Department of Electrical and Computer Engineering
ECE 673 - Random Signal Analysis I

Reading

Shanmugan & Breipohl, Chapter 6.1, 6.2.

Homework 8

1. **Problem 6.1**
   Given $P(H_0) = 1/2$ (i.e. $P(H_1) = 1/2$ also) and

$$\begin{aligned} f_{Y|H_0}(y|H_0) &= 1, \quad 0 \le y \le 1 \\ f_{Y|H_1}(y|H_1) &= 2y, \quad 0 \le y \le 1 \end{aligned}$$

a. The MAP decision rule is

$$\begin{aligned} \Pr[H_0|y] &\underset{H_1}{\overset{H_0}{\gtrless}} \Pr[H_1|y] \\ f_{Y|H_0}(y|H_0)\Pr[H_0] &\underset{H_1}{\overset{H_0}{\gtrless}} f_{Y|H_1}(y|H_1)\Pr[H_1] \\ \frac{f_{Y|H_0}(y|H_0)}{f_{Y|H_1}(y|H_1)} &\underset{H_1}{\overset{H_0}{\gtrless}} \frac{\Pr[H_1]}{\Pr[H_0]} \\ \frac{1}{2y} &\underset{H_1}{\overset{H_0}{\gtrless}} 1 \\ \frac{1}{2} &\underset{H_1}{\overset{H_0}{\gtrless}} y \end{aligned}$$

b. The average probability of error is

$$\begin{aligned} \Pr(D_1|H_0) &= \Pr(y \ge \frac{1}{2}|H_0) = \int_{\frac{1}{2}}^{1} f_{Y|H_0}(y|H_0)dy = \int_{\frac{1}{2}}^{1} 1dy = \frac{1}{2} \\ \Pr(D_0|H_1) &= \Pr(y \le \frac{1}{2}|H_1) = \int_{0}^{\frac{1}{2}} f_{Y|H_1}(y|H_1)dy = \int_{0}^{\frac{1}{2}} 2ydy = \frac{1}{4} \\ P_e &= \Pr(H_0)\Pr(D_1|H_0) + \Pr(H_1)\Pr(D_0|H_1) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{3}{8} \end{aligned}$$

2. **Problem 6.2**
A coin is tossed 8 times to decide if the coin is fair or not. The two hypotheses are

$$\begin{aligned} H_0 : &\quad \text{Fair coin}, \Pr[head] = 0.5 \\ H_1 : &\quad \text{Unfair coin}, \Pr[head] = 0.4 \end{aligned}$$

Therefore, the conditional pmfs are

$$\begin{aligned} p(n|H_0) \sim \text{Binomial}(8, 0.5) &= \binom{8}{n} 0.5^n (1-0.5)^{8-n} = \frac{8!}{n!(8-n)!} 0.5^8 \\ p(n|H_1) \sim \text{Binomial}(8, 0.4) &= \binom{8}{n} 0.4^n (1-0.4)^{8-n} = \frac{8!}{n!(8-n)!} 0.4^n \ 0.6^{8-n} \end{aligned}$$

a. Assumeing $\Pr(H_0) = 0.5$ (i.e. $\Pr(H_1) = 0.5$ also), the MAP decision rule is

$$\begin{aligned} \Pr[H_0|n] &\underset{H_1}{\overset{H_0}{\gtrless}} \Pr[H_1|n] \\ p(n|H_0) \Pr[H_0] &\underset{H_1}{\overset{H_0}{\gtrless}} p(n|H_1) \Pr[H_1] \\ \frac{p(n|H_0)}{p(n|H_1)} &\underset{H_1}{\overset{H_0}{\gtrless}} \frac{\Pr[H_1]}{\Pr[H_0]} \\ \frac{0.5^8}{0.4^n \ 0.6^{8-n}} &\underset{H_1}{\overset{H_0}{\gtrless}} 1 \\ \left(\frac{0.5}{0.6}\right)^8 &\underset{H_1}{\overset{H_0}{\gtrless}} \left(\frac{0.4}{0.6}\right)^n \qquad \text{i.e. } n > 3 \text{ choose } H_0, \ n \le 3 \text{ choose } H_1. \end{aligned}$$

b.

$$\begin{aligned} \Pr(D_1|H_0) &= \Pr(n \le 3|H_0) = \sum_{n=0}^{3} p(n|H_0) = \sum_{n=0}^{3} \frac{8!}{n!(8-n)!} 0.5^8 = 0.3633 \\ \Pr(D_0|H_1) &= \Pr(n > 3|H_1) = \sum_{n=4}^{8} p(n|H_1) = \sum_{n=4}^{8} \frac{8!}{n!(8-n)!} 0.4^n \ 0.6^{8-n} = 0.4079 \end{aligned}$$

$$\begin{aligned} P_e &= \Pr(H_0)\Pr(D_1|H_0) + \Pr(H_1)\Pr(D_0|H_1) \\ &= (0.5)(0.3633) + (0.5)(0.4079) = 0.3846 \end{aligned}$$

3. **Problem 6.3**
Given $\Pr(H_0) = 0.999$ (i.e. $\Pr(H_1) = 0.001$) and

$$\begin{aligned} f_{Y|H_0}(y|H_0) &= \frac{y}{N_0} \exp\left(-\frac{y^2}{2N_0}\right), \quad y > 0 \\ f_{Y|H_1}(y|H_1) &\approx \sqrt{\frac{y}{2\pi A N_0}} \exp\left[-\frac{(y-A)^2}{2N_0}\right], \quad y > 0 \end{aligned}$$

The MAP decision rule is

$$
\begin{aligned}
\Pr[H_0|y] &\underset{H_1}{\overset{H_0}{\gtrless}} \Pr[H_1|y] \\
f_{Y|H_0}(y|H_0)\Pr[H_0] &\underset{H_1}{\overset{H_0}{\gtrless}} f_{Y|H_1}(y|H_1)\Pr[H_1] \\
\frac{f_{Y|H_0}(y|H_0)}{f_{Y|H_1}(y|H_1)} &\underset{H_1}{\overset{H_0}{\gtrless}} \frac{\Pr[H_1]}{\Pr[H_0]} \\
\frac{\frac{y}{N_0}\exp\left(-\frac{y^2}{2N_0}\right)}{\sqrt{\frac{y}{2\pi A N_0}}\exp\left[-\frac{(y-A)^2}{2N_0}\right]} &\underset{H_1}{\overset{H_0}{\gtrless}} \frac{0.001}{0.999} \\
\sqrt{\frac{2\pi A y}{N_0}}\exp\left(\frac{A^2-2Ay}{2N_0}\right) &\underset{H_1}{\overset{H_0}{\gtrless}} \frac{1}{999}
\end{aligned}
$$

4. **Problem 6.5**
   The noise N is $\mathcal{N}(0, 1/9)$, which means

$$f_N(n) = \frac{3}{\sqrt{2\pi}}\exp\left(-\frac{9n^2}{2}\right).$$

Given $\Pr(H_0) = \Pr(H_1) = 1/2$ and

$$
\begin{aligned}
H_0: &\quad Y = N \\
H_1: &\quad Y = 2 + N
\end{aligned}
$$

we have

$$
\begin{aligned}
f_{Y|H_0}(y|H_0) &= \frac{3}{\sqrt{2\pi}}\exp\left(-\frac{9n^2}{2}\right), \\
f_{Y|H_1}(y|H_1) &= \frac{3}{\sqrt{2\pi}}\exp\left(-\frac{9(n-2)^2}{2}\right)
\end{aligned}
$$

a. The MAP decision rule is

$$
\begin{aligned}
\Pr[H_0|y] &\underset{H_1}{\overset{H_0}{\gtrless}} \Pr[H_1|y] \\
f_{Y|H_0}(y|H_0)\Pr[H_0] &\underset{H_1}{\overset{H_0}{\gtrless}} f_{Y|H_1}(y|H_1)\Pr[H_1] \\
\frac{f_{Y|H_0}(y|H_0)}{f_{Y|H_1}(y|H_1)} &\underset{H_1}{\overset{H_0}{\gtrless}} \frac{\Pr[H_1]}{\Pr[H_0]} \\
\frac{\frac{3}{\sqrt{2\pi}}\exp\left(-\frac{9n^2}{2}\right)}{\frac{3}{\sqrt{2\pi}}\exp\left(-\frac{9(n-2)^2}{2}\right)} &\underset{H_1}{\overset{H_0}{\gtrless}} 1 \\
1 &\underset{H_1}{\overset{H_0}{\gtrless}} y
\end{aligned}
$$

b. The average probability of error is

$$
\begin{aligned}
\Pr(D_1|H_0) &= \Pr(y \geq 1|H_0) = \int_1^{\infty} f_{Y|H_0}(y|H_0)dy = Q\left(\frac{|0-1|}{1/3}\right) = Q(3) \\
\Pr(D_0|H_1) &= \Pr(y < 1|H_1) = \int_{-\infty}^{1} f_{Y|H_1}(y|H_1)dy = Q\left(\frac{|2-1|}{1/3}\right) = Q(3) \\
P_e &= \Pr(H_0)\Pr(D_1|H_0) + \Pr(H_1)\Pr(D_0|H_1) = \left(\frac{1}{2}\right)Q\left(3\right) + \left(\frac{1}{2}\right)Q\left(3\right) = Q(3)
\end{aligned}
$$