \geq \left| \frac{U_0 + c_0}{\beta - (c_1/3)} \right| (= 0.158). \tag{5.35}$$

The streamlines of a free solitary wave with amplitude $a = -0.2$ in a moving frame in which the eddy appears stationary are shown in Fig. 3 and such an eddy has been interpreted as a mild axisymmetric vortex breakdown [12].

In the critical case ($F \neq 0$ and $b_s = 0$), the fKdV equation (4.1) has the following symmetry. As the coefficients c_1, c_a and c_w in (5.29) and (5.34) are independent of the sign of $(U_0 + c_0)$ (the direction of propagation), we see from (5.30) that if $A(X, T)$ is a solution of the fKdV equation (4.1) for $U > 0$ and $(U_0 + c_0) > 0$, then $-A(-X, T)$ is also a solution when both U and $(U_0 + c_0)$ change sign. Therefore we need only consider the case of $U > 0$ and $(U_0 + c_0) > 0$ (the left-going waves) in this section.

For the forcing disturbance, we consider a slender cosine-shaped body moving along the tube axis, with the body radius

$$r_0(X) = r_m \cos^2\left[\frac{\pi}{L}X\right] \quad \text{for } -\frac{L}{2} \leq X \leq \frac{L}{2} \tag{5.36}$$

and $r_0(X) = 0$ elsewhere. We could choose a typical blunt body (like a sphere) as forcing agent which might be more effective in generating upstream-progressing solitons, but the additional jump discontinuities in body slope at the leading and trailing edges would require special consideration as such singularities of body geometry are not

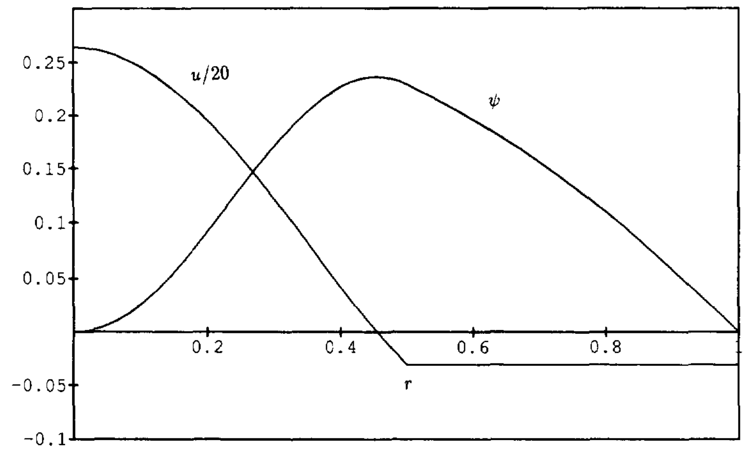

Fig. 2. The eigenfunction of stream function (ψ) and axial velocity (u) for the Rankine vortex with the core radius $r_c = 0.5$ with a uniform tube of radius $R = 1$.

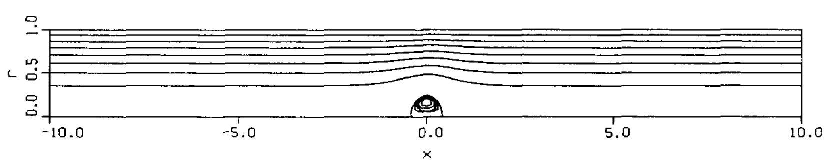

Fig. 3. The streamlines of a free solitary vortex wave solution of amplitude $a = -0.2$ in the Rankine vortex with the core radius $r_c = 0.5$ within a uniform tube of radius $R = 1$.

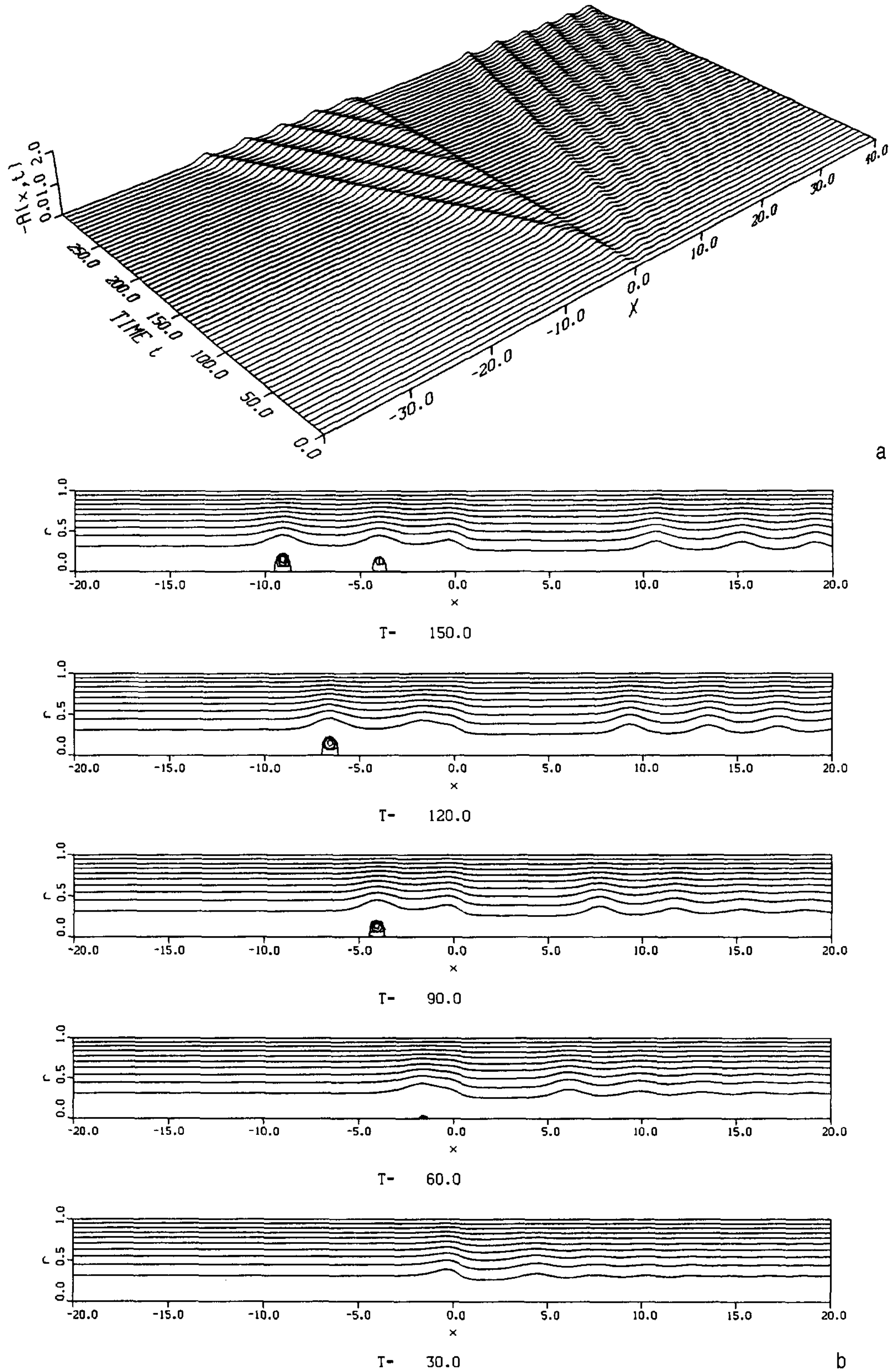

Fig. 4. (a) The numerical solution of the fKdV equation for the amplitude function $A(X, T)$ due to a body r_0 specified in (5.36) at the central axis with $r_m = 0.1$ and $L = 2$ moving at the critical speed ($\delta = 0$). The primary flow is the Rankine vortex with the core radius $r_c = 0.5$ and tube radius $R = 1$. (b) The streamlines corresponding to the solution in (a) at several time instants as specified.

consistent with our original assumptions. For the class of smooth body geometry as given by (5.36), with $r_{0X}/r_0 = O(\epsilon^{1/2})$ uniformly bounded say, we choose a typical maximum body radius $r_m = 0.1$ and a body length $L = 2$. The fKdV equation (4.1) is solved numerically using the scheme of Zabusky and Kruskal [23], i.e., the leap-frog method in time and the central difference in space. Fig. 4(a) shows a typical result of the time sequence of evolution of $-A(X, T)$ starting from rest, here with the detuning parameter $\delta = 0$. Typical of the generation of solitary waves by sustained resonant forcing, it shows that after a certain growth period a solitary wave emerges in front of the disturbance, breaking away to radiate ahead of the disturbance as a free wave and is followed by another new solitary wave similarly produced and radiated. This whole process of generation of upstream solitons seems to continue periodically and indefinitely while a uniform depression in vorticity amplitude just behind the disturbance is being prolonged and followed by a train of cnoidal-like trailing waves. Fig 4(b) shows the corresponding streamlines in a moving coordinate (X, T) at several time instants up to when two isolated recirculating vortons have been generated and radiated upstream. More numerical results of the fKdV equation will be discussed over a range of pertinent parameters in Section 5.2.

The present theory based on the fKdV model is expected to provide good approximations to real physical solutions since, in modeling axisymmetric free solitary wave phenomena, the KdV model has been shown [24] to be in good qualitative agreement with the exact numerical solutions of the Euler equations even for finite amplitudes.

5.2. The Burgers vortex

The Burgers vortex is well-established as a realistic model of non-uniformly rotating fluid in a long cylindrical tube for studying various vortex generation mechanisms [22]. Here we take the following Burgers vortex as a primary flow,

$$U_0(y) = U_m e^{-\mu_1 y}, \tag{5.37}$$

$$\Gamma_0(y) = \Gamma_m(1 - e^{-\mu_2 y}), \tag{5.38}$$

which is linearly stable with respect to axisymmetrical perturbation by the criterion of Howard and Gupta [18]. Here all the variables are non-dimensionalized with length scaled by the tube radius and velocity by the maximum azimuthal velocity at $y = 1.2565/\mu_2$ (or $r \doteq 1.1209/\sqrt{\mu_2}$), making this flow a three parameter family in (U_m, μ_1, μ_2). Noting that U_m is the ratio of the maximum axial velocity to the maximum azimuthal velocity, and for given μ_2 which is a measure of the axial vorticity distribution, Γ_m is scaled out. Substituting (5.37) and (5.38) into (3.15)–(3.18) and solving the resulting eigenvalue problem numerically, we can determine the first eigenvalue c_0 and the corresponding eigenfunction ϕ_1, which can be normalized. Since this eigenvalue problem has a regular singular point at $y = 0$, we may obtain a series solution about $y = 0$ with a suitable normalization and find ϕ_1 and ϕ_1' at $y = y_0 \ll 1$. Taking these as initial values, we find the solution using the 4th order Runge–Kutta method and the corresponding eigenvalue using the secant method. Subsequently, all the coefficients in the fKdV equation can be determined by using (3.33)–(3.35).

For numerical computations, we take $\mu_1 = 5$, $\mu_2 = 12$ and $U_m = 0.6$, for which the velocity profiles of the primary flow are shown in Fig 5. Since the basic axial velocity is not uniform, the right-going (RG) and the left-going (LG) waves are not symmetric with respect to each other. The absolute values of c_1, c_a and c_w are found to depend on the choice of normalization constant for the eigenfunction but the final values of the stream function are invariant. Here, the value of $\phi_1' = 1$ at $y = 0$ is used and the corresponding eigenfunction of the stream function and axial velocity is shown in Fig 6. The coefficients of the fKdV equations are given in Table 3.

Numerical solutions of the fKdV equation are obtained for the same axisymmetric forcing as (5.36). With $r_m = 0.1$, $L = 2$, and with the detuning parameter $\delta = 0$, the resulting perspective view of $A(X, T)$ and the drag D_w given by (4.10) are shown in Fig. 7(a) and (b) with the corresponding streamlines shown in Fig. 8(a) and (b). The distance traversed by the body, denoted by ξ_b, in units of the tube radius is $\xi_b = c_0 t$. At $t = 12$ ($\xi_b = 5.35$ for LG waves accompanying the left-going body and $\xi_b = 17.89$ for RG waves with the right-going body), the first

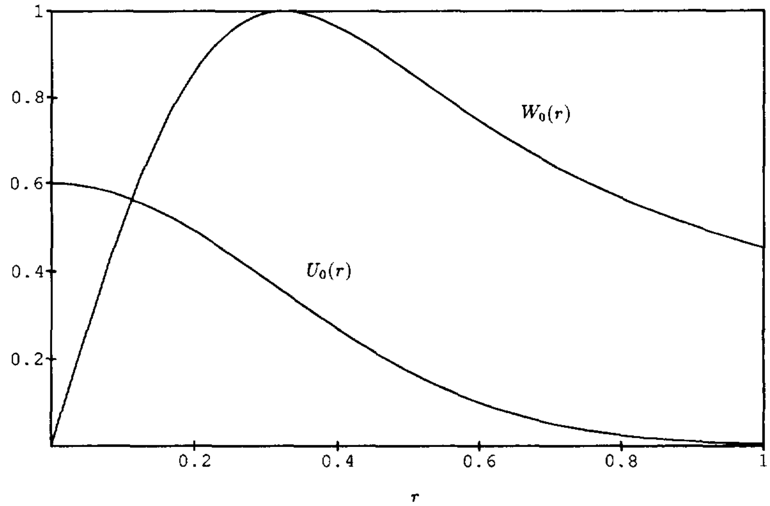

Fig. 5. The velocity profiles of $U_0(r)$ and $W_0(r)$ of the Burgers vortex, (5.37) and (5.38), with $\mu_1 = 5$, $\mu_2 = 12$ and $U_m = 0.6$.

forward-radiating eddy has just emerged free of the body forcing effect; and by $t = 30$ ($\xi_b = 13.37$ for LG waves and $\xi_b = 44.72$ for RG waves), the first three identical forward-radiating vortons have already been generated and the periodicity, as evidenced in the wave resistance data, is remarkably uniform. As can be seen, the LG vortons, which propagate opposite in direction to the basic axial velocity, are generated in much shorter traveling distance of the body as compared with that of the RG ones. More details of numerical solutions for a broad range of the parameters in the fKdV equation can be found in [2] for the analogous shallow water problem. Generally, as the velocity of the disturbance relative to linear long wave speed increases, the amplitude and the generation period of upstream-advancing solitons will both increase.

To examine the effects of basic axial vorticity on the process of generation of vortons, let us consider for simplicity the case with $U_0 = 0$ and a body held along the axis. Pertinent numerical results for the Burgers vortex flow show that the process of generating forward-radiating vortons becomes more effective when the axial vorticity is more concentrated near the axis of rotation since the absolute value of c_a increases with increasing μ_2 which increases with the concentration of axial vorticity. (This also can be seen from Table 1 for the Rankine vortex as the core radius r_c decreases.) Due to the presence of a body held on the axis, a vortexline near the axis becomes displaced radially outward to create vorticity in a radial direction and, in turn, a negative azimuthal vorticity is generated since a material point on the vortexline moves around the axis more slowly than around the same vortexline on the upstream side due to the conservation of circulation [25]. Then, from the linear vorticity equation for the radial and azimuthal vorticity components $(\omega_r, \omega_\theta)$, which reads

$$\frac{\partial \omega_r}{\partial t} = (D_* W_0) v_x, \qquad \frac{\partial \omega_\theta}{\partial t} = -2(W_0/r)\omega_r, \tag{5.39}$$

Table 3
The coefficients of the fKdV equation associated with the Burgers vortex

	μ_1	μ_2	U_m	c_0	c_1	c_2	c_a	c_w
LG	5.0	12.0	0.6	0.4457	0.7316	−0.0137	−11.5056	0.1530
RG	5.0	12.0	0.6	−1.4906	0.7790	0.0195	−15.4165	1.0225

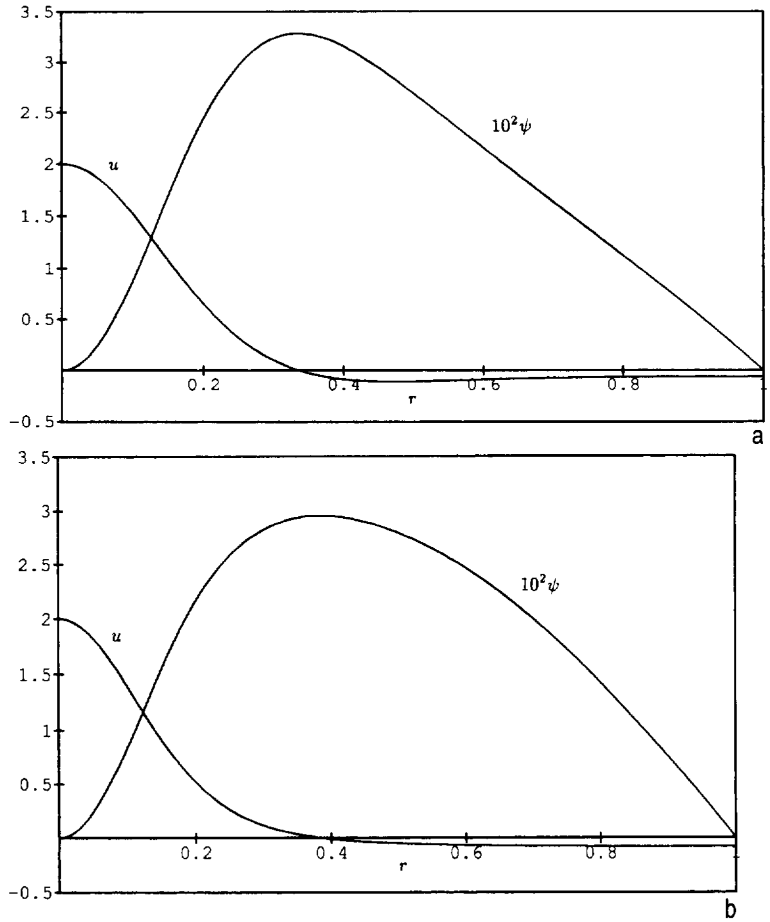

Fig. 6. The eigenfunction of stream function (ψ) and axial velocity (u) for the Burgers vortex with $\mu_1 = 5$, $\mu_2 = 12$ and $U_m = 0.6$ in a uniform tube of radius $R = 1$, pertaining to: (a) the left-going waves; (b) the right-going waves.

it can be seen that the azimuthal vorticity ω_θ can be generated more effectively as the basic axial vorticity $(D_* W_0)$ is more highly sheared.

Also, as can be seen in Table 3, a moving body at the tube axis can be more effective than a tube wall constriction (of the same magnitude) in generating vortons, especially with a highly concentrated vortical flow near the axis, since the absolute value of c_a is much greater than that of c_w. Although a tube wall constriction of $O(\epsilon^2)$ can generate upstream-radiating waves of $O(\epsilon)$, the power of concentrated axial vorticity makes a comparable body at the axis the more effective of the two forcing agents in producing waves of large amplitude.

6. Discussion

To explore the basic mechanism underlying the phenomenon of periodic production of upstream-radiating solitary waves, the hydrodynamic stability analysis of several forced steady solitons of the fKdV equation which has been developed by Camassa and Wu [6] can be directly applied to our problem. For convenience, we transform the present fKdV equation (4.1) into one for open channel flow using the following new variables:

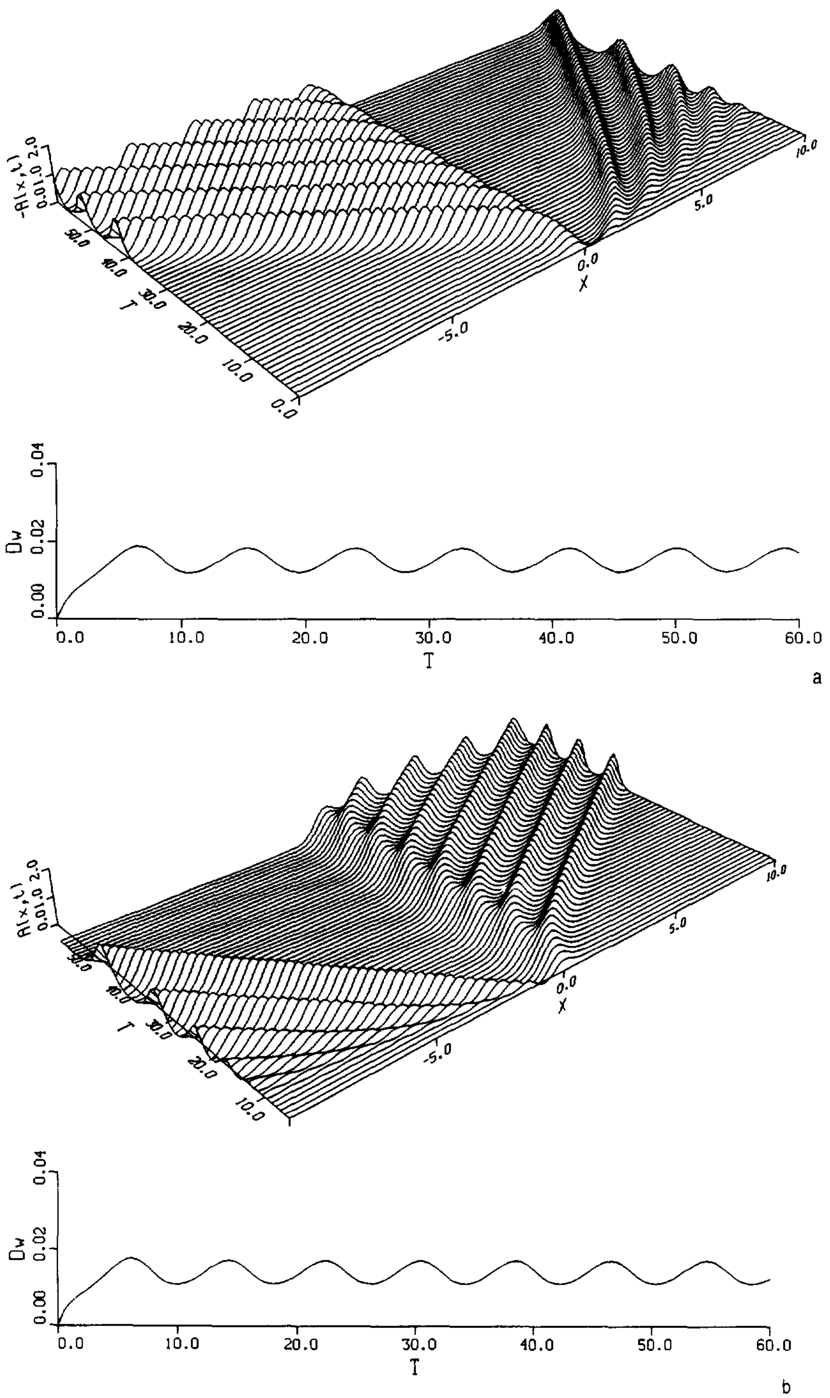

Fig. 7. The numerical solution of the fKdV equation for the amplitude function $A(X, T)$ and the wave resistance D_w induced by the body r_0 in (5.36) moving along the central axis, with $r_m = 0.1$, $L = 2$ and with the critical speed ($\delta = 0$). The primary flow is the Burgers vortex with $\mu_1 = 5$, $\mu_2 = 12$ and $U_m = 0.6$: (a) the left-going waves; (b) the right-going waves.

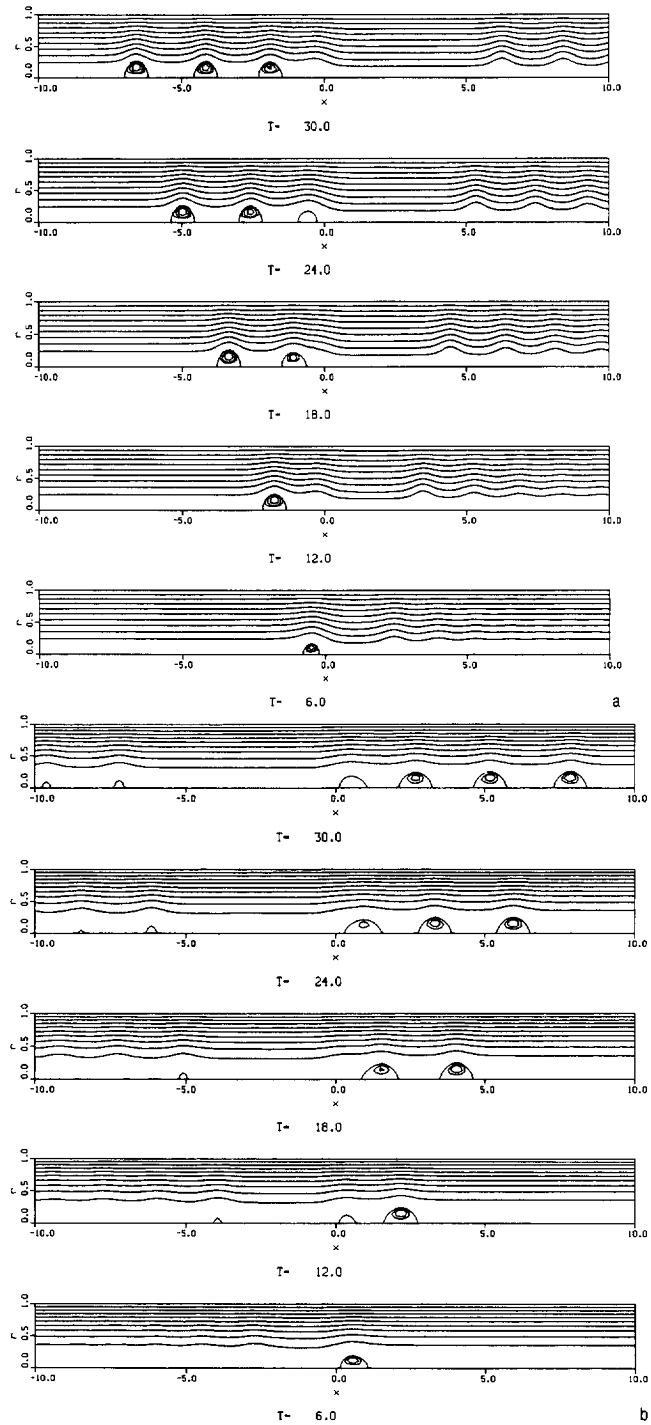

Fig. 8. The streamlines corresponding to the solution shown in Fig. 7, for (a) the left-going waves; (b) the right-going waves.

$$\hat{t} = \gamma_1 T, \qquad \hat{x} = \gamma_2 X, \qquad \eta = \gamma_3 A, \tag{6.1}$$

$$\gamma_1 = c_0 \gamma_2, \qquad \gamma_2^2 = -\frac{1}{6}\frac{c_0}{c_2}, \qquad \gamma_3 = -\frac{2}{3}\frac{c_1}{c_0}, \tag{6.2}$$

where c_0/c_2 is always negative as shown in Table 1 or Table 3. Then the fKdV equation can be written as (after dropping the hat)

$$\eta_t + (F_r - 1)\eta_x - \tfrac{3}{2}\eta\eta_x - \tfrac{1}{6}\eta_{xxx} = P_x, \tag{6.3}$$

where

$$F_r = \frac{U}{c_0}, \qquad P = -\frac{2}{3}\frac{c_1}{c_0^2} F. \tag{6.4}$$

The stationary solutions of (6.3) that vanish at infinity satisfy the following equation:

$$(F_r - 1)\eta - \tfrac{3}{4}\eta^2 - \tfrac{1}{6}\eta_{xx} = P(x), \tag{6.5}$$

which is the first integral of (6.3). One solution of (6.5) whose instability has been investigated in detail in [6] is

$$\eta_s(x) = \tfrac{4}{3}k^2 \operatorname{sech}^2(kx), \qquad P(x) = \tfrac{4}{3}k^2\left(F_r - 1 - \tfrac{2}{3}k^2\right)\operatorname{sech}^2(kx), \tag{6.6}$$

where the Froude number F_r and wave number k (of order $O(\epsilon^{1/2})$ characterizing the length of disturbance) are two parameters. By considering small and finite perturbations of this particular steady solution, Camassa and Wu [6] showed how instability near the critical speed gives rise to the phenomena of periodic generation of upstream-advancing solitons when the eigenvalues are complex with a positive real part.

Another question of hydrodynamic stability is in regard to that of free solitary waves after they have left the area of axisymmetric forcing since the physical significance of these axisymmetric vortons naturally relies on how stable they are. But the stability of such free solitary waves of the KdV family has been shown by Benjamin [26] to be robust. We may therefore draw the conclusion that the vortex soliton is stable under axisymmetric perturbations although the stability characteristics under non-axisymmetrical perturbations is still an open question which will be addressed in a forthcoming publication.

References

[1] D.M. Wu and T.Y. Wu, "Three dimensional nonlinear long waves due to moving surface pressure", *Proc. 14th Symp. on Naval Hydrodynamics*, National Academy Press, Washington, DC (1982) 103–125.

[2] S.J. Lee, G.T. Yates and T.Y. Wu, "Experiments and analyses of upstream-advancing solitary waves generated by moving disturbances", *J. Fluid Mech. 199*, 569–593 (1989).

[3] T.R. Akylas, "On the excitation of long nonlinear waves by a moving pressure distribution", *J. Fluid Mech. 141*, 455–466 (1984).

[4] S.L. Cole, "Transient waves produced by flow past a bump", *Wave Motions 7*, 579–587 (1985).

[5] T.Y. Wu, "On generation of solitary waves by moving disturbances", *J. Fluid Mech. 184*, 75–99 (1987).

[6] R. Camassa and T.Y. Wu, "Stability of forced steady solitary waves", *Philos. Trans. Roy. Soc. London Ser. A 337*, 429–466 (1991).

[7] G.T. Yates and T.Y. Wu, "Stability of solitary waves under skewed forcing", Volume Festschrift in honor of Marshall P. Tulin, in: T. Miloh, ed., *Mathematical Approaches in Hydrodynamics*, SIAM Philadelphia, PA (1991) 193–206.

[8] R.H.J. Grimshaw and N. Smyth, "Resonant flow of a stratified fluid over topography", *J. Fluid Mech. 169*, 429–464 (1986).

[9] J. Zhu, T.Y. Wu, and G.T. Yates "Generation of internal runaway solitons by moving disturbances", *Proc. 16th Symp. on Naval Hydrodynamics*, National Academy Press, Washington, DC (1986) 186–197.

[10] R. Grimshaw, "Resonance flow of a rotating fluid past an obstacle: The general case", *Stud. Appl. Math. 83*, 249–269 (1990).

[11] H. Hanazaki, "Upstream-advancing nonlinear waves in an axisymmetric resonant flow of rotating fluid past an obstacle", *Phys. Fluids A 3*, 3117-3120 (1991).

[12] T.B. Benjamin, "Some developments in the theory of vortex breakdown", *J. Fluid Mech. 28*, 65–84 (1967).

[13] S. Leibovich, "Weakly non-linear waves in rotating fluids", *J. Fluid Mech. 42*, 803–822 (1970).
[14] T. Kakutani, "Effect of an uneven bottom on gravity waves", *J. Phy. Soc. Japan 30*, 272–275 (1971).
[15] R.S. Johnson, "Some numerical solutions of a variable-coefficient KdV equation", *J. Fluid Mech. 54*, 81–91 (1972).
[16] M.H. Teng and T.Y. Wu, "Nonlinear waves in channels of arbitrary shape", *J. Fluid Mech. 242*, 211–233 (1992).
[17] S. Leibovich and J.D. Randall, "Amplification and decay of long nonlinear waves", *J. Fluid Mech. 53*, 481–493 (1973).
[18] L.N. Howard, and A.S. Gupta, "On the hydrodynamic and hydromagnetic stability of swirling flows", *J. Fluid Mech. 14*, 463–476 (1962).
[19] C.S. Gardner and G.M. Morikawa, "Similarity in the asymptotic behaviour of collision free hydromagnetic wave and water waves", Report *NYO–9082*, Courant Inst. of Math. Sciences (1969).
[20] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Oxford University Oxford Press, (1961).
[21] J.W. Miles, "On the Korteweg–de Vries equation for a gradually varying channel", *J. Fluid Mech. 91*, 181–190 (1979).
[22] S. Leibovich, "Vortex breakdown : Survey and extension", *AIAA J. 22*, 1192–1206 (1984).
[23] N.J. Zabusky and M.D. Kruskal, "Interaction of solitons in a collisionless plasma and the recurrence of initial states", *Phys. Rev. Lett. 15*, 241–243 (1965).
[24] S. Leibovich and A. Kribus, "Large amplitude wavetrains and solitary waves in vortices", *J. Fluid Mech. 216*, 459–504 (1990).
[25] G.K. Batchelor, *An introduction to fluid dynamics*, Cambridge University Cambridge Press, (1967).
[26] T.B. Benjamin, "The stability of solitary waves", *Proc. Roy. Soc. London Ser A 328*, 153–183 (1972).