weakly nonlinear Oscillator :

$$\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0$$

two - timing substitutions

$O(1)$: $\partial_{\tau\tau} x_0 + x_0 = 0$

$O(\varepsilon)$: $\partial_{\tau\tau} x_1 + x_1 = -2\,\partial_{\tau T} x_0 - h$

$$x_0 = r(T)\cos(\tau + \phi(T))$$

$$2\left[r'\sin(\tau+\phi) + r\phi'\cos(\tau+\phi)\right] - h = \partial_{\tau\tau} x_1 + x_1$$

$$h(\theta) = \sum_{k=0}^{\infty} a_k \cos k\theta + \sum_{k=1}^{\infty} b_k \sin\theta$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} h \, \cancel{\cos\theta} \, d\theta$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} h \cdot \cos k\theta \, d\theta \qquad k \geq 1$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} h \sin k\theta \, d\theta \qquad k \geq 1$$

for van der Pol oscillator

$$h = (x^2-1)\dot{x} = (r^2\cos^2\theta - 1)(-r\sin\theta)$$

by demanding terms involving $\sin(\tau+\phi)$ & $\cos(\tau+\phi)$ to vanish,

we have $\quad r' = \langle h \sin\theta \rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\sin\theta\, d\theta$

$$= \langle (r^2\cos^2\theta - 1)(-r\sin\theta)\sin\theta \rangle$$

$$= r \langle \sin^2\theta \rangle - r^3 \langle \cos^2\theta \sin^2\theta \rangle$$

$$= \frac{1}{2} r - \frac{1}{8} r^3 \qquad r(0) = 1$$

$$r\phi' = \langle h \cdot \cos\theta \rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\cos\theta\, d\theta$$

$$= \langle (r^2\cos^2\theta - 1)(-r\sin\theta)\cos\theta \rangle$$

$$= r\langle \sin\theta\cos\theta \rangle - r^3 \langle \cos^3\theta \sin\theta \rangle$$

$$= 0 - 0 = 0$$

$$\int \frac{8\,dr}{r(4-r^2)} = \int dT \qquad r(T) = 2(1+3e^{-T})^{-1/2}$$

$$x(t,\varepsilon) \sim x_0(\tau, T) + O(\varepsilon)$$

$$= \frac{2}{\sqrt{1+3e^{-\varepsilon t}}} \cos t + O(\varepsilon)$$

Now consider the Duffing oscillator:

$$\ddot{x} + x + \varepsilon x^3 = 0$$

$$h = \cancel{\ } x^3 = r^3 \cos^3\theta$$

$$r' = \langle h \sin\theta \rangle = r^3 \langle \cos^3\theta \sin\theta \rangle = 0$$

$$r\phi' = \langle h \cos\theta \rangle = r^3 \langle \cos^4\theta \rangle = \frac{3}{8} r^3$$

$$r = a = \text{constant}$$

$$\phi' = \frac{3}{8} a^2$$

$$\omega = 1 + \varepsilon \phi' = 1 + \frac{3}{8} a^2 \cdot \varepsilon$$

$\varepsilon > 0$ hardening spring

$\varepsilon < 0$ softening spring