Note 9: Multistage Models

In the **serial structure**, materials enter the first stage and progressively pass through a sequence of production stages until final products exit at the last stage. An example is a metal fabrication and coating process. In the **assembly structure**, each production stage has at most one successor but may have several predecessor stages. An example is the assembly of a computer. In the **general structure**, each stage can have multiple successors and multiple predecessors. An example is the production of 1/4 hp. submersible pumps, which can be used as part of residential fountains, hot tubs, small boat options, and are sold directly from hardware stores.

Notation:

$i = 1, ..., N$ stages where stage 1 denotes the end product and the stage numbers increase as we move further down the bill of materials structure;

$a(i)$ = the set of stages that are immediate successors of stage i: $j \in a(i)$ if j is "after" stage i in the production process;

$b(i)$ = the set of stages that are immediate predecessors of stage i: $j \in b(i)$ if j is "before" stage i in the production process;

$A(i)$ = the set of all stages that are successors of stage i;

$B(i)$ = the set of all stages that are predecessors of stage i;

r_{ij} = number of stage i units required per unit of immediate successor stage j;

ρ_{ij} = number of stage i units required per unit of successor stage j.

A serial structure is such that $a(i)$ and $b(i)$ contain at most one element, $A(i)$ are all stages from i to the end-product stage 1, and $B(i)$ are all stages from stage i back to the lowest-level stage N. Assembly structure stages have unique successors, therefore, $a(i)$ contains one element, $b(i)$ is the set of components/subassemblies that immediately precede stage i in assembly, $A(i)$ is the unique path of stages from i to the end item, and $B(i)$ resembles an assembly structure in which stage i is the end item. For serial and assembly structures, ρ_{ij} is easily computed by taking the product of r_{ij} values on the unique path from item i to item j. General structures are such that $a(i)$ and $b(i)$ have multiple elements and $A(i)$ and $B(i)$ can be general structures also. Finally, it is easy to show that $i \in B(j)$ if and only if $j \in A(i)$.

Example: For 1(2,3,4), 2(5,10), 3(6,7), 4(8,9), 6(11), with $r_{52} = 3$, $r_{21} = 2$, $r_{73} = 3$, and $r_{ij} = 1$ for all other (i, j) pairs, compute $a(4)$, $b(2)$, $A(9)$, $B(3)$, and ρ_{51}.

We use I_i to denote installation inventory at stage i. Echelon inventory E_i of stage i is a measure of how much of stage i production is still in the system:

$$E_i = I_i + \sum_{k \in A(i)} \rho_{ik} I_k = I_i + \sum_{j \in a(i)} r_{ij} E_j.$$

The cost for holding a unit of product i installation inventory for one time period is denoted h_i and generally is a percentage of the total dollars invested in the item. In multistage systems, successor items have a higher installation holding cost. As more operations are performed on an item and more parts are combined, more dollars are invested in that item. Define the echelon holding cost e_i at stage i to be the value added at stage i:

$$e_i = h_i - \sum_{k \in B(i)} \rho_{ki} e_k = h_i - \sum_{j \in b(i)} r_{ji} h_j.$$

The total system inventory cost using echelon inventory costs is equivalent to the total system inventory cost using installation inventory costs.

$$\begin{aligned} &\textstyle\sum_i h_i I_i = \sum_i (\sum_{k \in B(i)} \rho_{ki} e_k + e_i) I_i = \sum_i \sum_{k \in B(i)} \rho_{ki} e_k I_i + \sum_i e_i I_i \\ &\textstyle= \sum_i \sum_{k \in A(i)} \rho_{ik} e_i I_k + \sum_i e_i I_i = \sum_i e_i (\sum_{k \in A(i)} \rho_{ik} I_k + I_i) = \sum_i e_i E_i. \end{aligned}$$

Linear Models for Low Setup Time

Data Parameters:

$k = 1, ..., K$ resources;

$i = 1, ..., N$ stages;

$t = 1, ..., T$ time periods;

J is the set of end items: $i \in J$ is a stage that is an end item;

π_j = shortage cost per unit per period for end-item $j \in J$;

h_i = holding cost per period per unit of item i;

p_{ik} = the number of units of resource k used per unit of stage i produced;

P_{kt} = the amount of resource k available in period t;

r_{il} = the number of units of stage i needed per unit of stage l;

τ_i = minimum lead time for stage i;

D_{it} = external demand for stage i in period t.

Decision Variables:

X_{it} = the number of units of stage i begun in period t;

I_{it}^+ = on-hand inventory of stage i at the end-of-period t;

I_{it}^- = on-hand backorders of stage i at the end-of-period t.

$$\min \sum_{t=1}^{T} (\sum_{i=1}^{N} h_i I_{it}^+ + \sum_{j \in J} \pi_j I_{jt}^-)$$

subject to

$$\sum_{i=1}^{N} p_{ik} X_{it} \le P_{kt}, k = 1, ..., K, t = 1, ..., T$$

$$I_{it}^+ - I_{it}^- = I_{i,t-1}^+ - I_{i,t-1}^- + X_{i,t-\tau_i} - \sum_{l \neq i} r_{il} X_{lt} - D_{it}, i = 1, ..., N, t = 1, ..., T$$

$$I_{it}^- = 0, i \notin J, t = 1, ..., T$$

$$X_{it} \geq 0, I_{it}^+ \geq 0, 0 \leq I_{it}^- \leq \sum_{l=1}^{t} D_{il}, i = 1, ..., N, t = 1, ..., T$$

Example: Lamps are made from a shade, a base assembly and an electrical wiring/fixture assembly. The electrical wiring assembly goes into the base and then a shade is attached to the base/wiring assembly component. Finally, the lamp is tested in quality control. The lead time for each operation is one day. Skilled electricians construct the electrical wiring assembly and do the testing. This is the key resource and we have 200 electrician hours per day. The lamp requires 1.5 electrician hours per unit for assembly and 1.2 hours per unit for testing. The master production schedule calls for 50 lamps in day 4 and 100 lamps in day 6. Backorders of the end item cost \$5 per occurrence, and external demand exists only for the end item. Inventory holding costs are \$1 per unit per period of wiring assembly, \$1.25 per unit per period of the base/wiring assembly component, \$1.5 per unit per period of the attached shade assembly, and \$2 per unit per period of the tested finished product.

Stage 1–test product, stage 2–attach shade, stage 3–attach base to assembly, stage 4–construct assembly. $J = \{1\}$.

$$\min \sum_{t=1}^{6} (1I_{4t}^+ + 1.25I_{3t}^+ + 1.5I_{2t}^+ + 2I_{1t}^+ + 5I_{1t}^-)$$

subject to

$$1.5X_{4t} + 1.2X_{1t} \leq 200 \quad \forall t = 1, .., 6$$

$$I_{1t}^+ - I_{1t}^- = I_{1,t-1}^+ - I_{1,t-1}^- + X_{i,t-1} - D_{1t} \quad \forall t = 1, ..., 6$$

$$D_{14} = 50, D_{16} = 100, D_{11} = D_{12} = D_{13} = D_{15} = 0$$

$$I_{it}^+ - I_{it}^- = I_{i,t-1}^+ - I_{i,t-1}^- + X_{i,t-1} - X_{i-1,t} \quad \forall i = 2, 3, 4, t = 1, ..., 6$$

$$X_{it} \geq 0, I_{it}^+ \geq 0, I_{it}^- = 0 \text{ for } i = 2, 3, 4, I_{1t}^- \leq \sum_{l=1}^{t} D_{1l}.$$

Models with Setup Costs

Model ASI:

$$\min \sum_{t=1}^{T} \sum_{i=1}^{N} (A_{it}\delta_{it} + h_{it}I_{it})$$

subject to

$$X_{it} \leq M\delta_{it} \quad \text{for each } (i, t) \text{ pair}$$

$$I_{1,t-1} + X_{1t} - D_t = I_{1t} \quad \text{for each } t$$

$$I_{i,t-1} + X_{it} - r_{ia(i)}X_{a(i)t} = I_{it} \quad \text{for each } (i, t) \text{ pair}$$

$$X_{it} \geq 0, I_{it} \geq 0, \delta_{it} \in \{0, 1\}$$

Model ASE:

$$\min \sum_{t=1}^{T}\sum_{i=1}^{N}(A_{it}\delta_{it} + e_{it}E_{it})$$

subject to

$$X_{it} \leq M\delta_{it} \quad \text{for each } (i,t) \text{ pair}$$

$$E_{i,t-1} + X_{it} - \rho_{i1}D_t = E_{it} \quad \text{for each } (i,t) \text{ pair}$$

$$r_{ia(i)}E_{a(i)t} \leq E_{it} \quad \text{for each } (i,t) \text{ pair}$$

$$X_{it} \geq 0, E_{it} \geq 0, \delta_{it} \in \{0,1\}$$

Nesting: There exists an optimal solution to model ASE with the property if $X_{it} > 0$ then $X_{a(i)t} > 0$. This leads to **Path Production:** Suppose that $X_{it} > 0$ at optimality. Then, there exists an optimal solution such that $X_{jt} > 0$ for all $j \in A(i)$ (all stages on the path from stage i to the end item).

Echelon Inventory Level: There exists an optimal solution where $X_{it} \cdot E_{i,t-1} = 0$ for all (i,t) pairs.

The Lagrangian relaxation of ASE is

$$f(\lambda) = \min \sum_{i=1}^{N}[\sum_{t=1}^{T}(A_{it}\delta_{it} + (e_{it} + \sum_{k \in b(i)} r_{ki}\lambda_{kt} - \lambda_{it})E_{it}]$$

subject to

$$X_{it} \leq M\delta_{it} \quad \text{for each } (i,t) \text{ pair}$$

$$E_{i,t-1} + X_{it} - \rho_{i1}D_t = E_{it} \quad \text{for each } (i,t) \text{ pair}$$

$$X_{it} \geq 0, E_{it} \geq 0, \delta_{it} \in \{0,1\}$$

Solving the relaxed problem is similar to solving N independent Wagner-Whitin problems. The tightest lower bound is found by solving the following problem:

$$\max f(\lambda)$$

$$\text{subject to } \lambda \geq \mathbf{0}.$$

Subgradient optimization method can be used to solve the problem.