$$\frac{\partial u}{\partial t} = u(1-u) + \frac{\partial^2 u}{\partial x^2}$$

$$F(u) = u(1-u)$$

$$\mathcal{L}_{ij} = \mathcal{L}_{11} = 1 - 2u_0$$

$$(Muu)_i = \sum_{j,k} \frac{1}{2!} \frac{\partial^2 F_i}{\partial X_{0j} \partial X_{0k}} u_j u_k = 0$$

$$(Nuuu)_i = \sum_{j,k,l} \frac{1}{3!} \frac{\partial^3 F_i}{\partial X_{0j} \partial X_{0k} \partial X_{0l}} u_j u_k u_l = 0$$

$$u = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\frac{\partial X}{\partial t} = A - (B+1) X + X^2 Y + D_X \nabla^2 X$$

$$(M_0 uu)_1 = Y_0 a a + 2X_0 ab + X_0 ba$$

$$\frac{\partial Y}{\partial t} = B X - X^2 Y + D_Y \nabla^2 Y$$

$$(M_0 uu)_2 = -Y_0 a a - 2X_0 ab - X_0 ba$$

$$(X_0, Y_0) = (A, B/A)$$

$$(X, Y) = (X_0 + \xi, Y_0 + \eta)$$

$$\frac{\partial \xi}{\partial t} = (B-1)\xi + A^2 \eta + D_X \nabla^2 \xi + f(\xi, \eta)$$

$$\frac{\partial \eta}{\partial t} = -B\xi - A^2 \eta + D_Y \nabla^2 \eta - f(\xi, \eta)$$

$$f(\xi, \eta) = \frac{B}{A}\xi^2 + 2A\xi \cdot \eta + \xi^2 \cdot \eta$$

assuming $\xi, \eta \sim e^{i \underline{q} \cdot \underline{r} + \lambda t}$

$$\lambda^2 + \alpha(q)\lambda + \beta(q) = 0$$

$$\alpha(q) = 1 + A^2 - B + (D_x + D_y) q^2$$

$$\beta(q) = A^2 + [A^2 D_x + (1-B) D_y] q^2 + D_x D_y q^4$$

(X_0, Y_0) linearly stable iff $\alpha(q) \geq 0$, $\beta(q) \geq 0$ for all q

(i) $\alpha(q) \to 0$ for some q,

(ii) $\beta(q)$ vanishes for some q

Type I: $1 + A^2 - B + (D_x + D_y) q^2 = 0$

Critical wave #: $q_c = 0$ $\quad 1 + A^2 - B_c = 0$

$$\boxed{B_c = 1 + A^2}$$

Type II: $\beta(q_c) = 0$, $\left.\dfrac{d\beta}{dq}\right|_{q_c} = 0$ $\qquad B_c' = \left(1 + A\sqrt{D_x/D_y}\right)^2$

$$q_c^2 = \frac{A}{\sqrt{D_x D_y}}$$

Assume $B_c < B_c'$

$$\gamma \equiv \sqrt{D_x/D_y} > A^{-1}(\sqrt{1+A^2}-1)$$

to ensure that Type I instability occurs first as B is increased

$\Rightarrow$ $\frac{D_x}{D_y} > 1$ is sufficient in this inequality b.c. $\frac{\sqrt{1+A^2}-1}{A} < 1$ for all A

at criticality, eigenvalues $\lambda = \pm iA$ @ $B = B_c$

further define $\mu = \frac{B - B_c}{B_c}$ $\quad B \to B_c + \mu B_c$

$$\mathcal{L}_0 = \begin{pmatrix} A^2 & A^2 \\ -(1+A^2) & -A^2 \end{pmatrix} \qquad \mathcal{L}_1 = (1+A^2)\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

$$u_0 = \begin{pmatrix} 1 \\ -1 + iA^{-1} \end{pmatrix}$$

$$D = \begin{pmatrix} D_x & 0 \\ 0 & D_y \end{pmatrix}$$

$$u_0^* = \frac{1}{2}(1 - iA, \ -iA)$$

$$\lambda_1 = \frac{1+A^2}{2}$$

$$d = \frac{1}{2}\left[D_x + D_y - iA(D_x - D_y) \right]$$

$$(M_0 ab)_\xi = -(M_0 ab)_\eta = \frac{1+A^2}{A} a_\xi b_\xi + A(a_\xi b_\eta + a_\eta b_\xi)$$

$$(N_0 abc)_\xi = -(N_0 abc)_\eta = \frac{1}{3}(a_\xi b_\xi c_\eta + a_\xi b_\eta c_\xi + a_\eta b_\xi c_\xi)$$

$$V_+ = \bar{V}_- = \frac{(1+iA)^3}{3A^3}\begin{pmatrix} -2iA \\ 1+2iA \end{pmatrix}$$

$$V_0 = \frac{2(A^2-1)}{A^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$g = \frac{1}{2}\left(\frac{2+A^2}{A^2} + i \cdot \frac{4-7A^2+4A^4}{3A^3} \right)$$

$$\Rightarrow \quad W \to W \cdot e^{iC_0 t}$$

$$\frac{\partial W}{\partial t} = W + (1+iC_1)\nabla^2 W - (1+iC_2)|W|^2 W$$

$$C_1 = -A \cdot \frac{D_x - D_y}{D_x + D_y} = -A \frac{\gamma^2-1}{\gamma^2+1}$$

$$C_2 = \frac{4-7A^2+4A^4}{3A(2+A^2)}$$

Plane wave soln. of G-L eq.

$$\frac{\partial W}{\partial t} = W - (1+iC_2)|W|^2 W + (1+iC_1)\frac{\partial^2 W}{\partial x^2}$$

$$C_1 = d''/d' \qquad d' + id'' = U^* D U$$

$$C_2 = g''/g' \qquad g = g' + ig'' = -2U^* M_0 U V_0 - 2U^* M_0 \bar{U} V_+ - 3U^* N_0 U U \bar{U}$$

Plane wave soln. $\quad W_Q(x,t) = R_Q\, e^{i(Qx - \omega_Q t)} \qquad |Q| < 1$

$$R_Q = \sqrt{1-Q^2} \qquad \omega_Q = C_1 Q^2 + (1-Q^2) C_2$$

Stability of plane wave soln.

$$W(x,t) = W_Q(x,t) + u(x,t)\, e^{i(Qx - \omega_Q t)}$$

$$\frac{\partial u}{\partial t} = \left[-(1+iC_2)(1-Q^2) - 2(C_1 - i) Q \frac{\partial}{\partial x} + (1+iC_1)\frac{\partial^2}{\partial x^2}\right] u - (1+iC_2)(1-Q^2)\bar{u}$$