Department of Electrical and Computer Engineering
ECE 673 - Random Signal Analysis I

Reading

Shanmugan & Breipohl, Chapter 5.1, 5.2, 5.3, 5.4.

Homework 7

1. **Problem 5.3**
   For stationarity, we want

$$
\begin{aligned}
& E[X(1)] = E[(1/2)X(0) + e(1)] = \mu_X \\
\Leftrightarrow \quad & \mu_X = 0 \\
\\
& E[X(1)^2] = E[(\frac{1}{2}X(0) + e(1))^2] = \sigma_X^2 \\
\Leftrightarrow \quad & \frac{1}{4}\sigma_X^2 + \sigma_N^2 = \sigma_X^2 \\
\Leftrightarrow \quad & \sigma_X^2 = \frac{4}{3}
\end{aligned}
$$

2. **Problem 5.4**
   (a)

$$\mu_X = 0$$

   (b)

$$\sigma_X^2 = \frac{4}{3}$$

   (c)

$$R_{XX}(k) = \phi_1^k R_{XX}(0) = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^{k-2}$$

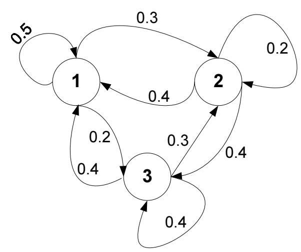
   (d)

$$r_{XX}(k) = \frac{R_{XX}(k)}{R_{XX}(0)} = \left(\frac{1}{2}\right)^k$$

   (e)

$$S_{XX}(f) = |H(f)|^2 S_{ee}(f) = \frac{\sigma_N^2}{1 - 2\phi_1 \cos(2\pi f) + \phi_1^2} = \frac{1}{\frac{5}{4} - \cos(2\pi f)}$$

3. **Problem 5.32**
The state diagram is shown below.



$$\mathbf{p}^T(4) = \mathbf{p}^T(0)A^4 = \begin{bmatrix} 0.3 & 0.3 & 0.4 \end{bmatrix} \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.2 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}^4$$
$$= \begin{bmatrix} 0.4089 & 0.2727 & 0.3183 \end{bmatrix}$$

4. **Problem 5.35**
Using Equation (5.81), it can shown that

$$\mathbf{P}(2) = \mathbf{P}^2(1) = \begin{bmatrix} \frac{r+q(1-r-q)^2}{r+q} & \frac{q-q(1-r-q)^2}{r+q} \\ \frac{r-r(1-r-q)^2}{r+q} & \frac{q+r(1-r-q)^2}{r+q} \end{bmatrix}$$
$$\mathbf{P}(4) = \mathbf{P}^4(1) = \begin{bmatrix} \frac{r+q(1-r-q)^4}{r+q} & \frac{q-q(1-r-q)^4}{r+q} \\ \frac{r-r(1-r-q)^4}{r+q} & \frac{q+r(1-r-q)^4}{r+q} \end{bmatrix}$$
$$\lim_{n\to\infty} \mathbf{P}(n) = \lim_{n\to\infty} \mathbf{P}^n(1) = \begin{bmatrix} \frac{r}{r+q} & \frac{q}{r+q} \\ \frac{r}{r+q} & \frac{q}{r+q} \end{bmatrix}$$
$$\mathbf{p}^T(n) = \mathbf{p}^T(0)\mathbf{P}(n) = \begin{bmatrix} a & 1-a \end{bmatrix} \begin{bmatrix} \frac{r+q(1-r-q)^n}{r+q} & \frac{q-q(1-r-q)^n}{r+q} \\ \frac{r-r(1-r-q)^n}{r+q} & \frac{q+r(1-r-q)^n}{r+q} \end{bmatrix}$$

5. **Problem 5.46**
Define the call arrivals at the telephone lines as Poisson processes $X_1(t), \ldots, X_5(t)$. These Poisson processes are independent and all have arrival rate λ. Since they all start at 8am, we have $X_i(0) = 0$ for $i = 1, \ldots, 5$. Thus, we have

$$\Pr[X_i(t) - X_i(0) = k] = \frac{(\lambda t)^k}{k!} \exp(-\lambda t).$$

Define

$$X(t) = \sum_{i=1}^{5} X_i(t),$$

it can shown that $X(t)$ is Poisson process as well and

$$E[X(t)] = E[\sum_{i=1}^{5} X_i(t)] = 5\lambda t.$$

Thus we can show that

$$\Pr[X(t) = 1] = (5\lambda t) \exp(-5\lambda t).$$

6. **Problem 5.48**

a. 50 jobs per hour means arrival rate is $\lambda_a = 50$. Mean processing time is 1 minute means departure rate is $\lambda_d = 60$. The mean delay (waiting time) is

$$E[W] = \left[\frac{\rho}{1-\rho}\right] E[S] = \left[\frac{\lambda_a/\lambda_d}{1-\lambda_a/\lambda_d}\right]\left[\frac{1}{\lambda_d}\right] = \left[\frac{50/60}{1-50/60}\right]\left[\frac{1}{60}\right] = \left[\frac{5}{60}\right] \text{ hour}$$

b. $\lambda_a = 50$ and $\lambda_d = 120$. Thus

$$E[W] = \left[\frac{50/120}{1-50/120}\right]\left[\frac{1}{120}\right] = \left[\frac{50}{140}\frac{1}{60}\right] \text{ hour}$$

c. $\lambda_a = 100$ and $\lambda_d = 120$. Thus

$$E[W] = \left[\frac{100/120}{1-100/120}\right]\left[\frac{1}{120}\right] = \left[\frac{100}{40}\frac{1}{60}\right] \text{ hour}$$

7. **Problem 5.49**

There are two Poisson processes. One is the random process of passengers arriving, the other is the random process of buses departing. Define the random process for the number of passenger arriving at bus terminal as $X(t)$ with rate $\lambda_a = 120$ per hour, then

$$\Pr[X(t_2) - X(t_1) = k] = \frac{(\lambda_a t_2 - \lambda_a t_1)^k}{k!} \exp[-(\lambda_a t_2 - \lambda_a t_1)].$$

Define the waiting time for the bus departing process as W with rate $\lambda_w = 4$, then

$$f_W(w) = \lambda_w \exp(-\lambda_w w), w \geq 0.$$

a. Average number of passenger per bus is

$$E_W[\ E_X[X(t+W) - X(t)|W]\ ] = E_W[120W] = 120/4 = 30.$$

b. Define the waiting time before 9am as W_1 and the waiting time after as W_2. The average number of passenger on the first bus after 9am is

$$E[E[X(t+W_2) - X(t-W_1)|W_2, W_1]] = E[120(W_2+W_1)] = 120/4 + 120/4 = 60.$$