Department of Electrical and Computer Engineering
ECE 673 - Random Signal Analysis I

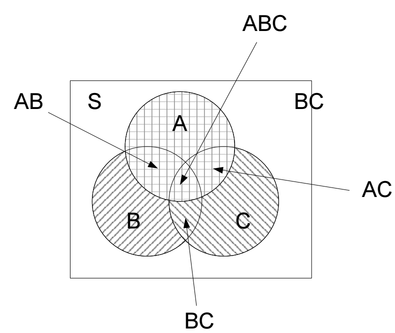
Reading

Shanmugan & Breipohl, Chapter 1, 2.

Homework 1

1. **Problem 2.5**

   a)
   $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC).$$

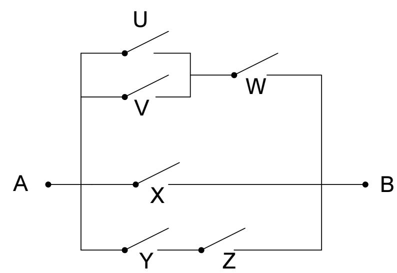


   b)
   $$\begin{aligned} & P(A|B) = P(A) \Leftrightarrow P(AB)/P(B) = P(A) \\ \Leftrightarrow \quad & P(AB) = P(A)P(B) \Leftrightarrow P(AB)/P(A) = P(B) \Leftrightarrow P(B|A) = P(B). \end{aligned}$$

   c)
   $$P(ABC) = P(C|AB)P(AB) = P(C|AB)P(B|A)P(A).$$

2. **Problem 2.8**

   Denote closed switch X as X=1, open switch X as X=0. Then



$$
\begin{aligned}
P(A \to B) &= P(A \to B|X=1)P(X=1) + P(A \to B|X \neq 1)P(X \neq 1) \\
&= p \ + \ (1-p)\ P(A \to B|X \neq 1) \\
&= p \ + \ (1-p)\ [P(A \to B| \neq 1, YZ=1)P(YZ=1) + P(A \to B|X \neq 1, YZ \neq 1)P(YZ \neq 1)] \\
&= p \ + \ (1-p)\ [p^2 \ + \ (1-p^2)\ P(A \to B|X \neq 1, YZ \neq 1)] \\
&= p \ + \ (1-p)\ p^2 \ + \ (1-p)(1-p^2)\ P(A \to B|X \neq 1, YZ \neq 1) \\
&= p \ + \ (1-p)\ p^2 \ + \\
&\qquad (1-p)(1-p^2)\ [P(U=1, W=1) + P(V=1, W=1) - P(U=1, V=1, W=1)] \\
&= p \ + \ (1-p)\ p^2 \ + \ (1-p)(1-p^2)\ [p^2 + p^2 - p^3]
\end{aligned}
$$

3. **Problem 2.10**

   a)

$$
\begin{aligned}
\{\text{outcomes}\} = \quad & \{\text{HHHH,HHHT,HHTH,HHTT,HTHH,HTHT,HTTH,HTTT,} \\
& \text{THHH,THHT,THTH,THTT,TTHH,TTHT,TTTH,TTTT}\}
\end{aligned}
$$

   b) X is number of head. Binomial, $n = 4, p = 0.5$.

$$
P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}, \quad k \in \{0,1,2,3,4\}
$$

$$
\begin{aligned}
P(X=0) &= 1/16, \\
P(X=1) &= 4/16, \\
P(X=2) &= 6/16, \\
P(X=3) &= 4/16, \\
P(X=4) &= 1/16.
\end{aligned}
$$

4. **Problem 2.17**

   a)

$$
P(Y=1|X=1) = P(Y=1, X=1)/P(X=1) = (1/4)\ /\ (1/4) = 1
$$

   b)

$$
P(X=1|Y=1) = P(X=1, Y=1)/P(Y=1) = (1/4)\ /\ (1/8 + 1/4) = 2/3
$$

c)

$$\mu_X = E[X] = 0, \qquad \mu_Y = E[Y] = 0,$$
$$\sigma_X^2 = E[(X-\mu_X)^2] = 1/2, \qquad \sigma_Y^2 = E[(Y-\mu_Y)^2] = 3/4,$$

$$\begin{aligned} \rho_{XY} &= \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y} \\ &= \frac{E[XY]}{\sigma_X \sigma_Y} = (1/2)/\sqrt{3/8} = \sqrt{2/3} \end{aligned}$$

5. **Problem 2.18**

a)
$$E[a + bX] = E[a] + E[bX] = a + bE[X]$$

b)
$$E[aX + bY] = E[aX] + E[bY] = aE[X] + bE[Y]$$

c)
$$\begin{aligned} \text{VAR}[aX + bY] &= E[(aX+bY)^2] - E[(aX+bY)]^2 \\ &= E[a^2X^2 + 2aXbY + b^2Y^2] - (E[aX]^2 + 2E[aX]E[bY] + E[bY]^2) \\ &= (E[a^2X^2] - E[aX]^2) + (E[b^2Y^2] - E[bY]^2) + (E[2aXbY] - 2E[aX]E[bY]) \\ &= \text{VAR}[aX] + \text{VAR}[bY] + 2ab\text{COV}[X,Y] \end{aligned}$$

6. **Problem 2.23**

If X is uniformly distributed in $[a, b]$, then

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & \text{otherwise} \end{cases}$$

Then,

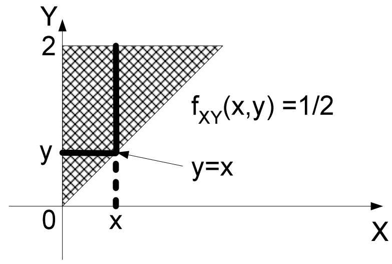
$$\begin{aligned} \mu_X &= E[X] = \int_a^b x \, \frac{1}{b-a} dx = \frac{1}{b-a} \left(\frac{x^2}{2} \right)_a^b = \frac{a+b}{2} \\ \sigma_X &= E[X^2] - E[X]^2 = \int_a^b x^2 \, \frac{1}{b-a} dx - \left(\frac{a+b}{2} \right)^2 = \frac{1}{b-a} \left(\frac{x^3}{3} \right)_a^b - \left(\frac{a+b}{2} \right)^2 = \frac{(b-a)^2}{12} \end{aligned}$$

7. **Problem 2.24**

$X \sim \mathcal{N}(2, 9)$ (i.e. $\mu_X = 2$, $\sigma_X^2 = 9$). Let $Y = (X-2)/3$, then $Y \sim \mathcal{N}(0,1)$.

$$P(-4 < X \leq 5) = P(-4 < 3Y + 2 \leq 5) = P(-2 < Y \leq 1) = Q(-2) - Q(1) = 1 - Q(2) - Q(1)$$

8. **Problem 2.29**



a)

$$
\begin{aligned}
f_X(x) &= \int_x^2 f_{XY}(x,y)dy = \frac{1}{2}\,(y)_x^2 = \frac{2-x}{2},\ x \in [0,2] \\
f_Y(y) &= \int_0^y f_{XY}(x,y)dx = \frac{1}{2}\,(x)_0^y = \frac{y}{2},\ y \in [0,2]
\end{aligned}
$$

b)

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1/2}{y/2} = \frac{1}{y}, \qquad 0 \le x \le y, 0 \le y \le 2 \\
f_{Y|X}(y|x) &= \frac{f_{XY}(x,y)}{f_X(x)} = \frac{1/2}{(2-x)/2} = \frac{1}{2-x}, \quad 0 \le x \le y, 0 \le y \le 2
\end{aligned}
$$

c)

$$
\begin{aligned}
E[X|Y=1] &= \int x f_{X|Y}(x|y=1)dx = \int_0^1 x\frac{1}{1}dx = \left(\frac{x^2}{2}\right)_0^1 = 0.5 \\
E[X|Y=0.5] &= \int x f_{X|Y}(x|y=0.5)dx = \int_0^{0.5} x\frac{1}{0.5}dx = 2\ \left(\frac{x^2}{2}\right)_0^{0.5} = 0.25
\end{aligned}
$$

d) X, Y are not independent since $f_{X|Y}(x|y) \neq f_X(x)$.

e)

$$\begin{aligned}
\mu_X &= E[X] = \int_0^2 x f_X(x)dx = \int_0^2 x(\frac{2-x}{2})dx = \left(\frac{x^2}{2} - \frac{x^3}{6}\right)_0^2 = 2/3 \\
\mu_Y &= E[Y] = \int_0^2 y f_Y(y)dy = \int_0^2 y(\frac{y}{2})dy = \left(\frac{y^3}{6}\right)_0^2 = 4/3 \\
\sigma_X^2 &= E[X^2] - \mu_X^2 = \int_0^2 x^2 f_X(x)dx - \mu_X^2 = \int_0^2 x^2(\frac{2-x}{2})dx - \mu_X^2 = \left(\frac{x^3}{3} - \frac{x^4}{8}\right)_0^2 - \mu_X^2 = 2/9 \\
\sigma_Y^2 &= E[Y^2] - \mu_Y^2 = \int_0^2 y^2 f_Y(y)dy - \mu_Y^2 = \int_0^2 y^2(\frac{y}{2})dy - \mu_Y^2 = \left(\frac{y^4}{8}\right)_0^2 - \mu_Y^2 = 2/9 \\
E[XY] &= \int_0^2 \int_0^y (xy) f_{XY}(x,y)dxdy = \int_0^2 \int_0^y \frac{xy}{2}dxdy = \int_0^2 \frac{y^3}{4}dy = \left(\frac{y^4}{16}\right)_0^2 = 1 \\
\rho_{XY} &= \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X\sigma_Y} \\
&= \frac{E[XY] - \mu_X\mu_Y}{\sigma_X\sigma_Y} = \frac{1-8/9}{2/9} = 1/2
\end{aligned}$$

9. **Problem 2.32**

$$\begin{aligned}
\text{VAR}(Z) &= E[(Z-\mu_Z)^2] \\
&= E[((X+Y-c) - (\mu_X + \mu_Y - c))^2] \\
&= E[((X-\mu_X) + (Y-\mu_Y))^2] \\
&= \sigma_X^2 + \sigma_Y^2 \quad \text{(due to independence of X and Y)}
\end{aligned}$$

10. **Problem 2.33**
a)

$$\begin{aligned}
\mu_Z &= E[Z] = E[(X+Y)/2] = 0 \\
\mu_W &= E[W] = E[(X-Y)/2] = 0 \\
\sigma_Z^2 &= E[Z^2] = E[(X+Y)^2/4] = (\sigma_X^2 + \sigma_Y^2)/4 \\
\sigma_W^2 &= E[W^2] = E[(X-Y)^2/4] = (\sigma_X^2 + \sigma_Y^2)/4
\end{aligned}$$

$$\begin{aligned}
\rho &= \frac{E[(Z-\mu_Z)(W-\mu_W)]}{\sigma_Z\sigma_W} = \frac{E[ZW] - \mu_Z\mu_W}{\sigma_Z\sigma_W} \\
&= \frac{E[(X+Y)(X-Y)/4]}{\sigma_Z\sigma_W} = \frac{(\sigma_X^2 - \sigma_Y^2)}{(\sigma_X^2 + \sigma_Y^2)}
\end{aligned}$$

$$f_{ZW}(z,w) = \frac{1}{2\pi\sigma_Z\sigma_W\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{z-\mu_Z}{\sigma_Z}\right)^2 + \left(\frac{w-\mu_W}{\sigma_W}\right)^2 - \frac{2\rho(z-\mu_Z)(w-\mu_W)}{\sigma_Z\sigma_W}\right]\right\}$$

b)

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma_Z^2}} \exp\left[-\frac{(z-\mu_Z)^2}{2\sigma_Z^2}\right]$$

c) Z and W are independent if $\rho = 0$, i.e. $\sigma_X^2 = \sigma_Y^2$.