$$\frac{\partial X}{\partial t} = F(X) + D\nabla^2 X$$

$$\left.\frac{\partial X}{\partial x}\right|_{\pm \xi/2} = 0 \quad \text{for} \quad X^\lambda \text{ in } \quad -\xi/2 \le x \le \xi/2$$

uniform steady soln. $X_0 \rightarrow X = X_0 + u$

$$\frac{\partial u}{\partial t} = \left(\mathcal{L} + D\nabla^2\right) u + Muu + Nuuu$$

$$\mathcal{L}_{ij} = \left.\frac{\partial F_i}{\partial X_{0j}}\right|_{X_0} = \frac{\partial F_i(X_0)}{\partial X_{0j}}$$

$$(Muu)_i \equiv \sum_{j,k} \frac{1}{2!} \frac{\partial^2 F_i(X_0)}{\partial X_{0j}\,\partial X_{0k}} u_j u_k$$

$$(Nuuu)_i \equiv \sum_{j,k,l} \frac{1}{3!} \frac{\partial^3 F_i(X_0)}{\partial X_{0j}\,\partial X_{0k}\,\partial X_{0l}} u_j u_k u_l$$

- two time scales $t, \tau \Rightarrow \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial \tau}$
- large wave length variation : $s = \varepsilon r$, $\nabla \rightarrow \varepsilon \nabla_s$
- $\mathcal{L} = \mathcal{L}_0 + \varepsilon^2 \chi \mathcal{L}_1 + \varepsilon^4 \mathcal{L}_2 + \cdots$

$M = M_0 + \varepsilon^2 \chi M_1 + \varepsilon^4 M_2 + \cdots$

$N = N_0 + \varepsilon^2 \chi N_1 + \varepsilon^4 N_2 + \cdots$

$$\frac{\partial}{\partial t} - \mathcal{L} - D\nabla^2 \longrightarrow \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial \tau} - \varepsilon^2 D \nabla_s^2 - \mathcal{L}_0 - \varepsilon^2 \chi \mathcal{L}_1 - \cdots$$

$$u \to \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$$

$$Muu \to \varepsilon^2 M_0 u_1 u_1 + \varepsilon^3 2 M_0 u_1 u_2$$ because M_0 is symmetric w.r.t. u_1 & u_2

$$Nuuu \to \varepsilon^3 N_0 u_1 u_1 u_1$$

$\therefore$

$$\frac{\partial u}{\partial t} = (\mathcal{L} + D\nabla^2) u + Muu + Nuuu$$

$$\left(\frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial \tau} - \varepsilon^2 D \nabla_s^2 - \mathcal{L}_0 - \varepsilon^2 \chi \mathcal{L}_1 \cdots \right)(\varepsilon u_1 + \varepsilon^2 u_2 + \cdots)$$

$$= \varepsilon^2 M_0 u_1 u_1 + \varepsilon^3 (2 M_0 u_1 u_2 + N_0 u_1 u_1 u_1) + \mathcal{O}(\varepsilon^4)$$

- 1st order in ε

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_0\right) u_1 = 0 \quad \Longrightarrow \quad \left(\frac{\partial}{\partial t} - \mathcal{L}_0\right) u_\nu = \tilde{B}_\nu$$

$\nu = 1, 2, 3, \cdots$

$$\tilde{B}_\nu = \sum_{l=-\infty}^{\infty} \hat{B}_\nu^{(l)}(\tau, s) e^{il\omega_0 t}$$

- 2nd order in ε

$$\left(\frac{\partial}{\partial \tau} \cancel{- D\nabla_s^2 - \chi \mathcal{L}_1}\right)\left(\frac{\partial}{\partial t} - \mathcal{L}_0\right) u_2 = M_0 u_1 u_1$$

$U^* \hat{B}_\nu^{(1)}(\tau, s) = 0$ is the solvability condition

- 3rd order in ε

$$\left(\frac{\partial}{\partial\tau} - D\nabla_s^2 - \chi\mathcal{L}_1\right) u_1 = 2M_0 u_1 u_2 + N_0 u_1 u_1 u_1$$
$$+\left(\frac{\partial}{\partial t} - \mathcal{L}_0\right) u_3$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_0\right) u_3 = -\left(\frac{\partial}{\partial\tau} - D\nabla_s^2 - \chi\mathcal{L}_1\right) u_1 + 2M_0 u_1 u_2 + N_0 u_1 u_1 u_1$$

$\chi \equiv \frac{\mu}{\varepsilon^2}$ where μ is a real scalar parameter s.t.

$F(X_0(\mu); \mu) = 0$ $\quad$ $\mu = 0$ is where the bifurcation occurs

- $\mathcal{L}_0$ is the linearized version of $F(X_0)$

and $\mathcal{L}_0 \cdot U = \lambda_0 U$, $\quad \mathcal{L}_0 \bar{U} = \bar{\lambda}_0 \bar{U}$
$\quad U^* \mathcal{L}_0 = \lambda_0 U^*$, $\quad \bar{U}^* \mathcal{L}_0$

$$U^* \bar{U} = \bar{U}^* U = 0$$
$$U^* U = \bar{U}^* \bar{U} = 1$$

$$u_1(t, \tau, s) = W(\tau, s)\, U \cdot e^{i\omega_0 t} + c.c.$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_0\right) u_2 = M_0 \cdot W U e^{i\omega_0 t}\, W U e^{i\omega_0 t} + c.c.$$

$$u_2 = V_+ W^2 e^{2i\omega_0 t} + V_- \bar{W}^2 e^{-2i\omega_0 t} + V_0 |W|^2 + v_0 u_1$$
$$V_+ = \bar{V}_- = -(\mathcal{L}_0 - 2i\omega_0)^{-1} M_0 U U \qquad V_0 = -2\mathcal{L}_0^{-1} M_0 U \bar{U}$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_0\right) u_\nu = \hat{B}_\nu$$

$$\int_0^{2\pi/\omega_0} U^* \cdot \tilde{B}_\nu e^{-i\omega_0 t} dt = \int_0^{2\pi/\omega_0} \left[U^* \left(\frac{\partial}{\partial t} - \mathcal{L}_0\right) u_\nu\right] e^{-i\omega_0 t} dt$$

$$= \int_0^{2\pi/\omega_0} \left(i\omega_0 U^* \cdot u_\nu - i\omega_0 U^* \cdot u_\nu\right) e^{-i\omega_0 t} dt = 0$$

$\therefore \displaystyle\int_0^{2\pi/\omega_0} U^* \cdot \tilde{B}_\nu e^{-i\omega_0 t} dt = 0$ is the solvability condition

$$\tilde{B}_3^{(1)} = -\left(\frac{\partial}{\partial \tau} - \chi \mathcal{L}_1 - D \nabla_s^2\right) WU$$
$$+ \left(2 M_0 U V_0 + 2 M_0 \bar{U} V_+ + 3 N_0 U U \bar{U}\right) |W|^2 W$$

$U^* \tilde{B}_3^{(1)} = 0$ solvability condition at $\nu = 3$

gives

$$\frac{\partial W}{\partial \tau} = \chi \lambda_1 W + d \nabla_s^2 W - g |W|^2 W$$

$$d \equiv d' + i d'' = U^* D U$$

$$\lambda_1 = \sigma_1 + i\omega_1 = U^* \mathcal{L}_1 U$$

$$g = g' + i g'' = -2 U^* M_0 U V_0 - 2 U^* M_0 \bar{U} V_+ - 3 U^* N_0 U U \bar{U}$$