Department of Electrical and Computer Engineering
ECE 673 - Random Signal Analysis I

Reading

Shanmugan & Breipohl, Chapter 2.

Homework 2

1. **Problem 2.30**
   a.

$$\begin{cases} Y_1 = X_1 X_2 \\ Y_2 = X_1 \end{cases} \quad \Rightarrow \quad \begin{cases} X_1 = Y_2 \\ X_2 = Y_1/Y_2 \end{cases}$$

The Jacobian is defined as

$$J(x_1, x_2) = \det \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \det \begin{vmatrix} x_2 & x_1 \\ 1 & 0 \end{vmatrix} = -x_1 = -y_2$$

So,

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{f_{X_1,X_2}(y_2, y_1/y_2)}{|J(x_1, x_2)|} = \frac{1}{y_2}, \qquad 0 \le y_1 \le y_2 \le 1.$$

The range is determined by setting

$$\begin{aligned} 0 &\le x_1 = y_2 \le 1 \\ 0 &\le x_2 = y_1/y_2 \le 1 \end{aligned}$$

b.

$$\begin{aligned} f_{Y_1}(y_1) &= \int_{y_1}^{1} f_{Y_1,Y_2}(y_1, y_2) dy_2 = \int_{y_1}^{1} \frac{1}{y_2}\, dy_2 = \ln(y_2)\Big|_{y_1}^{1} = -\ln(y_1), \quad 0 \le y_1 \le 1. \\ f_{Y_2}(y_2) &= \int_{0}^{y_2} f_{Y_1,Y_2}(y_1, y_2) dy_1 = \int_{0}^{y_2} \frac{1}{y_2}\, dy_1 = \frac{y_1}{y_2}\Big|_{0}^{y_2} = 1, \quad 0 \le y_2 \le 1. \end{aligned}$$

c. Y_1 and Y_2 are not independent because $f_{Y_1,Y_2}(y_1, y_2) \neq f_{Y_1}(y_1) f_{Y_2}(y_2)$.

2. **Problem 2.31**
a.

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} \right] \right\}$$

and

$$\begin{aligned} f_X(x) &= \int f_{X,Y}(x,y)\ dy \\ f_Y(y) &= \int f_{X,Y}(x,y)\ dx \end{aligned}$$

Since $\left(\frac{y-\mu_y}{\sigma_y}\right)^2 - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} = \left[\left(\frac{y-\mu_y}{\sigma_y}\right) - \rho\frac{(x-\mu_x)}{\sigma_x}\right]^2 - \left(\rho\frac{(x-\mu_x)}{\sigma_x}\right)^2$, then

$$\begin{aligned} f_X(x) &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp(-\frac{(x-\mu_x)^2}{2\sigma_x^2}) \int_{-\infty}^{+\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(y-\mu_y)}{\sigma_y} - \rho\frac{(x-\mu_x)}{\sigma_x}\right]^2} dy \\ &= \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp(-\frac{(x-\mu_x)^2}{2\sigma_x^2}) \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma_y^2(1-\rho^2)}} e^{-\frac{\left(y-\mu_y-\rho\frac{\sigma_y(x-\mu_x)}{\sigma_x}\right)^2}{2\sigma_y^2(1-\rho^2)}} dy \\ &= \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp(-\frac{(x-\mu_x)^2}{2\sigma_x^2}) \end{aligned}$$

Similarly,

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} \exp(-\frac{(y-\mu_y)^2}{2\sigma_y^2})$$

Therefore, both X and Y are Gaussian distribution.

b.

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{1}{\sqrt{2\pi\sigma_x^2(1-\rho^2)}} \\ &\quad \exp\left\{ -\frac{1}{2\sigma_x^2(1-\rho^2)} \left[(x-\mu_x)^2 + \left(\frac{\sigma_x}{\sigma_y}\right)^2 \rho^2 (y-\mu_y)^2 - 2\rho\frac{\sigma_x}{\sigma_y}(x-\mu_x)(y-\mu_y) \right] \right\} \end{aligned}$$

Let $t = \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)$, we have

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{1}{\sqrt{2\pi\sigma_x^2(1-\rho^2)}} \exp\left\{ -\frac{1}{2\sigma_x^2(1-\rho^2)} \left[(x-\mu_x)^2 + t^2 - t(x-\mu_x) \right] \right\} \\ &= \frac{1}{\sqrt{2\pi\sigma_x^2(1-\rho^2)}} \exp\left\{ -\frac{1}{2\sigma_x^2(1-\rho^2)} [x-\mu_x-t]^2 \right\} \end{aligned}$$

Thus, $E\{X|Y=y\} = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)$, and $\mathrm{VAR}\{X|Y=y\} = \sigma_x^2(1-\rho^2)$

3. **Problem 2.37**

a. Since $X_1 \sim \mathcal{N}(0, \sigma^2)$, $X_2 \sim \mathcal{N}(0, \sigma^2)$ and they are independent. Then,

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x_1^2 + x_2^2}{2\sigma^2}\right).$$

Given

$$\begin{cases} R &= \sqrt{X_1^2 + X_2^2} \\ \Theta &= \tan^{-1}[X_2/X_1] \end{cases} \Leftrightarrow \quad \begin{cases} X_1 &= R\cos(\Theta) \\ X_2 &= R\sin(\Theta) \end{cases}$$

The Jacobian is given as

$$J(x_1, x_2) = \det \begin{vmatrix} \frac{\partial r}{\partial x_1} & \frac{\partial r}{\partial x_2} \\ \frac{\partial \theta}{\partial x_1} & \frac{\partial \theta}{\partial x_2} \end{vmatrix} = \det \begin{vmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ -\frac{x_2}{x_1^2+x_2^2} & \frac{x_1}{x_1^2+x_2^2} \end{vmatrix} = \frac{1}{\sqrt{x_1^2 + x_2^2}} = \frac{1}{r}$$

So,

$$f_{R,\Theta}(r, \theta) = \frac{f_{X_1,X_2}(r\cos(\theta), r\sin(\theta))}{|J(x_1, x_2)|} = \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right), \qquad 0 \le r, 0 \le \theta \le 2\pi.$$

b.

$$f_R(r) = \int_0^{2\pi} f_{R,\Theta}(r, \theta)\ d\theta = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right), \qquad 0 \le r$$

$$f_\Theta(\theta) = \int_0^{\infty} f_{R,\Theta}(r, \theta)\ dr = \frac{1}{2\pi}, \qquad 0 \le \theta \le 2\pi.$$

c. Since $f_{R,\Theta}(r, \theta) = f_R(r) f_\Theta(\theta)$, R and Θ are indenpent.

4. **Problem 2.38**

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

a.

$$f_Y(y) = \frac{f_X(A^{-1}y)}{|\det(A)|} = 1/2.$$

The range is determined by setting

$$0 \le x_1 = (y_1 + y_2)/2 \le 1$$
$$0 \le x_2 = (y_1 - y_2)/2 \le 1$$

b.

$$\mu_{Y_1} = 1, \quad \mu_{Y_2} = 0$$
$$\sigma_{Y_1}^2 = 1/3, \quad \sigma_{Y_2}^2 = 1/3$$
$$\rho_{Y_1,Y_2} = \frac{E[(Y_1 - \mu_{Y_1})(Y_2 - \mu_{Y_2})]}{\sigma_{Y_1}\sigma_{Y_2}} = 0$$

$$E[Y_1 | Y_2 = 0.5] = \int y_1\ f_{Y_1,Y_2}(y_1, y_2 = 0.5)\ dy_1 = 1$$

5. **Problem 2.41**
Since $X \sim \text{unif}[-\pi, \pi]$ and $Y = a\sin(X)$, then

$$x_1 = \sin^{-1}(y/a), \quad x_2 = \sin^{-1}(y/a) - \pi$$
$$J(x) = dy/dx = a\cos(x).$$

$$f_Y(y) = \frac{f_X(x_1)}{|J(x_1)|} + \frac{f_X(x_2)}{|J(x_2)|} = \frac{1/\pi}{\sqrt{a^2 - y^2}}, \quad -|a| \le y \le |a|.$$

6. **Problem 2.42**
$X \sim \mathcal{N}(\mu_X, \Sigma_X)$, where

$$\mu_X = \begin{bmatrix} 6 \\ 0 \\ 8 \end{bmatrix}, \qquad \Sigma_X = \begin{bmatrix} 1/2 & 1/4 & 1/3 \\ 1/4 & 2 & 2/3 \\ 1/3 & 2/3 & 1 \end{bmatrix}$$

Given

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Then,

$$\begin{aligned} \mu_Y &= E[Y] = E[AX] = AE[X] = A\mu_X \\ \Sigma_Y &= E[(Y-\mu_Y)(Y-\mu_Y)^T] = E[A(X-\mu_X)(X-\mu_X)^T A^T] = A\Sigma_X A^T \end{aligned}$$

7. **Problem 2.44**

$$\Sigma_X = E[(X-\mu_X)(X-\mu_X)^T] \quad \Rightarrow \quad V^T\Sigma_X V = V^T E[(X-\mu_X)(X-\mu_X)^T]V = E[\, \|V^T(X-\mu_X)\|^2 \,] \ge 0.$$

8. **Problem 2.47**
a. Solve

$$\det(\Sigma_X - \lambda I) = (3-\lambda)^2 - 1 = 0,$$

we get the two eigenvalues $\lambda_1 = 2$, $\lambda_2 = 4$. Then, solve

$$\Sigma_X \; X_1 = \lambda_1 \; X_1, \quad \Sigma_X \; X_2 = \lambda_1 \; X_2,$$

we get the two eigenvectors $X_1 = [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T$, $X_2 = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T$. Then,

$$\Sigma_X = \underbrace{\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{U^T} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}}_{\Lambda_X} \underbrace{\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{U}$$

b. $Y = AX$ and we want Y be uncorrelated, then

$$E[YY^T] = E[AXX^TA^T] = A\Sigma_X A^T = AU^T\Lambda U A^T = \Lambda.$$

Thus, $AU^T = I$, which means $A = U$.