Department of Electrical and Computer Engineering
ECE 673 - Random Signal Analysis I

Reading

Shanmugan & Breipohl, Chapter 6.3, 6.4, 6.5.

Homework 9

1. **Problem 6.8**
   $\Pr(H_0) = 1/3$ and $\Pr(H_1) = 2/3$. Also given $Y|H_1 \sim \mathcal{N}(-1,1)$ and $Y|H_0 \sim \mathcal{N}(1,1)$, we have

$$\begin{aligned} f_{Y|H_0}(y|H_0) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y-1)^2}{2}\right), \\ f_{Y|H_1}(y|H_1) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y+1)^2}{2}\right). \end{aligned}$$

a. The MAP decision rule is

$$\begin{aligned} \Pr[H_0|y] &\underset{H_1}{\overset{H_0}{\gtrless}} \Pr[H_1|y] \\ f_{Y|H_0}(y|H_0)\Pr[H_0] &\underset{H_1}{\overset{H_0}{\gtrless}} f_{Y|H_1}(y|H_1)\Pr[H_1] \\ \frac{f_{Y|H_0}(y|H_0)}{f_{Y|H_1}(y|H_1)} &\underset{H_1}{\overset{H_0}{\gtrless}} \frac{\Pr[H_1]}{\Pr[H_0]} \\ \frac{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(y-1)^2}{2}\right)}{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(y+1)^2}{2}\right)} &\underset{H_1}{\overset{H_0}{\gtrless}} \frac{2/3}{1/3} \\ y &\underset{H_1}{\overset{H_0}{\gtrless}} \frac{\ln(2)}{2} \end{aligned}$$

b. With $C_{00} = C_{11} = 0$, $C_{01} = 6$ and $C_{10} = 1$, the Bayes' decision rule is

$$\begin{aligned} \frac{f_{Y|H_0}(y|H_0)}{f_{Y|H_1}(y|H_1)} &\underset{H_1}{\overset{H_0}{\gtrless}} \frac{\Pr[H_1](C_{01}-C_{11})}{\Pr[H_0](C_{10}-C_{00})} \\ \frac{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(y-1)^2}{2}\right)}{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(y+1)^2}{2}\right)} &\underset{H_1}{\overset{H_0}{\gtrless}} \frac{(2/3)(6-0)}{(1/3)(1-0)} \\ y &\underset{H_1}{\overset{H_0}{\gtrless}} \frac{\ln(12)}{2} \end{aligned}$$

$$\begin{aligned}
\bar{C}_{\min} &= C_{00}\Pr(H_0)\Pr(D_0|H_0) + C_{10}\Pr(H_0)\Pr(D_1|H_0) + C_{01}\Pr(H_1)\Pr(D_0|H_1) + C_{11}\Pr(H_1)\Pr(D_1|H_1) \\
&= (1)(1/3)\Pr(D_1|H_0) + (6)(2/3)\Pr(D_0|H_1) \\
&= (1/3)(1 - Q(\frac{\ln(12)}{2} - 1)) + 4Q(\frac{\ln(12)}{2} + 1) = 0.251
\end{aligned}$$

2. **Problem 6.10**
In Neyman-Pearson decision, the decision threshold is found by fixing $P_F = \alpha$ while minimizing P_M. Therefore, in term of Lagrange multiplier, it is to minimize

$$J = P_M + \lambda(P_F - \alpha), \quad \lambda \geq 0.$$

a.

$$\begin{aligned}
J &= P_M + \lambda(P_F - \alpha) \\
&= \Pr(D_0|H_1) + \lambda[\Pr(D_1|H_0) - \alpha] \\
&= \int_{R_0} f_{Y|H_1}(y|H_1)dy + \lambda[\int_{R_1} f_{Y|H_0}(y|H_0)dy - \alpha] \\
&= \int_{R_0} f_{Y|H_1}(y|H_1)dy + \lambda(1 - \int_{R_0} f_{Y|H_0}(y|H_0)dy) - \lambda\alpha \\
&= \lambda(1-\alpha) + \int_{R_0} [f_{Y|H_1}(y|H_1) - \lambda f_{Y|H_0}(y|H_0)]dy
\end{aligned}$$

b. To minimize J, we want the integrand to be negative ($f_{Y|H_1}(y|H_1) - \lambda f_{Y|H_0}(y|H_0) \leq 0$) in the decision region R_0. Hence, the decision rule is

$$\frac{f_{Y|H_0}(y|H_0)}{f_{Y|H_1}(y|H_1)} \underset{H_1}{\overset{H_0}{\gtrless}} \frac{1}{\lambda}.$$

To calculate λ, use

$$\alpha = P_F = \Pr(D_1|H_0) = \int_{R_1} f_{Y|H_0}(y|H_0)dy$$

to get the threshold γ , then $1/\lambda = \frac{f_{Y|H_0}(\gamma|H_0)}{f_{Y|H_1}(\gamma|H_1)}$.

3. **Problem 6.11**
Given $P_F = 0.001$, and $Y|H_1 \sim \mathcal{N}(-2, 4)$ and $Y|H_0 \sim \mathcal{N}(2, 4)$.
a. To find decision threshold,

$$\begin{aligned}
0.001 &= P_F = \Pr(D_1|H_0) = \int_{R_1} f_{Y|H_0}(y|H_0)dy = Q(\frac{|2-\gamma|}{2}) \\
\Rightarrow \quad \gamma &= -4.2
\end{aligned}$$

b. To find the probability of correct detection $P_D = 1 - P_M$, we have

$$\begin{aligned}
P_D &= \Pr(D_1|H_1) = \int_{R_1} f_{Y|H_1}(y|H_1)dy \\
&= Q(\frac{|-2-\gamma|}{2}) \\
&= Q(1.1) = 0.136
\end{aligned}$$

The probability of correct detection $P_D = 0.136$ is too low if false alarm probability is $P_F = 0.001$.

4. **Problem 6.14**
The following two waveforms are transmitted with equal probability (i.e. $\Pr[s_1(t)] = \Pr[s_0(t)] = 1/2$)

$$\begin{aligned} s_1(t) &= 4\sin(2\pi f_0 t), \quad 0 \le t \le T \\ s_0(t) &= -4\sin(2\pi f_0 t), \quad 0 \le t \le T \end{aligned}$$

where $T = 1\ ms$ and $f_0 = 10/T$. The noise process $N(t)$ is AWGN with power spectral density $S_{NN}(f) = 10^{-3} W/Hz$.

Reformulating the above problem in familiar notation, we get the following two hypotheses:

$$\begin{aligned} H_1: &\quad Y(t) = s_1(t) + N(t), \quad \Pr[s_1(t)] = 1/2 \\ H_0: &\quad Y(t) = s_0(t) + N(t), \quad \Pr[s_0(t)] = 1/2 \end{aligned}$$

a. The MAP decision rule is $\Pr[H_0|y(t)] \underset{H_1}{\overset{H_0}{\gtrless}} \Pr[H_1|y(t)]$. However, to compare the likelihood of waveform, we need to form a single decision variable $Z = \int_0^T Y(t)[s_1(t) - s_0(t)]dt$. Then, the orginal hypotheses can be rewritten as

$$\begin{aligned} H_1: &\quad Z = \int_0^T [s_1(t) + N(t)][s_1(t) - s_0(t)] = 16T + N, \quad \Pr[s_1(t)] = 1/2 \\ H_0: &\quad Z = \int_0^T [s_0(t) + N(t)][s_1(t) - s_0(t)] = -16T + N, \quad \Pr[s_0(t)] = 1/2 \end{aligned}$$

where $N = \int_0^T N(t)[s_1(t) - s_0(t)]dt \sim \mathcal{N}(0, 16 \times 10^{-3} T)$. The MAP decision rule is now transformed to

$$\begin{aligned} \Pr[H_0|Z] &\underset{H_1}{\overset{H_0}{\gtrless}} \Pr[H_1|Z] \\ \frac{f_{Z|H_0}(z|H_0)}{f_{Z|H_1}(z|H_1)} &\underset{H_1}{\overset{H_0}{\gtrless}} \frac{\Pr[H_1]}{\Pr[H_0]} \\ 0 &\underset{H_1}{\overset{H_0}{\gtrless}} z \end{aligned}$$

b. The error probability is

$$\begin{aligned} P_e &= \Pr(H_0)\Pr(D_1|H_0) + \Pr(H_1)\Pr(D_0|H_1) \\ &= (0.5)Q\left(\frac{|16T|}{\sqrt{16 \times 10^{-3} T}}\right) + (0.5)Q\left(\frac{|16T|}{\sqrt{16 \times 10^{-3} T}}\right) = Q(2) = 0.0228. \end{aligned}$$

5. **Problem 6.17**
$\vec{Y} = [Y_1, Y_2]^T$ is bivariate Guassian conditioning on H_0 or H_1:

$$\begin{aligned} H_1: &\quad \vec{Y} \sim \mathcal{N}\left(\begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right), \quad \Pr[H_1] = 1/2 \\ H_0: &\quad \vec{Y} \sim \mathcal{N}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right), \quad \Pr[H_0] = 1/2 \end{aligned}$$

Define $\mu_1 = E[\vec{Y}|H_1] = [4,4]^T$, $\mu_0 = E[\vec{Y}|H_0] = [1,1]^T$, and $\Sigma = \mathrm{Var}[\vec{Y}|H_1] = \mathrm{Var}[\vec{Y}|H_0] = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

a. The MAP decision rule is

$$
\begin{aligned}
\Pr[H_0|\vec{y}] &\underset{H_1}{\overset{H_0}{\gtrless}} \Pr[H_1|\vec{y}] \\
\frac{f_{\vec{Y}|H_0}(\vec{y}|H_0)}{f_{\vec{Y}|H_1}(\vec{y}|H_1)} &\underset{H_1}{\overset{H_0}{\gtrless}} \frac{\Pr[H_1]}{\Pr[H_0]} \\
\frac{\frac{1}{2\pi\det(\Sigma)}\exp\left(-\frac{(\vec{y}-\mu_0)^T\Sigma^{-1}(\vec{y}-\mu_0)}{2}\right)}{\frac{1}{2\pi\det(\Sigma)}\exp\left(-\frac{(\vec{y}-\mu_1)^T\Sigma^{-1}(\vec{y}-\mu_1)}{2}\right)} &\underset{H_1}{\overset{H_0}{\gtrless}} 1 \\
5 &\underset{H_1}{\overset{H_0}{\gtrless}} y_1 + y_2
\end{aligned}
$$

b. Define $Z = Y_1 + Y_2$, then Z is conditional Gaussian. $E[Z|H_0] = 2$, $E[Z|H_1] = 8$ and $\mathrm{Var}[Z|H_0] = \mathrm{Var}[Z|H_1] = E[(Y_1 - 1 + Y_2 - 1)^2|H_0] = 6$. Thus, the error probability is

$$
\begin{aligned}
P_e &= \Pr(H_0)\Pr(D_1|H_0) + \Pr(H_1)\Pr(D_0|H_1) \\
&= (0.5)Q\left(\frac{|2-5|}{\sqrt{6}}\right) + (0.5)Q\left(\frac{|8-5|}{\sqrt{6}}\right) = Q(\sqrt{1.5}) = 0.1151
\end{aligned}
$$