Department of Electrical and Computer Engineering
ECE 673 - Random Signal Analysis I

Reading

Shanmugan & Breipohl, Chapter 5.1, 5.5, 5.4.

Homework 6

1. **Problem 4.20**
   Given

$$\begin{aligned} S(t) &= A\sin(2\pi f_c t + \Theta), \quad \Theta \sim \text{unif}[-\pi, \pi] \\ X(t) &= S(t) + N(t), \quad S_{NN}(f) = \frac{\eta}{2} \\ H(f) &= \frac{1}{1 + j(f/f_0)} \\ Y(t) &= X(t) \otimes h(t) \end{aligned}$$

a. Find $S_{YY}(f)$.

$$\begin{aligned} R_{SS}(\tau) &= E[S(t)S(t+\tau)] = E[A\sin(2\pi f_c t + \Theta) \cdot A\sin(2\pi f_c(t+\tau) + \Theta)] \\ &= \frac{A^2}{2} E[\cos(2\pi f_c \tau) - \cos(2\pi f_c(2t+\tau) + 2\Theta)] \\ &= \frac{A^2}{2}\cos(2\pi f_c \tau) \\ S_{SS}(f) &= \mathcal{F}[R_{SS}(\tau)] = \frac{A^2}{4}[\delta(f - f_c) + \delta(f + f_c)] \end{aligned}$$

$$\begin{aligned} S_{XX}(f) &= S_{SS}(f) + S_{NN}(f) \\ &= \frac{A^2}{4}[\delta(f - f_c) + \delta(f + f_c)] + \frac{\eta}{2} \end{aligned}$$

$$\begin{aligned} S_{YY}(f) &= S_X X(f)|H(f)|^2 \\ &= \frac{1}{1 + (f/f_0)^2}\{\frac{A^2}{4}[\delta(f - f_c) + \delta(f + f_c)] + \frac{\eta}{2}\} \end{aligned}$$

b. Find Signal-to-Noise Ratio.

$$\begin{aligned}
S &= \int_{-\infty}^{\infty} \frac{1}{1+(f/f_0)^2}\frac{A^2}{4}[\delta(f-f_c)+\delta(f+f_c)]df \\
&= \left[\frac{1}{1+(f_c/f_0)^2}\right]\frac{A^2}{2} \\
N &= \int_{-\infty}^{\infty} \frac{1}{1+(f/f_0)^2}\frac{\eta}{2}df \\
&= \frac{\eta f_0}{2}\tan^{-1}(\frac{f}{f_0})|_{-\infty}^{\infty} = \frac{\eta f_0 \pi}{2} \\
S/N &= \frac{A^2}{\eta\pi[f_0+f_c^2/f_0]}
\end{aligned}$$

c. What f_0 will maximize SNR?

$$\frac{\partial}{\partial f_0}[f_0+f_c^2/f_0]=0 \qquad \Rightarrow \qquad f_0=f_c$$

2. **Problem 5.56**

a. Given $Z(t)=A\cos(2\pi f_c t)+N(t)$ and $N(t)$ is zero-mean Gaussian random process centered around f_c, we can write (as in example 5.10)

$$\begin{aligned}
Z(t) &= A\cos(2\pi f_c t)+N(t) \\
&= A\cos(2\pi f_c t)+N_c(t)\cos(2\pi f_c t)+N_s(t)\sin(2\pi f_c t) \\
&= \sqrt{[A+N_c(t)]^2+[N_s(t)]^2}\cdot\cos\left[2\pi f_c t+\tan^{-1}\left(\frac{N_s(t)}{A+N_c(t)}\right)\right]
\end{aligned}$$

Thus,

$$\begin{cases} R(t)=\sqrt{[A+N_c(t)]^2+[N_s(t)]^2} \\ \Theta(t)=\tan^{-1}\left(\frac{N_s(t)}{A+N_c(t)}\right)\end{cases} \qquad \Leftrightarrow \qquad \begin{cases} N_c(t)=R(t)\cos[\Theta(t)]-A \\ N_s(t)=R(t)\sin[\Theta(t)]\end{cases}$$

Since $f_{N_c,N_s}(n_c,n_s)=\frac{1}{2\pi\sigma^2}\exp\left(-\frac{n_c^2+n_s^2}{2\sigma^2}\right)$, we can apply the Jacobian method

$$\begin{aligned}
J(n_c,n_s) &= \det\begin{vmatrix} \frac{\partial r}{\partial n_c} & \frac{\partial r}{\partial n_s} \\ \frac{\partial \theta}{\partial n_c} & \frac{\partial \theta}{\partial n_s}\end{vmatrix} = -\frac{1}{r} \\
f_{R,\Theta}(r,\theta) &= \frac{f_{N_c,N_s}(n_c,n_s)}{|J(n_c,n_s)|} \\
&= \frac{r}{2\pi\sigma^2}\exp\left(-\frac{[r\cos\theta-A]^2+[r\sin\theta]^2}{2\sigma^2}\right), 0\le r<\infty, 0\le\theta<2\pi
\end{aligned}$$

b.

$$\begin{aligned}
f_R(r) &= \int_0^{2\pi} f_{R,\Theta}(r,\theta)d\theta \\
&= \frac{r}{\sigma^2}\exp\left(-\frac{r^2+A^2}{2\sigma^2}\right)\int_0^{2\pi}\frac{1}{2\pi}\exp\left(-\frac{Ar\cos\theta}{\sigma^2}\right)d\theta \\
&= \frac{r}{\sigma^2}\exp\left(-\frac{r^2+A^2}{2\sigma^2}\right)I_0\left(\frac{Ar}{\sigma^2}\right)
\end{aligned}$$

c. R and Θ are not independent.

3. **Problem 5.57**
Define $X(t) = A\cos(2\pi f_c t + \Theta)$, then $Y(t) = X(t) + N(t)$, where $X(t)$ and $N(t)$ are independent.

$$\begin{aligned} R_{XX}(t, t+\tau) &= E[X(t)X(t+\tau)] = A^2 E[\cos(2\pi f_c t + \Theta)\cos(2\pi f_c(t+\tau) + \Theta)] \\ &= \frac{A^2}{2} E[\cos(2\pi f_c \tau) + \cos(4\pi f_c t + 2\pi f_c \tau + \Theta)] \\ &= \frac{A^2}{2}\cos(2\pi f_c \tau) \end{aligned}$$

Then

$$\begin{aligned} R_{YY}(t, t+\tau) &= E[Y(t)Y(t+\tau)] = E[(X(t) + N(t))(X(t+\tau) + N(t+\tau))] \\ &= E[X(t)X(t+\tau)] + E[N(t)N(t+\tau)] = R_{XX}(t, t+\tau) + R_{NN}(t, t+\tau) \\ S_{YY}(f) &= S_{XX}(f) + S_{NN}(f) \end{aligned}$$