Department of Electrical and Computer Engineering
ECE 673 - Random Signal Analysis I

Reading

Shanmugan & Breipohl, Chapter 3.1, 3.2, 3.3, 3.5.

Homework 3

1. **Problem 2.35**
   $X \sim \mathcal{N}(0, \sigma_X^2)$.

   (a) $Y = g(X) = X^2$
   If $y < 0$, there is no solution. Hence, $f_Y(y) = 0$ for $y < 0$.
   If $y \geq 0$, there are two solutions. $x_1 = \sqrt{y}$ and $x_2 = -\sqrt{y}$.
   Thus, $g'(x_1) = 2\sqrt{y}$ and $g'(x_2) = -2\sqrt{y}$.

   $$f_Y(y) = \sum_{i=1}^{2} \frac{f_X(x_i)}{|g'(x_i)|} = \begin{cases} \frac{1}{\sqrt{2\pi\sigma_X^2 y}} e^{-\frac{y}{2\sigma_X^2}}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

   (b) $Y = g(X) = |X|$
   If $y < 0$, there is no solution. Hence, $f_Y(y) = 0$ for $y < 0$.
   If $y \geq 0$, there are two solutions. $x_1 = y$ and $x_2 = -y$.
   Thus, $g'(x_1) = 1$ and $g'(x_2) = -1$.

   $$f_Y(y) = \sum_{i=1}^{2} \frac{f_X(x_i)}{|g'(x_i)|} = \begin{cases} \frac{2}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{y^2}{2\sigma_X^2}}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

   (c) $Y = g(X) = \frac{1}{2}(X + |X|)$
   If $y < 0$, there is no solution. Hence, $f_Y(y) = 0$ for $y < 0$.
   If $y > 0$, there is one solution. $x_1 = y$ and $g'(x_1) = 1$.
   If $y = 0$, $\Pr(Y = 0) = \Pr(X \leq 0) = \frac{1}{2}$. Hence, $f_Y(y) = \frac{1}{2}\delta(y)$ for $y = 0$.

   $$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{y^2}{2\sigma_X^2}}, & y > 0 \\ \frac{1}{2}\delta(y), & y = 0 \\ 0, & y < 0 \end{cases}$$

   (d) $Y = g(X) = \begin{cases} 1, & X > \sigma_X \\ X, & X \leq |\sigma_X| \\ -1, & X < -\sigma_X \end{cases}$
   For $y = -1$, $\Pr(Y = -1) = \Pr(X < -\sigma_X) = Q(1)$.

For $y = 1$, $\Pr(Y = 1) = \Pr(X > \sigma_X) = Q(1)$.
For $-1 < y < 1$, $Y = X$ means $f_Y(y) = f_X(x)$.

$$f_Y(y) = \begin{cases} 0, & y < -1 \\ Q(1)\delta(y+1), & y = -1 \\ \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{y^2}{2\sigma_X^2}}, & |y| < \sigma_X \\ Q(1)\delta(y-1), & y = 1 \\ 0, & y > 1 \end{cases}$$

2. **Problem 2.43**

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}, \qquad \mu_X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \Sigma_X = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

By partition, we have

$$\vec{X}_1 = \begin{bmatrix} X_1 \end{bmatrix}, \qquad \mu_{\vec{X}_1} = \begin{bmatrix} 0 \end{bmatrix}, \qquad \Sigma_{11} = \begin{bmatrix} 4 \end{bmatrix}$$

$$\vec{X}_2 = \begin{bmatrix} X_2 \\ X_3 \\ X_4 \end{bmatrix}, \qquad \mu_{\vec{X}_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \Sigma_{22} = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 4 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

Then,

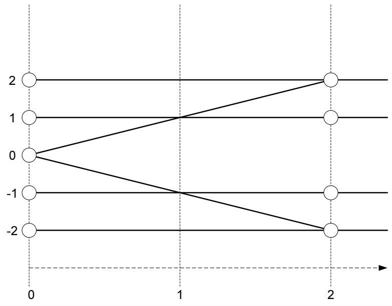
$$\begin{aligned} E[\vec{X}_1 | \vec{X}_2 = [0.5, 1.0, 2.0]^T] &= \mu_{\vec{X}_1} + \Sigma_{12}\Sigma_{22}^{-1}(\vec{X}_2 - \mu_{\vec{X}_2}) \\ &= 0 + \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 \\ 3 & 4 & 3 \\ 2 & 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 1.0 \\ 2.0 \end{bmatrix} \\ &= \frac{1}{12} \end{aligned}$$

$$\begin{aligned} \Sigma_{\vec{X}_1|\vec{X}_2} &= \Sigma_1 1 - \Sigma_1 2 \Sigma_2 2^{-1} \Sigma_2 1 \\ &= 4 + \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 \\ 3 & 4 & 3 \\ 2 & 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \\ &= \frac{5}{3} \end{aligned}$$

3. **Problem 3.1**

a.

	$X(0) = -2$	$X(0) = -1$	$X(0) = 0$	$X(0) = 1$	$X(0) = 2$
$X(2) = -2$	1/6	0	1/6	0	0
$X(2) = -1$	0	1/6	0	0	0
$X(2) = 0$	0	0	0	0	0
$X(2) = 1$	0	0	0	1/6	0
$X(2) = 2$	0	0	1/6	0	1/6



b.

$$\Pr(X(0)=n)\begin{cases}1/6, & n=-2\\ 1/6, & n=-1\\ 1/3, & n=0\\ 1/6, & n=1\\ 1/6, & n=2\end{cases} \qquad \Pr(X(2)=n)\begin{cases}1/3, & n=-2\\ 1/6, & n=-1\\ 0, & n=0\\ 1/6, & n=1\\ 1/3, & n=2\end{cases}$$

c.

$$\begin{aligned} E[X(0)] &= 0\\ E[X(2)] &= 0\\ E[X(0)X(2)] &= 5/3 \end{aligned}$$

4. **Problem 3.13**

$$E[X(t)]=0, \quad R_{XX}(\tau)=E[X(t)X(t-\tau)]$$
$$E[Y(t)]=0, \quad R_{YY}(\tau)=E[Y(t)Y(t-\tau)]$$

a. $Z(t)=a+bX(t)+cY(t)$.

$$\begin{aligned} R_{ZZ}(\tau) &= E[Z(t)Z(t-\tau)]\\ &= E[(a+bX(t)+cY(t))(a+bX(t-\tau)+cY(t-\tau))]\\ &= a^2+b^2R_{XX}(\tau)+c^2R_{YY}(\tau) \end{aligned}$$

b. $Z(t)=aX(t)Y(t)$.

$$\begin{aligned} R_{ZZ}(\tau) &= E[Z(t)Z(t-\tau)]\\ &= E[(aX(t)Y(t))(aX(t-\tau)Y(t-\tau))]\\ &= a^2R_{XX}(\tau)R_{YY}(\tau) \end{aligned}$$

5. **Problem 3.14**

$$Y(t)=X(t+a)-X(t-a)$$

a.

$$\begin{aligned} R_{YY}(\tau) &= E[Y(t)Y(t-\tau)] \\ &= E[(X(t+a)-X(t-a))(X(t-\tau+a)-X(t-\tau-a))] \\ &= 2R_{XX}(\tau) - R_{XX}(\tau+2a) - R_{XX}(\tau-2a) \end{aligned}$$

b.

$$\begin{aligned} S_{YY}(f) &= \mathcal{F}[R_{YY}(\tau)] \\ &= \mathcal{F}[2R_{XX}(\tau) - R_{XX}(\tau+2a) - R_{XX}(\tau-2a)] \\ &= 2S_{XX}(f) - S_{XX}(f)e^{2\pi\cdot 2af} - S_{XX}(f)e^{-2\pi\cdot 2af} \\ &= S_{XX}(f)\sin^2(2\pi af) \end{aligned}$$

6. **Problem 3.21**

a. For any n, any $\{t_1,\ldots,t_n\}$, it is obvious that $Z(t_1),\ldots,Z(t_n)$ is jointly Gaussian. Therefore, $Z(t)$ is a Gaussian random process.

b. The joint pdf is specified by mean $\vec{\mu}$ and covariance matrix K:

$$\begin{aligned} \vec{\mu} &= \begin{bmatrix} E[Z(t_1)] \\ E[Z(t_2)] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ K &= \begin{bmatrix} E[Z(t_1)Z(t_1)] & E[Z(t_1)Z(t_2)] \\ E[Z(t_2)Z(t_1)] & E[Z(t_2)Z(t_2)] \end{bmatrix} = \begin{bmatrix} 1 & \cos[2000\pi(t_1-t_2)] \\ \cos[2000\pi(t_1-t_2)] & 1 \end{bmatrix} \end{aligned}$$

c. Since $R_{ZZ}(\tau) = \cos[2000\pi\tau]$, the process is WSS.

d. If a Gaussian process is WSS, it is SSS.

e. $E[Z(t_2)|Z(t_1)] = E[\ E[Z(t_2)|Z(t_1)=z]\ ] = 0$.