$A \xrightarrow{k_1} X$ $\qquad \dfrac{\partial X}{\partial t} = k_1 A + k_2 X^2 Y - k_3 B X - k_4 X$

$2X + Y \xrightarrow{k_2} 3X$

$B + X \xrightarrow{k_3} Y + D$ $\qquad \dfrac{\partial Y}{\partial t} = - k_2 X^2 Y + k_3 B X$

$X \xrightarrow{k_4} E$

$2X + Y \xrightarrow{k_2} 3X$

the net rate of increase in X is $k_2 [X]^2 [Y]$
(from $2X$ to $3X$)

Add diffusion to take into account of mass conservation

$$\frac{\partial X}{\partial t} = k_1 A + k_2 X^2 Y - k_3 B X - k_4 X + D_X \frac{\partial^2 X}{\partial r^2}$$

$$\frac{\partial Y}{\partial t} = - k_2 X^2 Y + k_3 B X + D_Y \frac{\partial^2 Y}{\partial r^2}$$

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note that the over-all reaction is

$$A + B \rightarrow D + E$$

$\therefore$ constant A B $D E$
concentration for

Steady, uniform state : $\frac{\partial X}{\partial t} = 0 = \frac{\partial Y}{\partial t}$

$$\frac{\partial^2 X}{\partial r^2} = 0 = \frac{\partial^2 Y}{\partial r^2}$$

$$0 = k_1 A + k_2 X_0^2 Y_0 - k_3 B X_0 - k_4 X_0 + 0$$

$$0 = -k_2 X_0^2 Y_0 + k_3 B X_0 + 0$$

$$-k_2 X_0 \left(X_0 Y_0 - \frac{k_3 B}{k_2} \right) = 0$$

$$X_0 \cdot Y_0 = \frac{k_3}{k_2} B$$

$$0 = k_1 A + k_2 \cdot \frac{k_3}{k_2} \cdot B \cdot X_0 - k_3 B X_0 - k_4 X_0$$

$$0 = k_1 A - k_4 X_0$$

$$X_0 = \frac{k_1}{k_4} \cdot A \qquad \therefore \quad Y_0 = \frac{k_3}{k_2} B \Big/ \frac{k_1}{k_4} A$$

$$Y_0 = \frac{k_3 k_4}{k_1 k_2} \cdot \frac{B}{A}$$

scaling : $x = \frac{X}{X_0}$ $\qquad \tau = k_1 t$

$$y = \frac{Y}{Y_0} \qquad S = \frac{r}{\sqrt{D_x / k_1}} = \frac{r}{\sqrt{D / k_1}} \qquad D_x = D = D_y$$

$$X_0^2 Y_0 = \frac{k_1^2}{k_4^2} A^2 \cdot \frac{k_3 \cancel{k_4}}{\cancel{k_1} k_2} \frac{B}{A} = \frac{k_1 k_3}{k_2 k_4} \cdot AB$$

$$\left\{\begin{aligned} \frac{X_0}{\frac{1}{k_1}}\frac{\partial x}{\partial \tau} &= k_1 A + k_2 \cdot X_0^2 Y_0\, x^2 y - k_3 B X_0 x - k_4 X_0 x + \frac{D \cdot X_0}{D/k_1}\frac{\partial^2 x}{\partial s^2} \\ \frac{Y_0}{\frac{1}{k_1}}\frac{\partial y}{\partial \tau} &= -k_2 X_0^2 Y_0\, x^2 y + k_3 B X_0 x + \frac{D Y_0}{D/k_1}\frac{\partial^2 y}{\partial s^2} \end{aligned}\right.$$

$$k_1 X_0 \frac{\partial x}{\partial \tau} = k_1 A + k_2 \cdot \frac{k_1 k_3}{k_2 k_4} AB\, xy - k_3 B \cdot \frac{k_1}{k_4} A x - k_4 \cdot \frac{k_1}{k_4} A x + k_1 \frac{k_1}{k_4} A x''$$

$$k_1 \cdot \frac{k_1}{k_4} A \frac{\partial x}{\partial \tau} = k_1 A + \frac{k_1 k_3}{k_4} AB\, xy - \frac{k_1 k_3}{k_4} AB x - k_1 A x + \frac{k_1^2}{k_4} A \frac{\partial^2 x}{\partial s^2}$$

$$\frac{k_1^2}{k_4} A \frac{\partial x}{\partial \tau} = k_1 A (1 - x) + \frac{k_1 k_3}{k_4} AB (xy - x) + \frac{k_1^2}{k_4} A \frac{\partial^2 x}{\partial s^2}$$

$$\boxed{\frac{\partial x}{\partial \tau} = \frac{k_4}{k_1}(1 - x) + \frac{k_3}{k_1} B (x^2 y - x) + \frac{\partial^2 x}{\partial s^2}}$$

$$k_1 \frac{k_3 k_4}{k_1 k_2} \frac{B}{A} \frac{\partial y}{\partial \tau} = -k_2 \cdot \left(\frac{k_1}{k_4} A\right)^2 \cdot \frac{k_3 k_4}{k_1 k_2} \frac{B}{A}\, xy + k_3 \cdot B \cdot \frac{k_1}{k_4} A x + k_1 \cdot \frac{k_3 k_4}{k_1 k_2} \frac{B}{A} \frac{\partial^2 y}{\partial s^2}$$

$$\frac{k_3 k_4}{k_2} \frac{B}{A} \frac{\partial y}{\partial \tau} = -\frac{k_3 k_1}{k_4} AB\, xy + \frac{k_1 k_3}{k_4} AB x + \frac{k_3 k_4}{k_2} \frac{B}{A} \frac{\partial^2 y}{\partial s^2}$$

$$\frac{k_3 k_4}{k_2} \frac{B}{A} \frac{\partial y}{\partial \tau} = -\frac{k_1 k_3}{k_4} AB (xy - 1) + \frac{k_3 k_4}{k_2} \frac{B}{A} \frac{\partial^2 y}{\partial s^2}$$

$$\boxed{\frac{\partial y}{\partial \tau} = \frac{-k_1 k_2}{k_4^2} A^2 (x^2 y - 1) + \frac{\partial^2 y}{\partial s^2}}$$

$K = \frac{k_4}{k_1}$, $\beta \equiv B \cdot \frac{k_3}{k_1}$, $\gamma \equiv \frac{k_1 k_2}{k_4^2} A^2$

$$\begin{cases} \frac{\partial x}{\partial \tau} = \kappa (1-x) + \beta (x^2 y - x) + \frac{\partial^2 x}{\partial s^2} \\ \frac{\partial y}{\partial \tau} = - \gamma x^2 y + \gamma x + \frac{\partial^2 y}{\partial s^2} \end{cases}$$

Steady, homogeneous (uniform) soln. : $x = 1$, $y = 1$

$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$ perturbation of the uniform soln.

$$x = 1 + \bar{x} = 1 + \hat{x} \cos \mu s \cdot e^{\sigma \tau}$$
$$y = 1 + \bar{y} = 1 + \hat{y} \cos \mu s \cdot e^{\sigma \tau}$$

$$\frac{\partial \bar{x}}{\partial \tau} = - K \bar{x} + \beta (1+\bar{x}) \left[(1+\bar{x})(1+\bar{y}) - 1 \right] + \frac{\partial^2 \bar{x}}{\partial s^2}$$

$$= - K \bar{x} + \beta (1+\bar{x}) (\bar{x} + \bar{y} + \bar{x}\bar{y}) + \frac{\partial^2 \bar{x}}{\partial s^2}$$

$$\boxed{\frac{\partial \bar{x}}{\partial \tau} = - K \bar{x} + \beta (\bar{x} + \bar{y}) + \frac{\partial^2 \bar{x}}{\partial s^2}}$$

$$\frac{\partial \bar{y}}{\partial \tau} = - \gamma (1+\bar{x}) \left[(1+\bar{x})(1+\bar{y}) - 1 \right] + \frac{\partial^2 \bar{y}}{\partial s^2}$$

$$\frac{\partial \bar{y}}{\partial \tau} = - \gamma (1+\bar{x}) (\bar{x} + \bar{y} + \bar{x}\bar{y}) + \frac{\partial^2 \bar{y}}{\partial s^2}$$

$$\boxed{\frac{\partial \bar{y}}{\partial \tau} = - \gamma (\bar{x} + \bar{y}) + \frac{\partial^2 \bar{y}}{\partial s^2}}$$

$$\bar{x} = \hat{x}\cos\mu s \cdot e^{\sigma\tau} \qquad \frac{\partial \bar{x}}{\partial \tau} = \sigma\bar{x}, \qquad \frac{\partial^2 \bar{x}}{\partial s^2} = -\mu^2\bar{x}$$

$$\bar{y} = \hat{y}\cos\mu s\, e^{\sigma\tau} \qquad \frac{\partial \bar{y}}{\partial \tau} = \sigma\bar{y}, \qquad \frac{\partial^2 \bar{y}}{\partial s^2} = -\mu^2\bar{y}$$

$$\sigma\bar{x} = -\kappa\bar{x} + \beta\bar{x} + \beta\bar{y} - \mu^2\bar{x}$$

$$\sigma\bar{x} = (-\kappa + \beta - \mu^2)\bar{x} + \beta\bar{y}$$

$$\sigma\bar{y} = -\gamma(\bar{x}+\bar{y}) - \mu^2\bar{y}$$

$$\sigma\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}\beta-\kappa-\mu^2 & \beta \\ -\gamma & -\gamma-\mu^2\end{pmatrix}\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

$$0 = \begin{pmatrix}\beta-\kappa-\mu^2-\sigma & \beta \\ -\gamma & -\gamma-\mu^2-\sigma\end{pmatrix}\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

$$\det\begin{pmatrix}\beta-\kappa-\mu^2-\sigma & \beta \\ -\gamma & -\cancel{\beta}\gamma-\mu^2-\sigma\end{pmatrix} = 0$$ for non-trivial soln.

$$(\sigma-\beta+\kappa+\mu^2)(\sigma+\gamma+\mu^2)+\beta\gamma = 0$$

$$\sigma^2 + \underline{(-\beta+\kappa+\mu^2+\gamma+\mu^2)\sigma} + (-\beta+\kappa+\mu^2)(\gamma+\mu^2) + \beta\gamma = 0$$

$$b = -\beta+\kappa+\gamma+2\mu^2 \quad > 0$$

$$c = -\beta\mu^2 + (\kappa+\mu^2)(\gamma+\mu^2) \quad > 0$$

for stability

$$\sigma = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

if $b > 0$, $c > 0$ real(σ) < 0 $\Rightarrow$ perturbation dies out stability

$$-\beta\mu^2 + (K+\mu^2)(\gamma+\mu^2) > 0$$

$$(K+\mu^2)(\gamma+\mu^2)/\mu^2 > \beta$$

$$\|$$

$$\beta_c(\mu^2) = (K+\mu^2)(\gamma/\mu^2 + 1)$$

$C = 0$ $\quad -\beta\mu^2 + (K+\mu^2)(\gamma+\mu^2) = 0$

$$\beta_c = (K+\mu^2)(\gamma/\mu^2 + 1)$$

$$-\beta_c + (K + \gamma + 2\mu^2) > 0$$

$$(K+\mu^2) + (\gamma+\mu^2) > (K+\mu^2)(\gamma/\mu^2 + 1)$$

$$1 + \gamma + \mu^2 > \frac{\gamma}{\mu^2} + 1$$

$$\gamma + \mu^2 > \frac{\gamma}{\mu^2}$$

$$\gamma\mu^2 + \mu^4 > \gamma$$

$$\mu^4 + \gamma\mu^2 - \gamma > 0$$

$$\mu^2 = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\gamma}}{2}$$

$$\mu = \frac{-\gamma + \sqrt{\gamma^2 + 4\gamma}}{2}$$