Department of Electrical and Computer Engineering
ECE 673 - Random Signal Analysis I

Reading

Shanmugan & Breipohl, Chapter 7.1, 7.2, 7.3.

Homework 10

1. **Problem 7.2**
   Given X is an observation of signal S plus noise N, we have $X = S + N$, where $S \sim \mathcal{N}(1,1)$ and $N \sim \mathcal{N}(0,1)$. Thus $(X|S = s) \sim \mathcal{N}(s,1)$. Also, since S and N are independent, $X \sim \mathcal{N}(1,2)$ Thus, we can write

$$\begin{aligned} f_{S|X}(s|x) &= \frac{f_{X|S}(x|s)f_S(s)}{f_X(x)} \\ &= \frac{\frac{1}{\sqrt{2\pi}}\exp\left(-(x-s)^2/2\right) \cdot \frac{1}{\sqrt{2\pi}}\exp\left(-(s-1)^2/2\right)}{\frac{1}{\sqrt{2\pi 2}}\exp\left(-(x-1)^2/4\right)} \\ &= \frac{1}{\sqrt{\pi}}\exp\left(-(s-\frac{x+1}{2})^2\right) \end{aligned}$$

   Thus, we have $(S|X = x) \sim \mathcal{N}(\frac{x+1}{2}, \frac{1}{2})$. The Bayes' estimator is $E[S|X = x] = \frac{x+1}{2}$. For $X = 2$, the estimation is $3/2$.

2. **Problem 7.5**
   Given $Y = mX + N$, where X and N are independent, we have

$$\begin{aligned} \mu_Y &= m\mu_X + \mu_N \\ \sigma_Y^2 &= m^2\sigma_X^2 + \sigma_N^2 \\ \sigma_{XY} &= E[(X-\mu_X)(Y-\mu_Y)] = m\sigma_X^2 \end{aligned}$$

   Then, the best linear estimator is

$$\begin{aligned} \hat{X} &= a + bY \\ b &= \frac{\sigma_{XY}}{\sigma_Y^2} = \frac{m\sigma_X^2}{m^2\sigma_X^2 + \sigma_N^2} \\ a &= \mu_X - b\mu_Y = \mu_X - \frac{m\sigma_X^2}{m^2\sigma_X^2 + \sigma_N^2}(m\mu_X + \mu_N) \end{aligned}$$

3. **Problem 7.6**
   Since X and Y are jointly normal, we can write the joint Gaussian random vector as $\vec{V} = [Y, X]^T$ and

$$\vec{V} \sim \mathcal{N}\left(\begin{bmatrix} \mu_Y \\ \mu_X \end{bmatrix}, \begin{bmatrix} \sigma_Y^2 & \sigma_{YX} \\ \sigma_{XY} & \sigma_X^2 \end{bmatrix}\right)$$

Then, applying (2.66.a), we get

$$E[Y|X=x] = \mu_Y + \frac{\sigma_{XY}}{\sigma_X^2}(x-\mu_X) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X).$$

The MSE of the estimation is give as $E[(Y-a-bX)^2]$, applying (2.66.b), we get

$$E[(Y-a-bX)^2] = \sigma_Y^2 - \frac{\sigma_{XY}\sigma_{YX}}{\sigma_X^2} = \sigma_Y^2(1-\rho^2).$$

4. **Problem 7.10**
First since $f_X(x) = 1/2, -1 \le x \le 1$, we have $E[X] = 0$, $E[X^2] = 1/3$, $E[X^3] = 0$, $E[X^4] = 1/5$.
Applying the principle of orthogonality, we have

$$\begin{cases} E[(S-\hat{S})] &= 0 \\ E[(S-\hat{S})X] &= 0 \\ E[(S-\hat{S})X^2] &= 0 \end{cases} \Rightarrow \begin{cases} E[(X^2-h_0-h_1X-h_2X^2)] &= 0 \\ E[(X^2-h_0-h_1X-h_2X^2)X] &= 0 \\ E[(X^2-h_0-h_1X-h_2X^2)X^2] &= 0 \end{cases} \Rightarrow \begin{cases} \frac{1}{3}-h_0-\frac{1}{3}h_2 = 0 \\ -\frac{1}{3}h_1 = 0 \\ \frac{1}{5}-\frac{1}{3}h_0-\frac{1}{5}h_2 = 0 \end{cases}$$

Therefore $\hat{S} = X^2$, and $E[(S-\hat{S})^2] = 0$.

5. **Problem 7.11**
The innovation can be written as

$$\begin{aligned} V_X(1) &= X(1) \\ V_X(2) &= X(2) - \rho\frac{\sigma_2}{\sigma_1}X(1) \end{aligned}$$

The estimation based on innovation is $\hat{S}_2 = h_1V_X(1) + h_2V_X(2)$, where

$$\begin{aligned} h_1 &= \frac{E[SV_X(1)]}{E[V_X^2(1)]} = \frac{\sigma_{SX_1}}{\sigma_{X_1}^2} \\ h_2 &= \frac{E[SV_X(2)]}{E[V_X^2(2)]} = \frac{\sigma_{SX_2} - \frac{\sigma_{X_1X_2}}{\sigma_{X_1}^2}\sigma_{SX_1}}{\sigma_{X_2}^2 - 2\frac{\sigma_{X_1X_2}}{\sigma_{X_1}^2}\sigma_{X_1X_2} + \frac{\sigma_{X_1X_2}}{\sigma_{X_1}^2}\sigma_{X_1}^2} \end{aligned}$$

Plug in the corresponding $V_X(1), V_X(2), h_1, h_2$ into $\hat{S}_2$, we can find $\hat{S}_2 = \bar{h}_1X(1) + \bar{h}_2X(2)$. And it can be shown that

$$\begin{bmatrix} \bar{h}_1 \\ \bar{h}_2 \end{bmatrix} = \begin{bmatrix} \sigma_{X_1}^2 & \sigma_{X_1X_2} \\ \sigma_{X_1X_2} & \sigma_{X_1}^2\bar{h}_2 \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{SX_1} \\ \sigma_{SX_2} \end{bmatrix}$$

This shows that the innovation estimator is optimal as Linear MMSE estimator.

6. **Problem 7.14**
The innovation process can be written as

$$\begin{aligned} V_X(1) &= X(1) \\ V_X(2) &= X(2) - E[X(2)|V_X(1)] = X(2) - \frac{\sigma_{X_2V_X(1)}}{\sigma_{V_X(1)}^2}V_X(1) \\ V_X(3) &= X(3) - E[X(3)|V_X(1), V_X(2)] = X(3) - \frac{\sigma_{X_3V_X(1)}}{\sigma_{V_X(1)}^2}V_X(1) - \frac{\sigma_{X_3V_X(2)}}{\sigma_{V_X(2)}^2}V_X(2) \end{aligned}$$

Also, since $\vec{V}_X = \mathbf{\Gamma}\vec{X}$, we can write

$$\begin{aligned}
V_X(1) &= \gamma_{11}X(1) \\
V_X(2) &= \gamma_{21}X(1) + \gamma_{22}X(2) \\
V_X(3) &= \gamma_{31}X(1) + \gamma_{32}X(2) + \gamma_{33}X(3)
\end{aligned}$$

Compare the two sets of equation, we get $\gamma_{11} = \gamma_{22} = \gamma_{33} = 1$ and

$$\begin{aligned}
\gamma_{21} &= -\frac{\sigma_{X_2 V_X(1)}}{\sigma^2_{V_X(1)}} = -0.5 \\
\gamma_{32} &= -\frac{\sigma_{X_3 V_X(2)}}{\sigma^2_{V_X(2)}} = -\frac{1 - 1/4}{3 + 2/4 - 1} = -0.3 \\
\gamma_{31} &= \cdots \quad = -0.1
\end{aligned}$$

Thus,

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ -0.1 & -0.3 & 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{L} = \mathbf{\Gamma}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.25 & 0.3 & 1 \end{bmatrix}$$