$$\varepsilon \cdot \frac{dx}{dt} = y - f(x) \qquad 0 < \varepsilon << 1$$

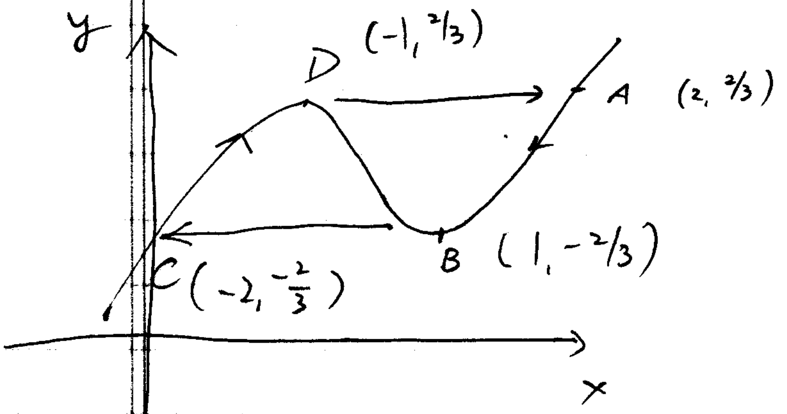
$$\frac{dy}{dt} = -x \qquad f(x) = \frac{1}{3}x^3 - x$$

$$y = f(x) \qquad \frac{dy}{dt} = f' \cdot \frac{dx}{dt} = -x$$

$$f' = x^2 - 1$$

$$(x^2 - 1)\frac{dx}{dt} = -x$$

$$-\left(x - \frac{1}{x}\right)dx = +\,dt$$



max : $x = 1,\ y = -\frac{2}{3}$

$x = -1,\ y = \frac{2}{3}$

$\cancel{-\frac{2}{3} = \frac{1}{3}x^3 - x}$

$\cancel{x^3 - 3x + 2 = 0}$

$\cancel{x^3 - 2x - x + 2 = 0}$

$\cancel{x^2(x^2 - 2)}$

$$-\frac{2}{3} = \frac{1}{3}x^3 - x$$
$$-2 = x^3 - 3x$$
$$x^3 - 3x + 2 = 0$$
$$x^3 - x - 2x + 2 = 0$$
$$x(x^2 - 1) - 2(x - 1) = 0$$
$$(x-1)[x(x+1) - 2] = 0$$
$$(x-1)(x+2)(x-1) = 0$$
$$x = 1, -2$$

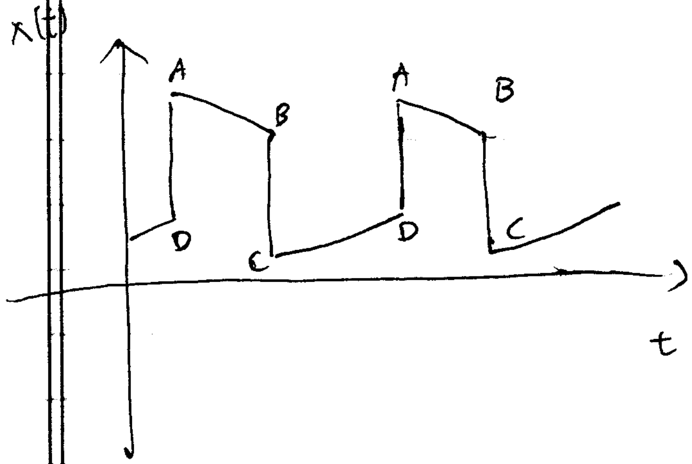
$$2 = x^3 - 3x$$
$$x^3 - 3x - 2 = 0$$
$$x^3 + 1 - 3(x+1) = 0$$
$$(x+1)(x^2 - x + 1 - 3) = 0$$
$$(x+1)(x^2 - x - 2) = 0$$
$$(x+1)(x+2)(x-1) = 0$$
$$x = 1, -1, -2$$

$$\int_2^1 \left(x + \frac{1}{x}\right) dx = -\int_0^{T/2} dt$$

$$T = 3 - 2\ln 2$$

(0) determine the maxima & minima on the nullcline

(1) from A ~~to~~ and C for $x(t)$



M451

$$\varepsilon \cdot \frac{dx}{dt} = qy - xy + x(1-x) \qquad \varepsilon << \delta$$

$$\delta \frac{dy}{dt} = -qy - xy + 2fz$$

$$\frac{dz}{dt} = x - z$$

by setting $\varepsilon \frac{dx}{dt} \approx 0$

$$qy - xy + x(1-x) = 0$$

$$-x^2 + (1-y)x + qy = 0$$

$$x = \frac{-(1-y) \pm \sqrt{(1-y)^2 + 4qy}}{-2} \Rightarrow \frac{(1-y) \pm \sqrt{(1-y)^2 + 4qy}}{2}$$

only the positive root makes physical sense

$$x(y) = \frac{1}{2}\left[(1-y) + \sqrt{(1-y)^2 + 4qy}\right] \quad q << 1$$

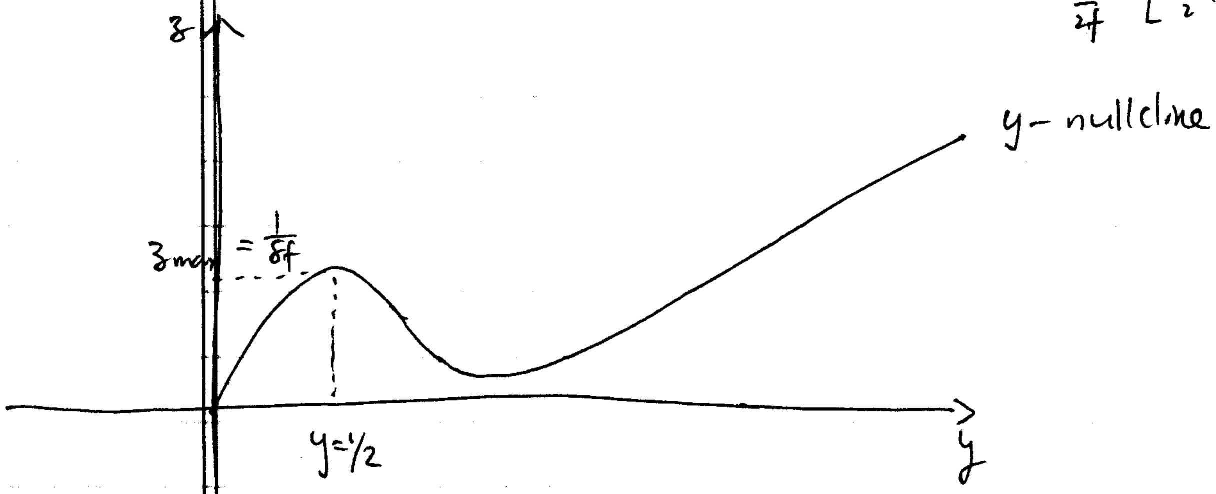
z nullcline $\Rightarrow$ $z = x(y) \approx \begin{cases} 1-y & \text{for } q << 1-y \leq 1 \\ \frac{qy}{y-1} & q << y-1 \end{cases}$

y nullcline $\Rightarrow$ $\cancel{y = \frac{x(1-x)}{q-x}}$ $z = \frac{y(q+x)}{2f}$

$$z = \frac{y(q+x)}{2f} \approx \begin{cases} \dfrac{y(1-y)}{2f} & q \ll 1-y < 1 \\[2ex] \dfrac{y\cdot\left(\frac{qy}{y-1}+q\right)}{2f} & q \ll y-1 \\[2ex] \dfrac{yq}{2f} & 1 \ll y \end{cases}$$

$$z = \frac{y\left(q + \frac{1}{2}\left[(1-y)+\sqrt{(1-y)^2+4qy}\,\right]\right)}{2f}$$

$$\frac{dz}{dy} = \frac{q + \frac{1}{2}\left[(1-y)+\sqrt{(1-y)^2+4qy}\,\right]}{2f} + \frac{y}{2f}\cdot\left[\frac{1}{2}\left(-1 + \frac{2(1-y)+4q}{2\sqrt{(1-y)^2+4qy}}\right)\right]$$



$$\frac{dz}{dy} = \frac{1}{2f}(x(y)+q) + \frac{y}{2f}\frac{dx}{dy}$$

$$\frac{dx}{dy} = \frac{1}{2}\left\{(-1) + \frac{1}{2}\cdot\frac{-2(1-y)+4q}{\sqrt{(1-y)^2+4qy}}\right\}$$

$$\frac{dz}{dy} = \frac{1}{2f}\left[\, x(y)+q + \frac{y}{2}\cdot\left\{(-1) + \frac{-2(1-y)+4q}{2\sqrt{(1-y)^2+4qy}}\right\}\right]$$

$$= \frac{1}{2f}\left[(1-y) + \left[(1-y)^2+4qy\right]^{1/2} + q + \frac{y}{2}\left\{(-1) + \frac{-2(1-y)+4q}{2\sqrt{(1-y)^2+4qy}}\right\}\right]$$

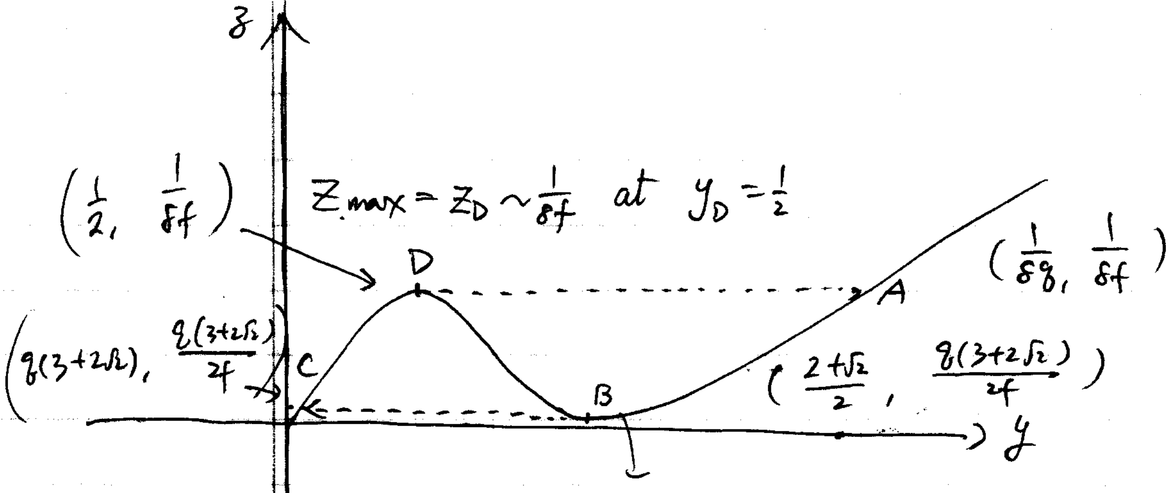
$$z = \frac{y}{2f}\left[\frac{gy}{y-1} + g\right]$$

$$\frac{dz}{dy} = \left[\frac{gy}{y-1} + g + y\cdot\left(\frac{g}{y-1} - \frac{gy}{(y-1)^2}\right)\right]\frac{1}{2f}$$

$$= \frac{gy(y-1) + g(y-1)^2 + gy(y-1) - gy^2}{2f(y-1)^2} = \left\{\frac{g}{2f(y-1)^2}\right\}(2y^2 - 4y + 1)$$

$$2y^2 - 4y + 1 = 0 \qquad y_{min} = \frac{2+\sqrt{2}}{2}$$

$$Z_{min} = \frac{g}{2f}(1+\sqrt{2})^2 = \frac{g(3+2\sqrt{2})}{2f}$$



$$Z_{min} = Z_B \sim \frac{g(3+2\sqrt{2})}{2f} \quad \text{at} \quad y_B = \frac{2+\sqrt{2}}{2}$$

$$Z_c = Z_{min} = \cancel{\frac{gy_c}{y_c-1}} \; \frac{y_c(1-y_c)}{2f}$$

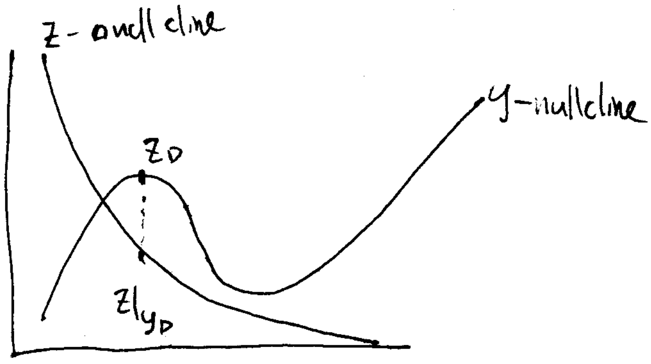
$$y_c^2 - y_c + 2f z_c = 0$$

$$y_c = \frac{1 \pm \sqrt{1 - 8f z_c}}{2} \quad \text{choose the "$-$" soln.}$$

$$y_c = \frac{1}{2}\left(1 - \sqrt{8f z_c}\right) \approx g(3+2\sqrt{2})$$

$$\frac{q\, y_A}{f} = z_{max} = \frac{1}{8f}$$

$$y_A = \frac{1}{8q}$$

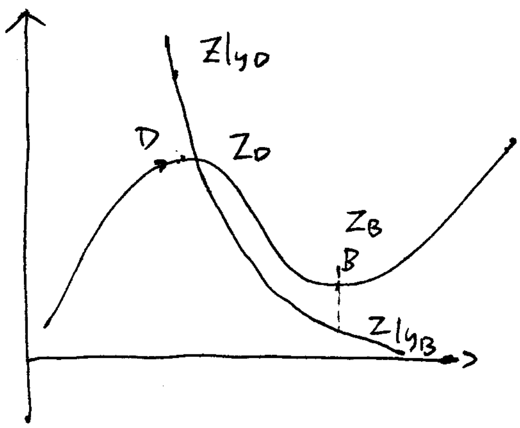


Case I : on the z null cline

$$z|_{y_D} < z_D$$

$$1 - y_D < z_D$$

$$1 - \frac{1}{2} < \frac{1}{8f} , \quad \boxed{f < \frac{1}{4}}$$



Case II : on the z null cline

$$z|_{y_D} > z_D$$

$$z|_{y_B} < z_B$$

$$z|_{y_B} \approx \frac{q\, y_B}{y_B - 1} < \frac{q(3+2\sqrt{2})}{2f}$$

$$1 - y_D > \frac{1}{8f}$$

$$\boxed{\frac{1+\sqrt{2}}{2} > f > \frac{1}{4}}$$

Case III: on the z nullcline

$$z|_{y_B} > z_B$$

$$\frac{q\, y_B}{y_B - 1} > \frac{q\,(3+2\sqrt{2})}{2f} \Rightarrow \boxed{f > \frac{1+\sqrt{2}}{2}}$$

$$y_B = \frac{2+\sqrt{2}}{2} \qquad \frac{2+\sqrt{2}}{\sqrt{2}} > \frac{3+2\sqrt{2}}{2f}$$

$$f > \frac{3+2\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2+\sqrt{2}}$$

With ① f in the range $\frac{1}{4} < f < \frac{1+\sqrt{2}}{2}$

② steady state unstable

case I w/ unstable steady state
$\Rightarrow$ limit cycle solutions

$$T \approx \int_{AB} dt + \int_{CD} dt \qquad \text{for } 0 < \delta \ll 1$$

T is the $O(1)$ estimate for periodicity

$$T \approx \underbrace{\int_{z_A}^{z_B} \left(\frac{dz}{dt}\right)^{-1} dz}_{T_{AB}} + \underbrace{\int_{z_C}^{z_D} \left(\frac{dz}{dt}\right)^{-1} dz}_{T_{CD}}$$

$$T_{AB} = \int_{z_A}^{z_B} \frac{1}{x(y)-z}\, dz$$

for $q \ll 1$ along most of AB (part of the nullcline) $\quad x(y) \approx \frac{qy}{y-1} \sim q$

$$T_{AB} \approx \int_{z_A}^{z_B} \frac{1}{q-z}\, dz$$

$$= \ln \frac{z_A - q}{z_B - q}$$

$$= \ln \frac{\frac{1}{8f} - q}{\frac{q(3+2\sqrt{2})}{2f} - q}$$

$$\approx -\ln 4(3-2f+2\sqrt{2})q$$

this is upper bound for T_{AB} $\quad q \ll 1$

$x(y_A) \sim q + O(q^2) \quad q \ll 1$

$$x(y_B) = \frac{q\left[\frac{2+\sqrt{2}}{2}\right]}{\left[\frac{2+\sqrt{2}}{2} - 1\right]} = q(1+\sqrt{2})$$

$$T_{AB} = \int_{z_A}^{z_B} \frac{1}{q(1+\sqrt{2}) - z}\, dz = \ln\left[4(3-2f+2\sqrt{2})q\right]$$

$$\ln\left[4(3-2f+2\sqrt{2})q\right) < T_{AB} < -\ln\left[4(3-2f+2\sqrt{2})(1+\sqrt{2})q\right]$$

$$-\ln\left[4(3-2f+2\sqrt{2})(1+\sqrt{2})q\right] < T_{AB} < -\ln\left[4(3-2f+2\sqrt{2})q\right]$$

$$T_{CD} = \int_{Z_C}^{Z_D} \frac{1}{X(y) - Z} \, dZ$$

Between C & D $\quad X(y) \sim 1-y$

$$z(y) \sim \frac{y(1-y)}{2f} \qquad \frac{dz}{dy} = \frac{1-2y}{2f}$$

$$T_{CD} \approx \int_{y_C}^{y_D} \frac{\frac{1-2y}{2f}}{1-y - \frac{y(1-y)}{2f}} \, dy$$

$$= -\left[\frac{4f-1}{2f-1}\right] \ln \left[2^{\frac{1}{4f-1}} \cdot \frac{4f-1}{4f} \right]$$

$T_{AB} + T_{CD}$ is the period of oscillation up to $O(1)$

---

on AB

$$\frac{dz}{dt} \approx z - z \qquad z(t) = z_A e^{-t} + z(1-e^{-t})$$

on CD

$$x(y) \sim 1-y \qquad z \sim \frac{y(1-y)}{2f}$$

$$\frac{1-2y}{2f} \frac{dy}{dt} \approx (1-y) - \frac{y(1-y)}{2f}$$

$$\ln\left[\frac{1-y}{(2f-y)^{4f-1}}\right] = K + (2f-1)t$$

$y(t_c) = y_c$

$t = t_c + t_B$

t_B is found when $z = z_B$ $\qquad z_B = z_A e^{-t_B} + \xi(1 - e^{-t_B})$

$$t_B = \ln\left[\frac{z_A(1-\xi)}{z_B - \xi}\right]$$

$t_c \approx t_B$ $\qquad \ln\left[\dfrac{1-y_c}{(2f-y_c)^{4f-1}}\right] = K + (2f-1)t_B$

$$K = \ln\left[\frac{1-y_c}{(2f-y_c)^{4f-1}}\right] - (2f-1)\,t_B$$

- Field-Körös - Noyes Model
  (FKN) model

problem 4 $\qquad$ p277

problem 3