Department of Electrical and Computer Engineering
ECE 673 - Random Signal Analysis I

Reading

Shanmugan & Breipohl, Chapter 3.4, 3.6, 3.9.

Homework 4

1. **Problem 3.4**
   Markove property is defined as

$$\begin{aligned}
& \Pr[X(t_k) \le x_k | X(t_{k-1}), \ldots, X(t_1)] = \Pr[X(t_k) \le x_k | X(t_{k-1})] \\
\Leftrightarrow \quad & F_{X(t_k)|X(t_{k-1}),\ldots,X(t_1)}(x_k|x_{k-1}, \ldots, x_1) = F_{X(t_k)|X(t_{k-1})}(x_k|x_{k-1}) \\
\Leftrightarrow \quad & f_{X(t_k)|X(t_{k-1}),\ldots,X(t_1)}(x_k|x_{k-1}, \ldots, x_1) = f_{X(t_k)|X(t_{k-1})}(x_k|x_{k-1})
\end{aligned}$$

   Then,

$$\begin{aligned}
E[X(t_k)|X(t_{k-1}), \ldots, X(t_1)] &= \int x_k f_{X(t_k)|X(t_{k-1}),\ldots,X(t_1)}(x_k|x_{k-1}, \ldots, x_1)\ dx_k \\
&= \int x_k f_{X(t_k)|X(t_{k-1})}(x_k|x_{k-1})\ dx_k \\
&= E[X(t_k)|X(t_{k-1})]
\end{aligned}$$

2. **Problem 3.7**
   The PMF is given as

$$\Pr[X(n) = m] = \binom{n}{k} \left(\frac{1}{2}\right)^n, \quad k = 0, 1, \ldots, n;\ m = 2k - n$$

   a. $n = 2$, $m = 0$, and $k = 1$, then

$$\Pr[X(2) = 0] = \binom{2}{1} \left(\frac{1}{2}\right)^2 = \frac{1}{2}.$$

   Alternatively,

$$\Pr[X(2) = 0] = \Pr[J_1 + J_2 = 0] = \Pr[J_1 = 1, J_2 = -1] + \Pr[J_1 = -1, J_2 = 1] = \frac{1}{2}.$$

   b.

$$\begin{aligned}
\Pr[X(8) = 0|X(6) = 2] &= \Pr[X(8) - X(6) = -2|X(6) = 2] \\
&= \Pr[X(8) - X(6) = -2] \\
&= \Pr[J_7 + J_8 = -2] = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}
\end{aligned}$$

c.

$$E[X(10)] = E[\sum_{i=1}^{10} J_i] = 0$$

d.

$$E[X(10)|X(4)=4] = E[\sum_{i=1}^{10} J_i | \sum_{i=1}^{4} J_i = 4] = E[\sum_{i=5}^{10} J_i] + 4 = 4$$

3. **Problem 3.10**
First of all, $N(t)$ is a Poisson random process with parameter λ, then (with notation $0^0 = 1$), we can write

$$\Pr[N(t)=k] = \frac{(\lambda t)^k}{k!} \exp(-\lambda t), \quad k = 0, 1, 2, \ldots$$

In this case, the random telegraph signal $X(t)$ can be written as

$$X(t) = (-1)^{N(t)+1} = \begin{cases} 1, & N(t) \text{ is odd} \\ -1, & N(t) \text{ is even} \end{cases}$$

The probability mass function $\Pr[X(t) = \pm 1]$ can be written as

$$\Pr[X(t)=1] = \Pr[N(t) \text{ is odd}] = \sum_{k \text{ odd}} \frac{(\lambda t)^k}{k!} \exp(-\lambda t) = \exp(-\lambda t)\sinh(\lambda t)$$
$$\Pr[X(t)=-1] = \Pr[N(t) \text{ is even}] = \sum_{k \text{ even}} \frac{(\lambda t)^k}{k!} \exp(-\lambda t) = \exp(-\lambda t)\cosh(\lambda t)$$

a. To show that $X(t)$ is Markov, we can show that

$$\begin{aligned}
\Pr[X(t_{n+1})|X(t_n),\ldots,X(t_1)] &= \Pr[(-1)^{N(t_{n+1})+1}|X(t_n),\ldots,X(t_1)] \\
&= \Pr[(-1)^{N(t_{n+1})-N(t_n)}\ X(t_n)|X(t_n),\ldots,X(t_1)] \\
&= \Pr[(-1)^{N(t_{n+1})-N(t_n)}\ X(t_n)|X(t_n)] \\
&= \Pr[X(t_{n+1})|X(t_n)].
\end{aligned}$$

b.

$$\begin{aligned}
\mu_X(t) &= E[X(t)] = (-1)\exp(-\lambda t)\cosh(\lambda t) + (1)\exp(-\lambda t)\sinh(\lambda t) \\
&= -\exp(-2\lambda t)
\end{aligned}$$

$$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)] = \sum_{x_1=\pm 1} \sum_{x_2=\pm 1} (x_1 x_2)\Pr[X(t_1)=x_1, X(t_2)=x_2]$$

For the moment, assume $t_2 \geq t_1$,

$$R_{XX}(t_1,t_2) = \sum_{x_1=\pm 1} \sum_{x_2=\pm 1} (x_1 x_2)\Pr[X(t_1)=x_1]\Pr[X(t_2)=x_2|X(t_1)=x_1]$$

$\Pr[X(t_1) = x_1]$ is given by the PMF at the beginning. We need to work out $\Pr[X(t_2) = x_2|X(t_1) = x_1]$, which can be worked out similarly as the PMF

$$\begin{aligned}
\Pr[X(t_2) = 1|X(t_1) = 1] &= \Pr[X(t_2) = -1|X(t_1) = -1] \\
&= \Pr[N(t_2) - N(t_1) \text{ is even}] \\
&= \sum_{k \text{ even}} \frac{(\lambda(t_2 - t_1))^k}{k!} \exp(-\lambda(t_2 - t_1)) = \exp(-\lambda(t_2 - t_1)) \cosh(\lambda(t_2 - t_1)) \\
\Pr[X(t_2) = -1|X(t_1) = 1] &= \Pr[X(t_2) = -1|X(t_1) = 1] \\
&= \Pr[N(t_2) - N(t_1) \text{ is odd}] \\
&= \sum_{k \text{ odd}} \frac{(\lambda(t_2 - t_1))^k}{k!} \exp(-\lambda(t_2 - t_1)) = \exp(-\lambda(t_2 - t_1)) \sinh(\lambda(t_2 - t_1))
\end{aligned}$$

Then, substitute into $R_{XX}(t_1, t_2)$, we get

$$R_{XX}(t_1, t_2) = \exp(-\lambda(t_2 - t_1)), \quad t_2 \geq t_1.$$

For$t_1 \geq t_2$, we get similarly

$$R_{XX}(t_1, t_2) = \exp(-\lambda(t_1 - t_2)), \quad t_1 \geq t_2.$$

Thus,

$$R_{XX}(t_1, t_2) = \exp(-\lambda|t_2 - t_1|).$$

4. **Problem 3.19**

a.

$$\begin{aligned}
R_{XX}(m, n) &= E[X(m)X(n)] = \begin{cases} 1, & m = n \text{ (unit variance)} \\ 1, & m \neq n \text{ (i.i.d. sequence)} \end{cases} \\
&= \delta(m - n)
\end{aligned}$$

$$S_{XX}(f) = \mathcal{F}[R_{XX}(m, n)] = \mathcal{F}[\delta(m - n)] = 1.$$

b. Given that $R_{XX}(m) = \exp(-\alpha|m|)$,

$$\begin{aligned}
S_{XX}(f) &= \mathcal{F}[R_{XX}(m)] = \mathcal{F}[\exp(-\alpha|m|)] = \sum_m \exp(-\alpha|m|) \exp(-j2\pi mf) \\
&= \sum_{m=0}^{\infty} \exp(-(\alpha + j2\pi f)m) + \sum_{m=-\infty}^{0} \exp(-(-\alpha + j2\pi f)m) - 1 \\
&= \frac{1}{1 - \exp(-(\alpha + j2\pi f))} + \frac{1}{1 - \exp(-(\alpha - j2\pi f))} - 1 \\
&= \frac{1 - e^{-2\alpha}}{1 + e^{-2\alpha} - \frac{1}{2} e^{-\alpha} \cos(2\pi f)}
\end{aligned}$$

c. Given

$$R_{XX}(m) = \begin{cases} 1, & m = 0 \\ -\frac{1}{2}, & m = \pm 1 \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$S_{XX}(f) = \mathcal{F}[R_{XX}(m)] = 1 - \frac{1}{2}\exp(-j2\pi f) - \frac{1}{2}\exp(+j2\pi f)$$
$$= 1 - \cos(2\pi f)$$

5. **Problem 3.20**

a. The Power In DC term is

$$P_{dc} = \int_{0^-}^{0^+} S_{XX}(f)df = 100.$$

b.

$$P(-\infty,\infty) = E[X(t)^2] = \int S_{XX}(f)df = 10100.$$

c.

$$P[0,100] = \int_{0^-}^{100^+} S_{XX}(f)df = 100 + 0.5 * (10 + 9) * 100 = 1050.$$

6. **Problem 3.26**

When $Z(t) = X(t)Y(t)$,

$$R_{ZZ}(t_1,t_2) = E[Z(t_1)Z(t_2)] = E[X(t_1)Y(t_1)X(t_2)Y(t_2)]$$
$$= R_{XX}(t_1,t_2) \cdot R_{YY}(t_1,t_2)$$

Then,

$$S_{ZZ}(f) = \mathcal{F}[R_{ZZ}(\tau)] = S_{XX}(f) \otimes S_{YY}(f)$$

$$S_{ZZ}(0) = A\sigma_X^2$$