Department of Electrical and Computer Engineering
ECE 673 - Random Signal Analysis I

Reading

Shanmugan & Breipohl, Chapter 4.1, 4.2, 4.3.

Homework 5

1. **Problem 4.3**

$$\begin{aligned} R_{YX}(k) \otimes h(k) &= \sum_m R_{YX}(k-m)h(m) \\ &= \sum_m E[Y(0)X(k-m)]h(m) \\ &= E[Y(0)\sum_m X(k-m)h(m)] \\ &= E[Y(0)Y(k)] = R_{YY}(k) \end{aligned}$$

2. **Problem 4.4**
   a. The impulse response $h(n)$ is obtained by setting system input to $x(n) = \delta(n)$. Thus,

$$h(n) = \frac{1}{k}\sum_{i=1}^{k} \delta(n-i) \quad \Leftrightarrow \quad H(f) = \frac{1}{k}\sum_{i=1}^{k} e^{-j2\pi f i}$$

   b.

$$R_{XX}(n) = \begin{cases} 1, n = 0 \\ 0, n \neq 0 \end{cases} \qquad \Leftrightarrow \quad S_{XX}(f) = 1$$

$$S_{YY}(f) = S_{XX}(f)|H(f)|^2 = \frac{1}{k^2}\sum_{m=1}^{k} e^{-j2\pi f m}\sum_{n=1}^{k} e^{j2\pi f n} = \frac{1}{k^2}\sum_{m=1}^{k}\sum_{n=1}^{k} e^{-j2\pi f(m-n)}$$

$$E[Y^2(n)] = E[\left(\frac{1}{k}\sum_{a=1}^{k} X(n-a)\right)\left(\frac{1}{k}\sum_{b=1}^{k} X(n-b)\right)] = \frac{1}{k^2}\sum_{a=1}^{k}\sum_{b=1}^{k} R_{XX}(b-a) = \frac{1}{k}$$

3. **Problem 4.7**

a.

$$\begin{aligned}
X(0) &= 1 \\
X(1) &= \sqrt{0}X(0) + U(0) = U(0) \\
X(2) &= \sqrt{1}X(1) + U(1) = U(0) + U(1) \\
X(3) &= \sqrt{2}X(2) + U(2) = \sqrt{2}U(0) + \sqrt{2}U(1) + U(2) \\
&\cdots \\
\Rightarrow \quad \mu_X(n) &= \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}
\end{aligned}$$

b.

$$\begin{aligned}
R_{XX}(0,n) &= E[X(0)X(n)] = E[X(n)] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \\
R_{XX}(1,1) &= E[X(1)X(1)] = E[U(0)U(0)] = \sigma_{U(0)}^2 \\
R_{XX}(1,2) &= E[X(1)X(2)] = E[U(0)(U(0) + U(1))] = \sigma_{U(0)}^2 \\
R_{XX}(1,3) &= E[X(1)X(3)] = E[U(0)(\sqrt{2}U(0) + \sqrt{2}U(1) + U(2))] = \sqrt{2}\sigma_{U(0)}^2
\end{aligned}$$

4. **Problem 4.12**

a.

$$\begin{aligned}
Y(t) &= \frac{1}{T}\int_{t-T}^{t} X(\alpha)d\alpha \\
\Leftrightarrow \quad h(t) &= \frac{1}{T}\int_{t-T}^{t} \delta(\alpha)d\alpha = \begin{cases} \frac{1}{T}, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases} \\
\Leftrightarrow \quad H(f) &= \mathcal{F}[h(t)] = e^{-j2\pi f(T/2)}\text{sinc}(fT)
\end{aligned}$$

b.

$$\begin{aligned}
S_{XX}(f) &= \eta/2 \\
S_{YY}(f) &= S_{XX}(f)|H(f)|^2 = (\eta/2)\text{sinc}^2(fT)
\end{aligned}$$

$$E[Y^2(t)] = \int S_{YY}(f)df = \int (\eta/2)\text{sinc}^2(fT)df = (\eta/2T)$$

5. **Problem 4.14**

a.

$$\begin{aligned}
h(t) &= \begin{cases} \exp(-t), & t \geq 0 \\ 0, & \text{otherwise} \end{cases} \\
\Leftrightarrow \quad H(f) &= \mathcal{F}[h(t)] = \frac{1}{1 + j2\pi f}
\end{aligned}$$

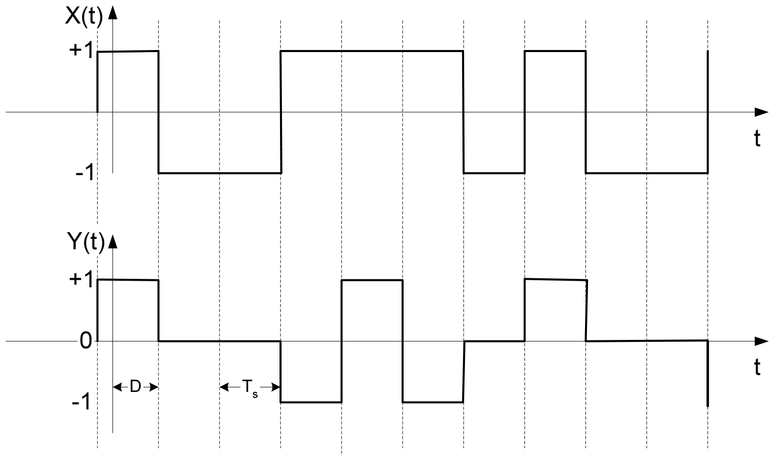
$$S_{YY}(f) = S_{XX}(f)|H(f)|^2 = (\eta/2)\frac{1}{1 + (2\pi f)^2}$$

b.

$$E[Y^2(t)] = \int S_{YY}(f)df = \int (\eta/2)\frac{1}{1 + (2\pi f)^2}df = (\eta/2)\left(\frac{1}{2\pi}\tan^{-1}(2\pi f)\right)_{f=-\infty}^{\infty} = (\eta/4)$$

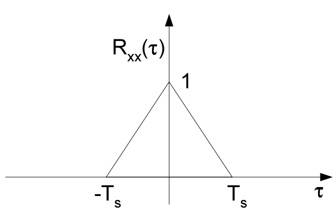
6. **Problem 4.15**

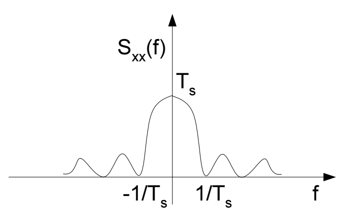
a. The sample function of $X(t)$ and corresponding $Y(t)$ are shown here:



b.

$$
\begin{aligned}
R_{XX}(t_1, t_2) &= E[X(t_1)X(t_2)] = E[\sum_k A(k)p(t_1 - kT_s - D)\sum_m A(m)p(t_2 - mT_s - D)] \\
&= \sum_k \sum_m E[A(k)p(t_1 - kT_s - D)A(m)p(t_2 - mT_s - D)] \\
&= \sum_{k=m} E[A(k)^2]E[p(t_1 - kT_s - D)p(t_2 - kT_s - D)] \\
&= \sum_{k=m} 1 \cdot E[p(t_1 - kT_s - D)p(t_2 - kT_s - D)] \\
&= \begin{cases} 1 - \frac{|t_1 - t_2|}{T_s}, & \tau \in [0, T_s] \\ 0, & \text{otherwise} \end{cases} \\
S_{XX}(f) &= \mathcal{F}[R_{XX}(\tau)] = T_s \mathrm{sinc}^2(fT_s)
\end{aligned}
$$





$$
\begin{aligned}
R_{YY}(t_1,t_2) &= E[Y(t_1)Y(t_2)] = E[\sum_k B(k)p(t_1-kT_s-D)\sum_m B(m)p(t_2-mT_s-D)] \\
&= \sum_k\sum_m E[B(k)B(m)]E[p(t_1-kT_s-D)p(t_2-mT_s-D)] \\
&= \sum_k\sum_m E[E[B(k)B(m)|A(k)A(m)]]E[p(t_1-kT_s-D)p(t_2-mT_s-D)] \\
&= \sum_k\sum_m \frac{1}{4}E[B(k)B(m)|A(k)=1,A(m)=1]E[p(t_1-kT_s-D)p(t_2-mT_s-D)] \\
&= \sum_{(k-m)=0}\frac{1}{4}E[p(t_1-kT_s-D)p(t_2-mT_s-D)] \\
&\quad - \sum_{(k-m)=\pm1}\frac{1}{4}E[p(t_1-kT_s-D)p(t_2-mT_s-D)] \\
&\quad + \underbrace{\sum_{|k-m|>1}\frac{1}{4}E[B(k)B(m)|A(k)=1,A(m)=1]E[p(t_1-kT_s-D)p(t_2-mT_s-D)]}_{0} \\
&= \sum_{(k-m)=0}\frac{1}{4}E[p(t_1-kT_s-D)p(t_2-mT_s-D)] \\
&\quad - \sum_{(k-m)=\pm1}\frac{1}{4}E[p(t_1-kT_s-D)p(t_2-mT_s-D)] \\
&= \frac{1}{4}R_{XX}(t_1-t_2) - \frac{1}{4}R_{XX}(t_1-t_2+T_s) - \frac{1}{4}R_{XX}(t_1-t_2-T_s)
\end{aligned}
$$

$$
S_{YY}(f) = \mathcal{F}[R_{YY}(\tau)] = \frac{1}{4}T_s\mathrm{sinc}^2(fT_s) - \frac{1}{2}T_s\mathrm{sinc}^2(fT_s)\cos(2\pi fT_s)
$$

Notice that the overall power is not reduced, but the DC-level power is reduced.